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Normality of Maximal Orbit Closures for Euclidean Quivers

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Abstract. Let Δ be a Euclidean quiver. We prove that the closures of the maximal orbits in the varieties of representations of Δ are normal and Cohen–Macaulay (even complete intersections). Moreover, we give a generalization of this result for the tame concealed-canonical algebras.

Introduction and the Main Results

Throughout the paper, k is a fixed algebraically closed field. By $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_+$, we denote the sets of the integers, the non-negative integers and the positive integers, respectively. Finally, if $i, j \in \mathbb{Z}$, then $[i, j] := \{l \in \mathbb{Z} \mid i \leq l \leq j\}$ (in particular, $[i, j] = \varnothing$ if $i > j$).

Let A be a finite-dimensional k-algebra. Given a non-negative integer d, one defines $\operatorname{mod}_A(d)$ as the set of all k-algebra homomorphisms from A to the algebra $\mathbb{M}_{d\times d}(k)$ of $d \times d$-matrices. This set has a structure of an affine variety and its points represent d-dimensional A-modules. Consequently, we call $\operatorname{mod}_A(d)$ the variety of A-modules of dimension d. The general linear group $\mathrm{GL}(d)$ acts on $\operatorname{mod}_A(d)$ by conjugation: $(g\cdot m)(a) := gm(a)g^{-1}$ for $g \in \mathrm{GL}(d)$, $m \in \operatorname{mod}_A(d)$ and $a \in A$. The orbits with respect to this action are in one-to-one correspondence with the isomorphism classes of the d-dimensional A-modules. Given a d-dimensional A-module M, we denote the orbit in $\operatorname{mod}_A(d)$ corresponding to the isomorphism class of M by $\mathcal{O}(M)$ and its Zariski-closure by $\overline{\mathcal{O}(M)}$.

Singularities appearing in the orbit closures of the above form are the object of intensive studies (see, for example, [1, 4, 12, 15, 33, 42, 48–51], and we also refer to a survey article of Zwara [52]). In particular, Zwara and the author [11] proved that if A is a hereditary algebra of Dynkin type $\mathbb{A}$ or $\mathbb{D}$, then $\overline{\mathcal{O}(M)}$ is a normal Cohen–Macaulay variety, which has rational singularities if the characteristic of k is 0. Recall that Gabriel [28] proved that the hereditary algebras of Dynkin type are precisely the hereditary algebras of finite representation type. Thus it is an interesting question if the orbit closures have good geometric properties for all hereditary algebras of finite representation type. The remaining case of hereditary algebras of type $\mathbb{E}$ is still open, but there are some partial results in this direction [46]. On the other hand, Zwara [47] exhibited an example of a module over the Kronecker algebra whose orbit closure is neither normal nor Cohen–Macaulay. This example generalizes easily to an

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arbitrary hereditary algebra of infinite representation type [19]. However, it is still an interesting problem to determine for which classes of modules over hereditary algebras of infinite representation type the corresponding orbit closures have good properties. In the paper, we study modules M such that $\mathcal{O}(M)$ is maximal, *i.e.*, there is no module N such that $\mathcal{O}(M) \subseteq \overline{\mathcal{O}(N)}$ and $\mathcal{O}(M) \neq \mathcal{O}(N)$.

According to Drozd's famous Tame and Wild Theorem [21, 26], the finite-dimensional algebras of infinite representation type can be divided into two disjoint classes. One class consists of the tame algebras, for which the indecomposable modules of a given dimension form a finite number of one-parameter families. The other class consists of the wild algebras, for which the classification of the indecomposable modules is as complicated as the classification of two non-commuting endomorphisms of a finite-dimensional vector space, hence is considered to be hopeless. There are examples showing that varieties of modules over tame algebras have often better properties than those over wild algebras (see for example [8, 20, 40, 41]). Consequently, we concentrate in the paper on the maximal orbits over the tame hereditary algebras. We recall that the tame hereditary algebras are precisely the hereditary algebras of Euclidean type.

The following theorem is the main result of the paper.

Theorem 1 *Let M be a module over a tame hereditary algebra. If $\mathcal{O}(M)$ is maximal, then $\overline{\mathcal{O}(M)}$ is a normal complete intersection (in particular, Cohen–Macaulay).*

It is known (see for example [35, Corollary 3.6]) that $\mathcal{O}(M)$ is maximal for each indecomposable module M over a tame hereditary algebra. Consequently, we get the following.

Corollary 2 *If M is an indecomposable module over a tame hereditary algebra, then $\overline{\mathcal{O}(M)}$ is a normal complete intersection (in particular, Cohen–Macaulay).*

Now we present the strategy of the proof of Theorem 1. Let M be a module over a tame hereditary algebra A such that $\mathcal{O}(M)$ is maximal. If $\operatorname{Ext}^1_A(M, M) = 0$, then it is well known that $\overline{\mathcal{O}(M)}$ is smoothly equivalent to an affine space, hence the claim is obvious in this case. Thus we may concentrate on the case $\operatorname{Ext}^1_A(M, M) \neq 0$. It follows from [35, proof of Corollary 3.6] that in this situation M is periodic with respect to the action of the Auslander–Reiten translation τ (see Section 1). Consequently, Theorem 1 follows from the next theorem.

Theorem 3 *Let M be a τ-periodic module over a tame hereditary algebra. If $\mathcal{O}(M)$ is maximal, then $\overline{\mathcal{O}(M)}$ is a complete intersection (in particular, Cohen–Macaulay).*

If A is a tame hereditary algebra, then the τ-periodic A-modules are direct sums of indecomposable modules, which lie in the sincere separating family of tubes in the Auslander–Reiten quiver of A. Existence of such families characterizes the concealed-canonical algebras [32, 39]. Recall [31] that an algebra A is called *concealed-canonical* if there exists a tilting bundle over a weighted projective line whose endomorphism ring is isomorphic to A. Thus it is natural to try to generalize Theorem 3 to the case of tame concealed-canonical algebras. Before we formulate this generalization, we present some necessary definitions.

Let A be a tame concealed-canonical algebra. For an A-module M, we denote by $\mathbf{dim}\, M$ its dimension vector, *i.e.*, the sequence indexed by the isomorphism classes of the simple A-modules that counts the multiplicities of the composition factors in the Jordan–Hölder filtration of M. In general, a sequence of non-negative integers indexed by the isomorphism classes of the simple A-modules is called a *dimension vector*. We call a dimension vector $\mathbf{d}$ *singular* if $\langle \mathbf{d}, \mathbf{d} \rangle_A = 0$ and there exists a dimension vector $\mathbf{x}$ such that $\mathbf{x} \leq \mathbf{d}$, $\langle \mathbf{x}, \mathbf{x} \rangle_A = 0$ and $|\langle \mathbf{x}, \mathbf{d} \rangle_A| = 2$, where $\langle -, - \rangle_A$ denotes the corresponding homological bilinear form (see Section 1). In Proposition 2.3 we describe the tame concealed-canonical algebras for which there exist singular dimension vectors. In particular, this description implies that singular dimension vectors do not exist for the tame hereditary algebras.

We have the following generalization of Theorem 3.

Theorem 4 *Let M be a τ-periodic module over a tame concealed-canonical algebra such that $\mathcal{O}(M)$ is maximal. Then $\overline{\mathcal{O}(M)}$ is a complete intersection (in particular, Cohen–Macaulay). Moreover, $\overline{\mathcal{O}(M)}$ is not normal if and only if* $\mathbf{dim}\, M$ *is singular and $\tau M \simeq M$.*

In this paper we concentrate on the proof of Theorem 4. Instead of using the framework of modules over algebras and the corresponding varieties, we use the framework of representations of quivers (and the corresponding varieties). Gabriel's Theorem [28] says that we may do this replacement on the level of modules and representations, while a result of Bongartz [14] justifies this passage on the level of varieties. For background on the representation theory of quivers we refer to [2,37,38].

The paper is organized as follows. In Section 1 we recall basic information about quivers and their representations. Next, in Section 2 we gather facts about the categories of modules over the tame concealed-canonical algebras. In Section 3 we introduce varieties of representations of quivers, while in Section 4 we review facts on semi-invariants with particular emphasis on the case of tame concealed-canonical algebras. Next, in Section 5 we present a series of facts that we later use in Sections 6 and 7 to study orbit closures for the non-singular and singular dimension vectors, respectively. Moreover, in Section 7 we make a remark about relationship between the degenerations and the hom-order for the tame concealed-canonical algebras. Finally, in Section 8 we give the proof of Theorem 4.

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1 Quivers and Their Representations

By a *quiver* Δ we mean a finite set Δ_0 (called the set of *vertices* of Δ) together with a finite set Δ_1 (called the set of *arrows* of Δ) and two maps $s, t \colon \Delta_1 \to \Delta_0$, which assign to each arrow α its *starting vertex* $s\alpha$ and *terminating vertex* $t\alpha$, respectively. By a *path of length* $n \in \mathbb{N}_+$ in a quiver Δ we mean a sequence $\sigma = (\alpha_1, \dots, \alpha_n)$ of arrows such that $s\alpha_i = t\alpha_{i+1}$ for each $i \in [1, n-1]$. In particular, we treat every arrow of Δ as a path of length 1. In the above situation we put $\ell\sigma := n$, $s\sigma := s\alpha_n$ and $t\sigma := t\alpha_1$. Moreover, for each vertex x we have a *trivial path* $\mathbf{1}_x$ at x such that $\ell\mathbf{1}_x := 0$ and $s\mathbf{1}_x := x =: t\mathbf{1}_x$. A subquiver Δ' of a quiver Δ is called *convex* if $\alpha_i \in \Delta_1'$ for each $i \in [1, n]$, provided $(\alpha_1, \dots, \alpha_n)$ is a path in Δ such that $t\alpha_1, s\alpha_n \in \Delta_0'$.

For the rest of the paper we assume that the considered quivers do not have oriented cycles, where by an oriented cycle we mean a path σ of positive length such that $s\sigma = t\sigma$.

Let Δ be a quiver. We define its *path category* $k\Delta$ to be the category whose objects are the vertices of Δ and, for $x, y \in \Delta_0$, the morphisms from x to y are the formal k-linear combinations of paths starting at x and terminating at y. For $x, y \in \Delta_0$ we denote by $k\Delta(x, y)$ the space of the morphisms from x to y in $k\Delta$. If $\omega \in k\Delta(x, y)$ for $x, y \in \Delta_0$, then we write $s\omega := x$ and $t\omega := y$. By a *representation* of Δ we mean a functor from $k\Delta$ to the category $\operatorname{mod} k$ of finite-dimensional vector spaces. We denote the category of representations of Δ by $\operatorname{rep} \Delta$. Observe that every representation of Δ is uniquely determined by its values on the vertices and the arrows. Given a representation M of Δ, we denote by $\mathbf{dim}\, M$ its *dimension vector* defined by $(\mathbf{dim}\, M)(x) := \dim_k M(x)$ for $x \in \Delta_0$. Observe the $\mathbf{dim}\, M \in \mathbb{N}^{\Delta_0}$ for each representation M of Δ. We call the elements of $\mathbb{N}^{\Delta_0}$ *dimension vectors*. A dimension vector $\mathbf{d}$ is called sincere if $\mathbf{d}(x) \neq 0$ for each $x \in \Delta_0$.

By a *relation* in a quiver Δ we mean a k-linear combination of paths of lengths at least 2 having a common starting vertex and a common terminating vertex. Note that each relation in a quiver Δ is a morphism in $k\Delta$. A set R of relations in a quiver Δ is called *minimal* if $\langle R \setminus \{\rho\} \rangle \neq \langle R \rangle$ for each $\rho \in R$, where for a set X of morphisms in $k\Delta$ we denote by $\langle X \rangle$ the ideal in $k\Delta$ generated by X. Observe that each minimal set of relations is finite. By a *bound quiver* $\boldsymbol{\Delta}$ we mean a quiver Δ together with a minimal set R of relations. Given a bound quiver $\boldsymbol{\Delta}$ we denote by $k\boldsymbol{\Delta}$ its path category, *i.e.*, $k\boldsymbol{\Delta} := k\Delta/\langle R \rangle$. Moreover, for $x, y \in \Delta_0$ we denote by $k\boldsymbol{\Delta}(x, y)$ the space of the morphisms from x to y in $k\boldsymbol{\Delta}$. By a *representation* of a bound quiver $\boldsymbol{\Delta}$ we mean a functor from $k\boldsymbol{\Delta}$ to $\operatorname{mod} k$. In other words, a representation of $\boldsymbol{\Delta}$ is a representation M of Δ such that $M(\rho) = 0$ for each $\rho \in R$. We denote the category of representations of a bound quiver $\boldsymbol{\Delta}$ by $\operatorname{rep} \boldsymbol{\Delta}$. Moreover, we denote by $\operatorname{ind} \boldsymbol{\Delta}$ the full subcategory of $\operatorname{rep} \boldsymbol{\Delta}$ consisting of the indecomposable representations. It is known that $\operatorname{rep} \boldsymbol{\Delta}$ is an abelian Krull–Schmidt category.

A bound quiver $\boldsymbol{\Delta}'$ is called a *convex subquiver* of a bound quiver $\boldsymbol{\Delta}$ if Δ' is a convex subquiver of Δ and $R' = R \cap k\Delta'$. If $\boldsymbol{\Delta}'$ is a convex subquiver of a bound quiver $\boldsymbol{\Delta}$, then $\operatorname{rep} \boldsymbol{\Delta}'$ can be naturally identified with an exact subcategory of $\operatorname{rep} \boldsymbol{\Delta}$, where by an exact subcategory of $\operatorname{rep} \boldsymbol{\Delta}$ we mean a full subcategory $\mathcal{X}$ of $\operatorname{rep} \boldsymbol{\Delta}$ such that $\mathcal{X}$ is an abelian category and the inclusion functor $\mathcal{X} \hookrightarrow \operatorname{rep} \boldsymbol{\Delta}$ is exact. In particular, if $\boldsymbol{\Delta}'$ is a convex subquiver of a tame bound quiver $\boldsymbol{\Delta}$, then $\boldsymbol{\Delta}'$ is either tame or representation-finite (we say that a bound quiver $\boldsymbol{\Delta}$ is *tame/representation-finite* if $\operatorname{rep} \boldsymbol{\Delta}$ is of tame/finite representation type, respectively).

Let $\boldsymbol{\Delta}$ be a bound quiver. For each vertex x of Δ we denote by S_x the simple representation at x, *i.e.*, $S_x(x) := k$, $S_x(y) := 0$ for $y \in \Delta_0 \setminus \{x\}$, and $S_x(\alpha) := 0$ for $\alpha \in \Delta_1$. More generally, if $\mathbf{d}$ is a dimension vector, then we put $S^{\mathbf{d}} := \bigoplus_{x \in \Delta_0} S_x^{\mathbf{d}(x)}$. Next, for each vertex x we denote by P_x the *projective representation at* x defined in the following way: $P_x(y) := k\boldsymbol{\Delta}(x, y)$ for $y \in \Delta_0$ and $P_x(\omega)$ is the composition (on the left) with ω for a morphism ω in $k\boldsymbol{\Delta}$. If M is a representation of $\boldsymbol{\Delta}$ and $x \in \Delta_0$, then according to Yoneda's Lemma the map

$$\operatorname{Hom}_{\boldsymbol{\Delta}}(P_x, M) \to M(x), \quad f \mapsto f(\mathbf{1}_x),$$

is an isomorphism. In particular, this implies that

$$\operatorname{Hom}_{\boldsymbol{\Delta}}(P_x, P_y) \simeq k\boldsymbol{\Delta}(y, x)$$

for any $x, y \in \Delta_0$. For $\omega \in k\boldsymbol{\Delta}(y, x)$ we denote the corresponding map $P_x \to P_y$ by P_ω. Observe that P_ω is the composition (on the right) with ω. Moreover, if M is a representation of $\boldsymbol{\Delta}$, then, under the Yoneda isomorphisms, the induced map

$$\operatorname{Hom}_{\boldsymbol{\Delta}}(P_\omega, M)\colon \operatorname{Hom}_{\boldsymbol{\Delta}}(P_y, M) \to \operatorname{Hom}_{\boldsymbol{\Delta}}(P_x, M), \quad f \mapsto f \circ P_\omega,$$

can be identified with $M(\omega)$.

Let $\boldsymbol{\Delta}$ be a bound quiver. An important role in the representation theory of quivers is played by the *Auslander–Reiten translations* τ and τ^- (see [2, Section IV.2] for the definition). We list their properties which we need in our proofs. First, $\tau M = 0$ ($\tau^- M = 0$) if and only if M is projective (injective, respectively). Moreover, $\tau^-\tau X \simeq X$ ($\tau\tau^- X \simeq X$) for each indecomposable representation X of $\boldsymbol{\Delta}$, which is not projective (injective, respectively). We say that a representation M of $\boldsymbol{\Delta}$ is *periodic* if there exists $n \in \mathbb{N}_+$ such that $\tau^n M \simeq M$. We have a celebrated Auslander–Reiten formula, which implies that

$$\dim_k \operatorname{Ext}^1_{\boldsymbol{\Delta}}(M, N) = \dim_k \operatorname{Hom}_{\boldsymbol{\Delta}}(N, \tau M)$$

for any representations M and N of $\boldsymbol{\Delta}$ such that $\operatorname{pdim}_{\boldsymbol{\Delta}} M \leq 1$. Dually, if M and N are representations of $\boldsymbol{\Delta}$ and $\operatorname{idim}_{\boldsymbol{\Delta}} N \leq 1$, then

$$\dim_k \operatorname{Ext}^1_{\boldsymbol{\Delta}}(M, N) = \dim_k \operatorname{Hom}_{\boldsymbol{\Delta}}(\tau^- N, M).$$

Let $\boldsymbol{\Delta}$ be a bound quiver. We define the corresponding Tits forms

$$\langle -, - \rangle_{\boldsymbol{\Delta}}\colon \mathbb{Z}^{\Delta_0} \times \mathbb{Z}^{\Delta_0} \to \mathbb{Z} \quad \text{and} \quad q_{\boldsymbol{\Delta}}\colon \mathbb{Z}^{\Delta_0} \to \mathbb{Z}$$

by

$$\langle \mathbf{d}', \mathbf{d}'' \rangle_{\boldsymbol{\Delta}} := \sum_{x \in \Delta_0} \mathbf{d}'(x)\mathbf{d}''(x) - \sum_{\alpha \in \Delta_1} \mathbf{d}'(s\alpha)\mathbf{d}''(t\alpha) + \sum_{\rho \in R} \mathbf{d}'(s\rho)\mathbf{d}''(t\rho),$$

for $\mathbf{d}', \mathbf{d}'' \in \mathbb{Z}^{\Delta_0}$, and $q_{\boldsymbol{\Delta}}(\mathbf{d}) := \langle \mathbf{d}, \mathbf{d} \rangle_{\boldsymbol{\Delta}}$, for $\mathbf{d} \in \mathbb{Z}^{\Delta_0}$. Bongartz [13, Proposition 2.2] proved that

$$\langle \mathbf{dim}\, M, \mathbf{dim}\, N \rangle_{\boldsymbol{\Delta}} = \dim_k \operatorname{Hom}_{\boldsymbol{\Delta}}(M, N) - \dim_k \operatorname{Ext}^1_{\boldsymbol{\Delta}}(M, N) + \dim_k \operatorname{Ext}^2_{\boldsymbol{\Delta}}(M, N)$$

for any $M, N \in \operatorname{rep} \boldsymbol{\Delta}$, provided $\operatorname{gl.dim} \boldsymbol{\Delta} \leq 2$.

2 Separating Exact Subcategories

In this section we present facts about sincere separating exact subcategories that we use in our considerations. For the proofs we refer to [32, 36].

Let $\mathbf{\Delta}$ be a bound quiver and $\mathcal{X}$ a full subcategory of $\operatorname{ind}\mathbf{\Delta}$. We denote by $\operatorname{add}\mathcal{X}$ the full subcategory of $\operatorname{rep}\mathbf{\Delta}$ formed by the direct sums of representations from $\mathcal{X}$. We say that $\mathcal{X}$ is an *exact subcategory* of $\operatorname{ind}\mathbf{\Delta}$ if $\operatorname{add}\mathcal{X}$ is an exact subcategory of $\operatorname{rep}\mathbf{\Delta}$. We put

$$\mathcal{X}_+ := \{X \in \operatorname{ind}\mathbf{\Delta} : \operatorname{Hom}_{\mathbf{\Delta}}(\mathcal{X}, X) = 0\}$$

and

$$\mathcal{X}_- := \{X \in \operatorname{ind}\mathbf{\Delta} : \operatorname{Hom}_{\mathbf{\Delta}}(X, \mathcal{X}) = 0\}.$$

Let $\mathbf{\Delta}$ be a bound quiver. Following [32] we say that $\mathcal{R}$ is a *sincere separating exact subcategory* of $\operatorname{ind}\mathbf{\Delta}$ provided the following conditions are satisfied:

(1) $\mathcal{R}$ is an exact subcategory of $\operatorname{ind}\mathbf{\Delta}$ consisting of periodic representations;
(2) $\operatorname{ind}\mathbf{\Delta} = \mathcal{R}_+ \cup \mathcal{R} \cup \mathcal{R}_-$;
(3) $\operatorname{Hom}_{\mathbf{\Delta}}(X, \mathcal{R}) \neq 0$ for each $X \in \mathcal{R}_+$ and $\operatorname{Hom}_{\mathbf{\Delta}}(\mathcal{R}, X) \neq 0$ for each $X \in \mathcal{R}_-$;
(4) $P \in \mathcal{R}_+$ for each indecomposable projective representation P of $\mathbf{\Delta}$ and $I \in \mathcal{R}_-$ for each indecomposable injective representation I of $\mathbf{\Delta}$.

Lenzing and de la Peña [32] proved that there exists a sincere separating exact subcategory $\mathcal{R}$ of $\operatorname{ind}\mathbf{\Delta}$ if and only if $\mathbf{\Delta}$ is concealed-canonical, *i.e.*, $\operatorname{rep}\mathbf{\Delta}$ is equivalent to the category of modules over a concealed-canonical algebra. In particular, if this the case, then $\operatorname{gl.dim}\mathbf{\Delta} \leq 2$.

For the rest of the section we fix a bound quiver $\mathbf{\Delta}$ and a sincere separating exact subcategory $\mathcal{R}$ of $\operatorname{ind}\mathbf{\Delta}$. Moreover, we put $\mathcal{P} := \mathcal{R}_+$ and $\mathcal{Q} := \mathcal{R}_-$. Finally, we denote by $\mathbf{P}$, $\mathbf{R}$ and $\mathbf{Q}$ the sets of the dimension vectors of the representations from $\operatorname{add}\mathcal{P}$, $\operatorname{add}\mathcal{R}$ and $\operatorname{add}\mathcal{Q}$, respectively.

It is known that $\operatorname{pdim}_{\mathbf{\Delta}} P \leq 1$ for each $P \in \mathcal{P}$ and $\operatorname{idim}_{\mathbf{\Delta}} Q \leq 1$ for each $Q \in \mathcal{Q}$. Next, $\operatorname{pdim}_{\mathbf{\Delta}} R = 1$ and $\operatorname{idim}_{\mathbf{\Delta}} R = 1$ for each $R \in \mathcal{R}$. Moreover, $\operatorname{Hom}_{\mathbf{\Delta}}(\mathcal{Q}, \mathcal{P}) = 0$. Since the categories $\mathcal{P}$ and $\mathcal{Q}$ are closed under the actions of τ and τ^-, using the Auslander–Reiten formulas we also obtain that $\operatorname{Ext}^1_{\mathbf{\Delta}}(\mathcal{P}, \mathcal{R} \cup \mathcal{Q}) = 0 = \operatorname{Ext}^1_{\mathbf{\Delta}}(\mathcal{P} \cup \mathcal{R}, \mathcal{Q})$. The above properties imply that $\langle \mathbf{d}', \mathbf{d}'' \rangle_{\mathbf{\Delta}} \geq 0$ if either $\mathbf{d}' \in \mathbf{P}$ and $\mathbf{d}'' \in \mathbf{R} + \mathbf{Q}$ or $\mathbf{d}' \in \mathbf{P} + \mathbf{R}$ and $\mathbf{d}'' \in \mathbf{Q}$. Similarly, $\langle \mathbf{d}'', \mathbf{d}' \rangle_{\mathbf{\Delta}} \leq 0$ if either $\mathbf{d}' \in \mathbf{P}$ and $\mathbf{d}'' \in \mathbf{R}$ or $\mathbf{d}' \in \mathbf{R}$ and $\mathbf{d}'' \in \mathbf{Q}$.

We have $\mathcal{R} = \coprod_{\lambda \in \mathbb{X}} \mathcal{R}_\lambda$ for some infinite set $\mathbb{X}$ and connected uniserial categories $\mathcal{R}_\lambda$, $\lambda \in \mathbb{X}$. For $\lambda \in \mathbb{X}$ we denote by r_λ the number of the pairwise non-isomorphic simple objects in $\operatorname{add}\mathcal{R}_\lambda$. Then $r_\lambda < \infty$ for each $\lambda \in \mathbb{X}$. Let $\mathbb{X}_0 := \{\lambda \in \mathbb{X} : r_\lambda > 1\}$. Then $|\mathbb{X}_0| < \infty$ and we call the sequence $(r_\lambda)_{\lambda \in \mathbb{X}_0}$ the *type* of $\mathbf{\Delta}$ (this definition does not depend on the choice of a sincere separating exact subcategory of $\operatorname{ind}\mathbf{\Delta}$). It is known that $\mathbf{\Delta}$ is tame if and only if $\sum_{\lambda \in \mathbb{X}_0} \frac{1}{r_\lambda} \geq |\mathbb{X}_0| - 2$, where by definition the empty sum equals 0. Observe that this implies that $|\mathbb{X}_0| \leq 4$ provided $\mathbf{\Delta}$ is tame. Moreover, if $\mathbf{\Delta}$ is tame and $|\mathbb{X}_0| = 4$, then $\mathbf{\Delta}$ is of type $(2, 2, 2, 2)$.

Fix $\lambda \in \mathbb{X}$. If $R_{\lambda,0}, \ldots, R_{\lambda,r_\lambda - 1}$ are chosen representatives of the isomorphism classes of the simple objects in $\operatorname{add}\mathcal{R}_\lambda$, then we may assume that $\tau R_{\lambda,i} = R_{\lambda,i-1}$ for each $i \in [0, r_\lambda - 1]$, where we put $R_{\lambda,i} := R_{\lambda, i \bmod r_\lambda}$ for $i \in \mathbb{Z}$. For $i \in \mathbb{Z}$ and $n \in \mathbb{N}_+$, there exists a unique (up to isomorphism) representation in $\mathcal{R}_\lambda$, whose top and length in add $\mathcal{R}_\lambda$ are $R_{\lambda,i}$ and n, respectively. We fix such representation and denote it by $R_{\lambda,i}^{(n)}$ and its dimension vector by $\mathbf{e}_{\lambda,i}^n$. Then the composition factors of $R_{\lambda,i}^{(n)}$ are (starting from the top): $R_{\lambda,i}, R_{\lambda,i-1}, \dots, R_{\lambda,i-(n-1)}$. Consequently, $\mathbf{e}_{\lambda,i}^n = \sum_{j\in[i-n+1,i]} \mathbf{e}_{\lambda,j}$, where $\mathbf{e}_{\lambda,j} := \mathbf{dim}\, R_{\lambda,j}$ for $j \in \mathbb{Z}$. Moreover, if $i \in \mathbb{Z}$ and $m, n \in \mathbb{N}_+$, then we have an exact sequence $0 \to R_{\lambda,i-n}^{(m)} \to R_{\lambda,i}^{(m+n)} \to R_{\lambda,i}^{(n)} \to 0$. Obviously, for each $R \in \mathcal{R}_\lambda$ there exist $i \in \mathbb{Z}$ and $n \in \mathbb{N}_+$ such that $R \simeq R_{\lambda,i}^{(n)}$. Moreover, it is known that the vectors $\mathbf{e}_{\lambda,0}, \dots, \mathbf{e}_{\lambda,r_\lambda-1}$ are linearly independent. Consequently, if $R \in \operatorname{add} \mathcal{R}_\lambda$, then there exist uniquely determined $q_0^R, \dots, q_{r_\lambda-1}^R \in \mathbb{N}$ such that $\mathbf{dim}\, R = \sum_{i\in[0,r_\lambda-1]} q_i^R \mathbf{e}_{\lambda,i}$. Observe that the numbers $q_0^R, \dots, q_{r_\lambda-1}^R$ count the multiplicities in which the modules $R_{\lambda,0}, \dots, R_{\lambda,r_\lambda-1}$ appear as composition factors in the Jordan–Hölder filtration of R in the category add $\mathcal{R}_\lambda$.

Let $R = \bigoplus_{\lambda\in\mathbb{X}} R_\lambda$ for $R_\lambda \in \operatorname{add} \mathcal{R}_\lambda$, $\lambda \in \mathbb{X}$. Then we put $q_{\lambda,i}^R := q_i^{R_\lambda}$ for $\lambda \in \mathbb{X}$ and $i \in [0, r_\lambda - 1]$. Next, we put $p_\lambda^R := \min\{q_{\lambda,i}^R : i \in [0, r_\lambda - 1]\}$ for $\lambda \in \mathbb{X}$, and $p_{\lambda,i}^R := q_{\lambda,i}^R - p_\lambda^R$ for $\lambda \in \mathbb{X}$ and $i \in [0, r_\lambda - 1]$. Then

$$\mathbf{dim}\, R = \sum_{\lambda\in\mathbb{X}} p_\lambda^R \mathbf{h}_\lambda + \sum_{\lambda\in\mathbb{X}} \sum_{i\in[0,r_\lambda-1]} p_{\lambda,i}^R \mathbf{e}_{\lambda,i},$$

where $\mathbf{h}_\lambda := \sum_{i\in[0,r_\lambda-1]} \mathbf{e}_{\lambda,i}$ for $\lambda \in \mathbb{X}$. It is known that $\mathbf{h}_\lambda = \mathbf{h}_\mu$ for any $\lambda, \mu \in \mathbb{X}$. We denote this common value by $\mathbf{h}$. Then

$$\mathbf{dim}\, R = p^R \mathbf{h} + \sum_{\lambda\in\mathbb{X}} \sum_{i\in[0,r_\lambda-1]} p_{\lambda,i}^R \mathbf{e}_{\lambda,i},$$

where $p^R := \sum_{\lambda\in\mathbb{X}} p_\lambda^R$. It is known that $p^R = p^{R'}$ and $p_{\lambda,i}^R = p_{\lambda,i}^{R'}$ for any $\lambda \in \mathbb{X}$ and $i \in [0, r_\lambda - 1]$, if $R, R' \in \operatorname{add} \mathcal{R}$ and $\mathbf{dim}\, R = \mathbf{dim}\, R'$. Consequently, for each $\mathbf{d} \in \mathbf{R}$ there exist uniquely determined $p^{\mathbf{d}} \in \mathbb{N}$ and $p_{\lambda,i}^{\mathbf{d}} \in \mathbb{N}$, $\lambda \in \mathbb{X}$, $i \in [0, r_\lambda - 1]$, such that for each $\lambda \in \mathbb{X}$ there exists $i \in [0, r_\lambda - 1]$ with $p_{\lambda,i}^{\mathbf{d}} = 0$ and

$$\mathbf{d} = p^{\mathbf{d}} \mathbf{h} + \sum_{\lambda\in\mathbb{X}} \sum_{i\in[0,r_\lambda-1]} p_{\lambda,i}^{\mathbf{d}} \mathbf{e}_{\lambda,i}.$$

Let $\lambda, \mu \in \mathbb{X}$, $i, j \in \mathbb{Z}$ and $m, n \in \mathbb{N}_+$. Then

$$\dim_k \operatorname{Hom}_{\mathbf{\Delta}}(R_{\lambda,i}^{(n)}, R_{\mu,j}^{(m)}) = \min\{q_{\lambda,i \bmod r_\lambda}^{R_{\mu,j}^{(m)}}, q_{\mu,(j-m+1) \bmod r_\lambda}^{R_{\lambda,i}^{(n)}}\}.$$

In particular, if $\lambda \in \mathbb{X}$, $i \in [0, r_\lambda - 1]$, $n \in \mathbb{N}_+$, $R \in \operatorname{add} \mathcal{R}$ and $\operatorname{Hom}_{\mathbf{\Delta}}(R_{\lambda,i}^{(n)}, R) \neq 0$, then $q_{\lambda,i}^R \neq 0$. Moreover, the above formula together with the Auslander–Reiten formula imply that

$$\langle \mathbf{e}_{i,\lambda}^n, \mathbf{d} \rangle_{\mathbf{\Delta}} = p_{\lambda,i \bmod r_\lambda}^{\mathbf{d}} - p_{\lambda,(i-n) \bmod r_\lambda}^{\mathbf{d}}$$

and

$$\langle \mathbf{d}, \mathbf{e}_{i,\lambda}^n \rangle_{\mathbf{\Delta}} = p_{\lambda,(i-n+1) \bmod r_\lambda}^{\mathbf{d}} - p_{\lambda,(i+1) \bmod r_\lambda}^{\mathbf{d}}$$

for any $\lambda \in \mathbb{X}$, $i \in \mathbb{Z}$, $n \in \mathbb{N}_+$, and $\mathbf{d} \in \mathbf{R}$. Consequently, $\langle \mathbf{h}, \mathbf{d} \rangle_{\mathbf{\Delta}} = 0 = \langle \mathbf{d}, \mathbf{h} \rangle_{\mathbf{\Delta}}$ for each $\mathbf{d} \in \mathbf{R}$. In particular, $q_{\mathbf{\Delta}}(\mathbf{h}) = 0$. On the other hand, if $\mathbf{d} \in \mathbf{R}$, then $q_{\mathbf{\Delta}}(\mathbf{d}) = 0$ if and only if $\mathbf{d} = p^{\mathbf{d}}\mathbf{h}$. One also shows that $\mathbf{h}$ is indivisible.

We also need some other properties of the Tits form, which we list now (the proofs can be found in [36, Sections 4.9 and 5.2]).

Proposition 2.1 *Assume that $\mathbf{\Delta}$ is tame. Then the following hold.*

(i) *$q_{\mathbf{\Delta}}(\mathbf{d}) \geq 0$ for each dimension vector $\mathbf{d}$.*

(ii) *If $q_{\mathbf{\Delta}}(\mathbf{d}) = 0$ for a dimension vector $\mathbf{d}$, then $\mathbf{d} \in \mathbf{P} \cup \mathbf{R} \cup \mathbf{Q}$ and $\langle \mathbf{d}, \mathbf{d}_0 \rangle_{\mathbf{\Delta}} + \langle \mathbf{d}_0, \mathbf{d} \rangle_{\mathbf{\Delta}} = 0$ for each dimension vector $\mathbf{d}_0$.*

(iii) *If $\mathbf{d} \in \mathbf{P} \cup \mathbf{Q}$ is non-zero, then $\langle \mathbf{d}, \mathbf{h} \rangle_{\mathbf{\Delta}} \neq 0$.*

(iv) *If $\mathbf{d} \in \mathbf{P} \cup \mathbf{Q}$ is non-zero and $q_{\mathbf{\Delta}}(\mathbf{d}) = 0$, then $\langle \mathbf{d}, \mathbf{d}_0 \rangle_{\mathbf{\Delta}} \neq 0$ for each non-zero vector $\mathbf{d}_0 \in \mathbf{R}$. In particular,*

$$|\langle \mathbf{d}, \mathbf{h} \rangle_{\mathbf{\Delta}}| \geq \max\{r_\lambda : \lambda \in \mathbb{X}\}.$$

(v) *If there exists non-zero $\mathbf{d} \in \mathbf{P} \cup \mathbf{Q}$ such that $q_{\mathbf{\Delta}}(\mathbf{d}) = 0$, then $\sum_{\lambda \in \mathbb{X}_0} \frac{1}{r_\lambda} = |\mathbb{X}_0| - 2$. In particular, if this is the case, then $\max\{r_\lambda : \lambda \in \mathbb{X}\} \geq 2$, and $\max\{r_\lambda : \lambda \in \mathbb{X}\} = 2$ if and only if $\mathbf{\Delta}$ is of type $(2, 2, 2, 2)$.*

As a consequence we obtain the following.

Corollary 2.2 *Let $\mathbf{d} \in \mathbf{R}$, $\mathbf{d}' \in \mathbf{P} + \mathbf{R}$ and $\mathbf{d}'' \in \mathbf{Q}$. If $p^{\mathbf{d}} > 0$, $\mathbf{d}' + \mathbf{d}'' = \mathbf{d}$ and $\mathbf{d}'' \neq 0$, then $\langle \mathbf{d}'', \mathbf{d}' \rangle_{\mathbf{\Delta}} \leq -p^{\mathbf{d}} - 1$. Moreover, $\langle \mathbf{d}'', \mathbf{d}' \rangle_{\mathbf{\Delta}} = -p^{\mathbf{d}} - 1$ if and only if one of the following conditions is satisfied:*

(i) *$q_{\mathbf{\Delta}}(\mathbf{d}'') = 1$ and $\langle \mathbf{d}'', \mathbf{d} \rangle_{\mathbf{\Delta}} = -p^{\mathbf{d}}$ (in particular, $\langle \mathbf{d}'', \mathbf{h} \rangle_{\mathbf{\Delta}} = -1$);*

(ii) *$q_{\mathbf{\Delta}}(\mathbf{d}'') = 0$ and $\langle \mathbf{d}'', \mathbf{d} \rangle_{\mathbf{\Delta}} = -2$.*

Proof Put $\mathbf{d}_0 := \mathbf{d} - p^{\mathbf{d}}\mathbf{h}$. Then $\mathbf{d}_0 \in \mathbf{R}$. We have

$$\langle \mathbf{d}'', \mathbf{d}' \rangle_{\mathbf{\Delta}} = \langle \mathbf{d}'', \mathbf{d} - \mathbf{d}'' \rangle_{\mathbf{\Delta}} = -q_{\mathbf{\Delta}}(\mathbf{d}'') + p^{\mathbf{d}} \langle \mathbf{d}'', \mathbf{h} \rangle_{\mathbf{\Delta}} + \langle \mathbf{d}'', \mathbf{d}_0 \rangle_{\mathbf{\Delta}}.$$

Now $\langle \mathbf{d}'', \mathbf{d}_0 \rangle_{\mathbf{\Delta}} \leq 0$. Moreover, $\langle \mathbf{d}'', \mathbf{h} \rangle_{\mathbf{\Delta}} \leq -1$ and $q_{\mathbf{\Delta}}(\mathbf{d}'') \geq 0$. Finally, if $q_{\mathbf{\Delta}}(\mathbf{d}'') = 0$, then $\langle \mathbf{d}'', \mathbf{h} \rangle_{\mathbf{\Delta}} \leq -2$, hence the inequality follows.

These considerations also imply that $\langle \mathbf{d}'', \mathbf{d}' \rangle_{\mathbf{\Delta}} = -p^{\mathbf{d}} - 1$ if and only if one of the following conditions is satisfied:

(i) $q_{\mathbf{\Delta}}(\mathbf{d}'') = 1$, $\langle \mathbf{d}'', \mathbf{h} \rangle_{\mathbf{\Delta}} = -1$ and $\langle \mathbf{d}'', \mathbf{d}_0 \rangle_{\mathbf{\Delta}} = 0$;

(ii) $q_{\mathbf{\Delta}}(\mathbf{d}'') = 0$, $p^{\mathbf{d}} = 1$, $\langle \mathbf{d}'', \mathbf{h} \rangle_{\mathbf{\Delta}} = -2$ and $\langle \mathbf{d}'', \mathbf{d}_0 \rangle_{\mathbf{\Delta}} = 0$.

These conditions immediately lead to (and follow from) the conditions given in the corollary. ∎

We call a dimension vector $\mathbf{d} \in \mathbf{R}$ singular if $p^{\mathbf{d}} > 0$ and there exists a dimension vector $\mathbf{x}$ such that $\mathbf{x} \leq \mathbf{d}$, $q_{\mathbf{\Delta}}(\mathbf{x}) = 0$ and $|\langle \mathbf{x}, \mathbf{d} \rangle_{\mathbf{\Delta}}| = 2$. It follows from the proposition below that this definition coincides with the definition given in the introduction.

Proposition 2.3 *Let $\mathbf{d} \in \mathbf{R}$ be such that $p^{\mathbf{d}} > 0$.*

(i) *If $\mathbf{d}$ is singular, then $\mathbf{d} = \mathbf{h}$ and $\mathbf{\Delta}$ is of type $(2, 2, 2, 2)$.*

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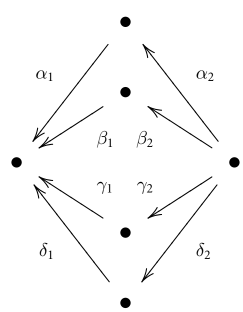
(ii) *There exist* $\mathbf{d}' \in \mathbf{P} + \mathbf{R}$ *and* $\mathbf{d}'' \in \mathbf{Q}$ *such that* $\mathbf{d}' + \mathbf{d}'' = \mathbf{d}$, $q_{\boldsymbol{\Delta}}(\mathbf{d}'') = 0$ *and* $\langle \mathbf{d}'', \mathbf{d} \rangle_{\boldsymbol{\Delta}} = -2$ *if and only if* $\mathbf{d}$ *is singular.*

Proof (i) Fix a dimension vector $\mathbf{x}$ such that $\mathbf{x} \leq \mathbf{d}$, $q_{\boldsymbol{\Delta}}(\mathbf{x}) = 0$ and $|\langle \mathbf{x}, \mathbf{d} \rangle_{\boldsymbol{\Delta}}| = 2$. Proposition 2.1(ii) implies that $\mathbf{x} \in \mathbf{P} \cup \mathbf{R} \cup \mathbf{Q}$. Since $\langle \mathbf{x}, \mathbf{d} \rangle_{\boldsymbol{\Delta}} \neq 0$, $\mathbf{x} \notin \mathbf{R}$. In particular, $\mathbf{x}$ is non-zero. By symmetry, we may assume $\mathbf{x} \in \mathbf{P}$. If $\mathbf{d}_0 := \mathbf{d} - p^{\mathbf{d}}\mathbf{h}$, then $2 = p^{\mathbf{d}} \langle \mathbf{x}, \mathbf{h} \rangle_{\boldsymbol{\Delta}} + \langle \mathbf{x}, \mathbf{d}_0 \rangle_{\boldsymbol{\Delta}}$. Using Proposition 2.1(iv) and (v) we obtain that $p^{\mathbf{d}} = 1$ and $\mathbf{d}_0 = 0$, *i.e.*, $\mathbf{d} = \mathbf{h}$. Moreover, $\boldsymbol{\Delta}$ must be of type $(2, 2, 2, 2)$ by Proposition 2.1(v).

(ii) One implication is obvious. Now assume $\mathbf{d}$ is singular, *i.e.*, there exists a dimension vector $\mathbf{x}$ such that $\mathbf{x} \leq \mathbf{d}$, $q_{\boldsymbol{\Delta}}(\mathbf{x}) = 0$ and $|\langle \mathbf{x}, \mathbf{d} \rangle_{\boldsymbol{\Delta}}| = 2$. From (i) we know that $\mathbf{d} = \mathbf{h}$. Easy calculations show that $\langle \mathbf{h}, \mathbf{h} - \mathbf{x} \rangle_{\boldsymbol{\Delta}} = -\langle \mathbf{h}, \mathbf{x} \rangle_{\boldsymbol{\Delta}}$ and $q_{\boldsymbol{\Delta}}(\mathbf{h} - \mathbf{x}) = 0$. Thus Proposition 2.1(ii) implies that, up to symmetry, $\mathbf{x} \in \mathbf{P}$ and $\mathbf{h} - \mathbf{x} \in \mathbf{Q}$, and the claim follows. ■

We finish this section with an example showing that singular dimension vectors exist. Fix $\lambda \in k \setminus \{0, 1\}$. Let Δ be the quiver



and $R := \{\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2, \alpha_1\alpha_2 + \beta_1\beta_2 + \lambda\delta_1\delta_2\}$. Then $\boldsymbol{\Delta}$ is a concealed-canonical algebra of type $(2, 2, 2, 2)$ (in fact, it is one of Ringel's canonical algebras [36]). Moreover, the vector

$$\begin{array}{ccc} & 2 & \\ & 2 & \\ 3 & & 1 \\ & 2 & \\ & 2 & \end{array}$$

is singular – the corresponding vector $\mathbf{x}$ can be taken to be

$$\begin{array}{ccc} & 1 & \\ & 1 & \\ 1 & & 1, \\ & 1 & \\ & 1 & \end{array}$$

the other choice being

$$\begin{array}{ccc} & 1 & \\ & 1 & \\ 2 & & 0. \\ & 1 & \\ & 1 & \end{array}$$

3 Varieties of Representations

First we recall some facts from algebraic geometry. Let $\mathcal{X}$ be a closed subvariety of an affine space $\mathbb{A}^n$, $n \in \mathbb{N}$. We say that $\mathcal{X}$ is a complete intersection if there exist polynomials $f_1, \dots, f_m \in k[\mathbb{A}^n]$ such that $\dim \mathcal{X} = n - m$ and

$$\{f \in k[\mathbb{A}^n] : f(x) = 0 \text{ for each } x \in \mathcal{X}\} = (f_1, \dots, f_m).$$

For $x \in \mathcal{X}$ we denote by $T_x\mathcal{X}$ the tangent space to $\mathcal{X}$ at x. We will use the following consequences of Serre's criterion (see for example [27, Theorem 18.15]).

Proposition 3.1 *Let $\mathcal{X}$ be a complete intersection and*

$$\mathcal{X}_{\mathrm{reg}} := \{x \in \mathcal{X} : \dim_k T_x\mathcal{X} = \dim \mathcal{X}\}.$$

(i) *The variety $\mathcal{X}$ is normal if and only if* $\dim(\mathcal{X} \setminus \mathcal{X}_{\mathrm{reg}}) < \dim \mathcal{X} - 1$.

(ii) *Let $f_1, \dots, f_m \in k[\mathcal{X}]$,*

$$\mathcal{Y} := \{x \in \mathcal{X} : f_i(x) = 0 \textit{ for each } i \in [1, m]\},$$

and

$$\mathcal{U} := \{x \in \mathcal{Y} \cap \mathcal{X}_{\mathrm{reg}} : \partial f_1(x), \dots, \partial f_m(x) \textit{ are linearly independent}\}.$$

If $\mathcal{U} \cap \mathcal{C} \neq \varnothing$ for each irreducible component $\mathcal{C}$ of $\mathcal{Y}$, then

$$\{f \in k[\mathcal{X}] : f(x) = 0 \textit{ for each } x \in \mathcal{Y}\} = (f_1, \dots, f_m).$$

In particular, $\mathcal{Y}$ is a complete intersection of dimension $\dim \mathcal{X} - m$.

Let Δ be a quiver and $\mathbf{d}$ a dimension vector. By $\operatorname{rep}_\Delta(\mathbf{d})$ we denote the set of the representations M of Δ such that $M(x) = k^{\mathbf{d}(x)}$ for each $x \in \Delta_0$. We may identify $\operatorname{rep}_\Delta(\mathbf{d})$ with the affine space $\prod_{\alpha \in \Delta_1} \mathbb{M}_{\mathbf{d}(t\alpha) \times \mathbf{d}(s\alpha)}(k)$. The group $\mathrm{GL}(\mathbf{d}) := \prod_{x \in \Delta_0} \mathrm{GL}\big(\mathbf{d}(x)\big)$ acts on $\operatorname{rep}_\Delta(\mathbf{d})$ by conjugation: $(g \cdot M)(\alpha) := g(t\alpha)M(\alpha)g(s\alpha)^{-1}$ for $g \in \mathrm{GL}(\mathbf{d})$, $M \in \operatorname{rep}_\Delta(\mathbf{d})$ and $\alpha \in \Delta_1$. Under this action the $\mathrm{GL}(\mathbf{d})$-orbits in $\operatorname{rep}_\Delta(\mathbf{d})$ correspond to the isomorphism classes of the representations of Δ with dimension vector $\mathbf{d}$. We denote the $\mathrm{GL}(\mathbf{d})$-orbit of a representation $M \in \operatorname{rep}_\Delta(\mathbf{d})$ by $\mathcal{O}(M)$.

Now let $\boldsymbol{\Delta}$ be a bound quiver and $\mathbf{d}$ a dimension vector. By $\operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d})$ we denote the intersection of $\operatorname{rep}_\Delta(\mathbf{d})$ with $\operatorname{rep} \boldsymbol{\Delta}$. Then $\operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d})$ is a closed $\mathrm{GL}(\mathbf{d})$-invariant subset of $\operatorname{rep}_\Delta(\mathbf{d})$ and we call it the *variety of representations of $\boldsymbol{\Delta}$ of dimension vector* $\mathbf{d}$. If $M \in \operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d})$, then there exists a canonical map $\pi_M \colon T_M \operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d}) \to \operatorname{Ext}^1_{\boldsymbol{\Delta}}(M, M)$ with kernel $T_M\mathcal{O}(M)$ [43, Section 3]. This map is an epimorphism if we consider $\operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d})$ as a scheme with a natural, but not necessarily reduced, structure. If we view $\operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d})$ as a variety with its reduced structure (as we do in this paper), then it does not have to be an epimorphism in general. However, easy arguments show (see for example [9, proof of Proposition 2.2]), that if $\operatorname{gl.dim} \boldsymbol{\Delta} \leq 2$ and $\operatorname{Ext}^2_{\boldsymbol{\Delta}}(M, M) = 0$, then π_M is an epimorphism.

Let $\mathbf{\Delta}$ be a bound quiver and $\mathbf{d}$ a dimension vector. If $M, N \in \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$ and there exists an exact sequence $0 \to N' \to M \to N'' \to 0$ such that $N \simeq N' \oplus N''$, then $N \in \overline{\mathcal{O}(M)}$ [17, Lemma 1.1]. In particular, $S^{\mathbf{d}} \in \overline{\mathcal{O}(M)}$ for each $M \in \operatorname{rep}_{\Delta}(\mathbf{d})$. If $\mathcal{V}$ is a $\mathrm{GL}(\mathbf{d})$-invariant subset of $\operatorname{rep}_{\Delta}(\mathbf{d})$ and $M \in \mathcal{V}$, then we say that the orbit $\mathcal{O}(M)$ is maximal in $\mathcal{V}$ if $\mathcal{O}(N) = \mathcal{O}(M)$ for each $N \in \mathcal{V}$ such that $\mathcal{O}(M) \subseteq \overline{\mathcal{O}(N)}$. If $\mathbf{\Delta}$ is tame concealed-canonical and $M \in \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$, then $\mathcal{O}(M)$ is maximal in $\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$ if and only if $\operatorname{Ext}^1_{\mathbf{\Delta}}(M', M'') = 0$ for each decomposition $M = M' \oplus M''$ [45, Corollary 6].

Put $a_{\mathbf{\Delta}}(\mathbf{d}) := \dim \mathrm{GL}(\mathbf{d}) - q_{\mathbf{\Delta}}(\mathbf{d})$ for a bound quiver $\mathbf{\Delta}$ and a dimension vector $\mathbf{d}$. The following facts were proved in [10].

Proposition 3.2 *Let $\mathbf{d}$ be the dimension vector of a periodic representation over a tame concealed-canonical bound quiver $\mathbf{\Delta}$. Then the following hold.*

(i) *The variety $\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$ is a normal complete intersection of dimension $a_{\mathbf{\Delta}}(\mathbf{d})$.*
(ii) *If there exists $M \in \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$ such that $\operatorname{Ext}^1_{\mathbf{\Delta}}(M, M) = 0$, then $\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d}) = \overline{\mathcal{O}(M)}$.*
(iii) *If $\operatorname{Ext}^1_{\mathbf{\Delta}}(M, M) \neq 0$ for each $M \in \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$, then there exists a convex subquiver $\mathbf{\Delta}'$ of $\mathbf{\Delta}$ and a sincere separating exact subcategory $\mathcal{R}'$ of $\operatorname{ind} \mathbf{\Delta}'$ such that $M \in \operatorname{add} \mathcal{R}'$ for each maximal orbit $\mathcal{O}(M)$ in $\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$.*
(iv) *If $M \in \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$, then π_M is an epimorphism.*
(v) *If $M \in \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$, then $\dim_k T_M \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d}) = \dim_k \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$ if and only if $\operatorname{Ext}^2_{\mathbf{\Delta}}(M, M) = 0$.*

Let $\mathbf{d}$ be the dimension vector of a periodic module over a tame concealed-canonical bound quiver $\mathbf{\Delta}$. The above theorem implies that in order to prove that $\overline{\mathcal{O}(M)}$ is a normal complete intersection for each maximal orbit $\mathcal{O}(M)$ in $\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$, we may assume that $\mathbf{d}$ is the dimension vector of a direct sum of modules from a sincere separating exact subcategory of $\operatorname{ind} \mathbf{\Delta}$. Thus we fix a tame bound quiver $\mathbf{\Delta}$ and a sincere separating exact subcategory $\mathcal{R}$ of $\operatorname{ind} \mathbf{\Delta}$. We will use freely notation introduced in Section 2. It follows from [10, Section 3] that if $\mathbf{d} \in \mathbf{R}$, then $M \in \operatorname{add} \mathcal{R}$ for each maximal orbit $\mathcal{O}(M)$ in $\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$.

For a full subcategory $\mathcal{X}$ of $\operatorname{ind} \mathbf{\Delta}$ and a dimension vector $\mathbf{d}$ we denote by $\mathcal{X}(\mathbf{d})$ the intersection of $\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$ with $\operatorname{add} \mathcal{X}$. If $\mathbf{d}', \mathbf{d}' \in \mathbb{N}^{\Delta_0}$, $C' \subseteq \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d}')$ and $C'' \subseteq \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d}'')$, then we denote by $C' \oplus C''$ the subset of $\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d}' + \mathbf{d}'')$ consisting of all M such that $M \simeq M' \oplus M''$ for some $M' \in C'$ and $M'' \in C''$. The following fact follows from [5, Section 3].

Proposition 3.3 *If $\mathbf{d}' \in \mathbf{P}+\mathbf{R}$ and $\mathbf{d}'' \in \mathbf{Q}$, then $(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')$ is an irreducible constructible subset of $\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d}' + \mathbf{d}'')$ of dimension $a_{\mathbf{\Delta}}(\mathbf{d}' + \mathbf{d}'') + \langle \mathbf{d}'', \mathbf{d}' \rangle_{\mathbf{\Delta}}$.*

Using Corollary 2.2 and Proposition 2.3, we immediately get the following.

Corollary 3.4 *Let $\mathbf{d} \in \mathbf{R}$, $\mathbf{d}' \in \mathbf{P} + \mathbf{R}$ and $\mathbf{d}'' \in \mathbf{Q}$. If $p^{\mathbf{d}} > 0$, $\mathbf{d}' + \mathbf{d}'' = \mathbf{d}$ and $\mathbf{d}'' \neq 0$, then*

$$\dim\big((\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')\big) \leq a_{\mathbf{\Delta}}(\mathbf{d}) - p^{\mathbf{d}} - 1.$$

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Moreover, the equality holds if and only if one of the following conditions is satisfied:

(i) $q_{\mathbf{\Delta}}(\mathbf{d}'') = 1$ *and* $\langle \mathbf{d}'', \mathbf{d} \rangle_{\mathbf{\Delta}} = -p^{\mathbf{d}}$ *(in particular,* $\langle \mathbf{d}'', \mathbf{h} \rangle_{\mathbf{\Delta}} = -1$*), or*

(ii) $q_{\mathbf{\Delta}}(\mathbf{d}'') = 0$ *and* $\langle \mathbf{d}'', \mathbf{d} \rangle_{\mathbf{\Delta}} = -2$ *(in particular,* $\mathbf{\Delta}$ *is of type* $(2, 2, 2, 2)$ *and* $\mathbf{d} = \mathbf{h}$*).*

Observe that

$$\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d}) = \mathcal{R}(\mathbf{d}) \cup \bigcup_{\substack{\mathbf{d}' \in \mathbf{P}+\mathbf{R}, \mathbf{d}'' \in \mathbf{Q} \\ \mathbf{d}'+\mathbf{d}''=\mathbf{d}, \mathbf{d}'' \neq 0}} (\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')$$

for each $\mathbf{d} \in \mathbf{R}$. Indeed, if $M \in (\mathcal{P} \cup \mathcal{R})(\mathbf{d})$ and we write $M = M' \oplus M''$ for $M' \in \operatorname{add} \mathcal{P}$ and $M'' \in \operatorname{add} \mathcal{R}$, then $\langle \operatorname{\mathbf{dim}} M', \mathbf{h} \rangle_{\mathbf{\Delta}} = \langle \mathbf{d}, \mathbf{h} \rangle_{\mathbf{\Delta}} = 0$, hence $M' = 0$ by Proposition 2.1(iii). The above formula together with Corollary 3.4 implies that $\dim\big(\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d}) \setminus \mathcal{R}(\mathbf{d})\big) \leq a_{\mathbf{\Delta}}(\mathbf{d}) - p^{\mathbf{d}} - 1$.

4 Stability and Semi-invariants

Let Δ be a quiver and $\theta \in \mathbb{Z}^{\Delta_0}$. We treat θ as a $\mathbb{Z}$-linear function $\mathbb{Z}^{\Delta_0} \to \mathbb{Z}$ in the usual way. A representation M of Δ is called *θ-semi-stable* if $\theta(\operatorname{\mathbf{dim}} M) = 0$ and $\theta(\operatorname{\mathbf{dim}} N) \geq 0$ for each subrepresentation N of M. The full subcategory of θ-semi-stable representations of Δ is an exact subcategory of $\operatorname{rep} \Delta$. Two θ-semi-stable representations are called *S-equivalent* if they have the same composition factors within this category. If $\mathbf{d}$ is a dimension vector, then by a *semi-invariant of weight θ* we mean a function $c \in k[\operatorname{rep}_{\Delta}(\mathbf{d})]$ such that $c(g \cdot M) = \chi^{\theta}(g) c(M)$ for any $g \in \operatorname{GL}(\mathbf{d})$ and $M \in \operatorname{rep}_{\Delta}(\mathbf{d})$, where $\chi^{\theta}(g) := \prod_{x \in \Delta_0} \big(\det g(x)\big)^{\theta(x)}$ for $g \in \operatorname{GL}(\mathbf{d})$.

Now let $\mathbf{\Delta}$ be a bound quiver and $\mathbf{d}$ a dimension vector. If $\theta \in \mathbb{Z}^{\Delta_0}$, then a function $c \in k[\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})]$ is called a *semi-invariant of weight θ* if c is a restriction of a semi-invariant of weight θ from $k[\operatorname{rep}_{\Delta}(\mathbf{d})]$. This definition differs from the definition used in other papers on the subject (see for example [7, 23–25]), however it is consistent with King's approach [29]. We denote the space of the semi-invariants of weight θ by $\operatorname{SI}[\mathbf{\Delta}, \mathbf{d}]_{\theta}$. If $\theta \in \mathbb{Z}^{\Delta_0}$, then we put $\Lambda_{\theta}(\mathbf{d}) := \bigoplus_{n \in \mathbb{N}} \operatorname{SI}[\mathbf{\Delta}, \mathbf{d}]_{n\theta}$. Note that $\Lambda_{\theta}(\mathbf{d})$ is a graded ring. For $M \in \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$ we denote by $\mathcal{I}_{\theta}(M)$ the ideal in $\Lambda_{\theta}(\mathbf{d})$ generated by the homogeneous elements c such that $c(M) = 0$.

The following results were proved in [29].

Proposition 4.1 *Let $\mathbf{\Delta}$ be a bound quiver and $\mathbf{d}$ a dimension vector and $\theta \in \mathbb{Z}^{\Delta_0}$.*

(i) *If $M \in \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$, then M is θ-semi-stable if and only if there exists a semi-invariant c of weight $n\theta$ for some $n \in \mathbb{N}_+$ such that $c(M) \neq 0$.*

(ii) *If $M, N \in \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$ are θ-semi-stable, then M and N are S-equivalent if and only if $\mathcal{I}_{\theta}(M) = \mathcal{I}_{\theta}(N)$.*

Now we recall a construction from [24]. Let $\mathbf{\Delta}$ be a bound quiver. Fix a representation V of $\mathbf{\Delta}$. We define $\theta^V : \mathbb{Z}^{\Delta_0} \to \mathbb{Z}$ by the condition

$$\theta^V(\operatorname{\mathbf{dim}} M) = -\dim_k \operatorname{Hom}_{\mathbf{\Delta}}(V, M) + \dim_k \operatorname{Hom}_{\mathbf{\Delta}}(M, \tau V)$$

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for each representation M of $\boldsymbol{\Delta}$ (it follows from [3, Corollary IV.4.3] that θ^V is well-defined). The Auslander–Reiten formula implies that $\theta^V = -\langle \mathbf{dim}\, V, -\rangle_{\boldsymbol{\Delta}}$ if $\operatorname{pdim}_{\boldsymbol{\Delta}} V \leq 1$. Dually, if $\operatorname{idim}_{\boldsymbol{\Delta}} V \leq 1$, then $\theta^V = \langle -, \mathbf{dim}\, \tau V\rangle_{\boldsymbol{\Delta}}$.

Now let $\mathbf{d}$ be a dimension vector. If $\theta^V(\mathbf{d}) = 0$, then we define a function $c^V \in k[\operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d})]$ in the following way. Let $P_1 \xrightarrow{f} P_0 \to V \to 0$ be a minimal projective presentation of V. There exist vertices $x_1, \dots, x_n, y_1, \dots, y_m$ of Δ such that $P_1 = \bigoplus_{i\in[1,n]} P_{x_i}$ and $P_0 = \bigoplus_{j\in[1,m]} P_{y_j}$. Moreover, there exist $\omega_{i,j} \in k\boldsymbol{\Delta}(y_j, x_i)$, $i \in [1,n]$, $j \in [1,m]$, such that $f = \left[P_{\omega_{i,j}}\right]_{\substack{j\in[1,m]\\ i\in[1,n]}}$. Consequently, if $M \in \operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d})$, then

$$\operatorname{Hom}_{\boldsymbol{\Delta}}(f, M) = \left[M(\omega_{i,j})\right]_{\substack{i\in[1,n]\\ j\in[1,m]}} : \bigoplus_{j\in[1,m]} M(y_j) \to \bigoplus_{i\in[1,n]} M(x_i).$$

In addition, one shows that

$$\dim_k \operatorname{Ker} \operatorname{Hom}_{\boldsymbol{\Delta}}(f, M) = \dim_k \operatorname{Hom}_{\boldsymbol{\Delta}}(V, M),$$
$$\dim_k \operatorname{Coker} \operatorname{Hom}_{\boldsymbol{\Delta}}(f, M) = \dim_k \operatorname{Hom}_{\boldsymbol{\Delta}}(M, \tau V).$$

Consequently,

$$\sum_{i\in[1,n]} \dim_k M(x_i) - \sum_{j\in[1,m]} \dim_k M(y_j)$$
$$= \dim_k \operatorname{Hom}_{\boldsymbol{\Delta}}(M, \tau V) - \dim_k \operatorname{Hom}_{\boldsymbol{\Delta}}(V, M) = \theta^V(\mathbf{d}) = 0.$$

Thus it makes sense to define $c^V \in k[\operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d})]$ by

$$c^V(M) := \det \operatorname{Hom}_{\boldsymbol{\Delta}}(f, M)$$

for $M \in \operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d})$. Note that $c^V(M) = 0$ if and only if

$$\operatorname{Hom}_{\boldsymbol{\Delta}}(V, M) = \operatorname{Ker} \operatorname{Hom}_{\boldsymbol{\Delta}}(f, M) \neq 0.$$

It is known that $c^V \in \operatorname{SI}[\boldsymbol{\Delta}, \mathbf{d}]_{\theta^V}$ (see [24, Section 3]). This function depends on the choice of f, but functions obtained for different fs differ only by non-zero scalars. In fact, we could start with an arbitrary projective presentation

$$P_1 \xrightarrow{f} P_0 \to V \to 0$$

of V such that $\dim_k \operatorname{Hom}_{\boldsymbol{\Delta}}(P_1, M) = \dim_k \operatorname{Hom}_{\boldsymbol{\Delta}}(P_0, M)$. As an easy consequence we obtain the following (see [23, Proposition 2] and [24, Lemma 3.3]).

Lemma 4.2 *Let $\boldsymbol{\Delta}$ be a bound quiver and $\mathbf{d}$ a dimension vector.*

(i) *If $V = V_1 \oplus V_2$, $\theta^V(\mathbf{d}) = 0$ and $c^V \neq 0$, then $\theta^{V_1}(\mathbf{d}) = 0 = \theta^{V_2}(\mathbf{d})$.*

(ii) *If $0 \to V_1 \to V \to V_2 \to 0$ is an exact sequence and $\theta^V(\mathbf{d}) = \theta^{V_1}(\mathbf{d}) = \theta^{V_2}(\mathbf{d}) = 0$, then $c^V = c^{V_1} c^{V_2}$.*

The following result follows from the proof of [24, Theorem 3.2] (note that the assumption about the characteristic of k made in [24, Theorem 3.2] is only necessary for surjectivity of the restriction morphism, which we have for free with our definition of semi-invariants).

Proposition 4.3 *Let $\mathbf{\Delta}$ be a bound quiver and $\mathbf{d}$ a dimension vector. If $\theta \in \mathbb{Z}^{\Delta_0}$, then the space $\mathrm{SI}[\mathbf{\Delta}, \mathbf{d}]_\theta$ is spanned by the functions c^V for $V \in \operatorname{rep} \mathbf{\Delta}$ such that $\theta^V = \theta$.*

Now we apply our considerations in the case of tame concealed-canonical quivers. For the rest of the section we fix a tame bound quiver $\mathbf{\Delta}$ and a sincere separating exact subcategory $\mathcal{R}$ of $\operatorname{ind} \mathbf{\Delta}$. We will use notation introduced in Section 2. We fix $\mathbf{d} \in \mathbf{R}$ such that $p^{\mathbf{d}} > 0$ and put $\theta := -\langle \mathbf{h}, - \rangle_{\mathbf{\Delta}}$. In particular, in the rest of the paper when we speak about S-equivalence, it is always the one induced by this θ.

First observe that $M \in \operatorname{rep} \mathbf{\Delta}$ is θ-semi-stable if and only if $M \in \operatorname{add} \mathcal{R}$ (recall that $\theta(\mathbf{dim}\, X) \neq 0$ for $X \in \mathcal{P} \cup \mathcal{Q}$ according to Proposition 2.1(iii)). Consequently, if M and N are θ-semi-stable, then M and N are S-equivalent if and only if $q^M_{\lambda,i} = q^N_{\lambda,i}$ for any $\lambda \in \mathbb{X}$ and $i \in [0, r_\lambda - 1]$. In particular, we obtain the following.

Proposition 4.4 *There are only finitely many isomorphism classes in each S-equivalence class.*

Now we fix $V \in \operatorname{rep} \mathbf{\Delta}$ such that $\theta^V = n\theta$ for some $n \in \mathbb{N}$ and $c^V \neq 0$. We show that $V \in \operatorname{add} \mathcal{R}$ and $\mathbf{dim}\, V = n\mathbf{h}$. Indeed, write $V = P \oplus R \oplus Q$ for $P \in \operatorname{add} \mathcal{P}$, $R \in \operatorname{add} \mathcal{R}$ and $Q \in \operatorname{add} \mathcal{Q}$. If $P \neq 0$, then $\theta^P(\mathbf{d}) \leq -\langle \mathbf{dim}\, P, \mathbf{h} \rangle_{\mathbf{\Delta}} < 0$ by Proposition 2.1(iii), hence $c^V = 0$ by Lemma 4.2(i). Consequently, $P = 0$ and, dually, $Q = 0$, thus $V = R \in \operatorname{add} \mathcal{R}$. In particular, $\operatorname{pdim}_{\mathbf{\Delta}} V = 1$, hence $-\langle n\mathbf{h}, - \rangle_{\mathbf{\Delta}} = \theta^V = -\langle \mathbf{dim}\, V, - \rangle_{\mathbf{\Delta}}$, and this implies that $\mathbf{dim}\, V = n\mathbf{h}$.

For $\lambda \in \mathbb{X}$ we denote by $\mathcal{A}_\lambda(\mathbf{d})$ the set of all $i \in [0, r_\lambda - 1]$ such that $p^{\mathbf{d}}_{\lambda,i} = 0$. Next, for $i \in \mathcal{A}_\lambda(\mathbf{d})$ we denote by $n_{\lambda,i}$ the minimal $n \in \mathbb{N}_+$ such that $p^{\mathbf{d}}_{\lambda,(i-n) \bmod r_\lambda} = 0$ and put $V_{\lambda,i} := R^{(n_{\lambda,i})}_{\lambda,i}$. Observe that $\theta^{V_{\lambda,i}}(\mathbf{d}) = -\langle \mathbf{dim}\, V_{\lambda,i}, \mathbf{d} \rangle_{\mathbf{\Delta}} = 0$ for any $\lambda \in \mathbb{X}$ and $i \in \mathcal{A}_\lambda(\mathbf{d})$. We put $c_{\lambda,i} := c^{V_{\lambda,i}}$ for $\lambda \in \mathbb{X}$ and $i \in \mathcal{A}_\lambda(\mathbf{d})$. More generally, if $\lambda \in \mathbb{X}$ and $J \subseteq \mathcal{A}_\lambda(\mathbf{d})$, then we put $V_{\lambda,J} := \bigoplus_{i \in J} V_{\lambda,i}$ and $c_{\lambda,J} := c^{V_{\lambda,J}} = \prod_{i \in J} c_{\lambda,i}$. In particular, we put $V_\lambda := V_{\lambda,\mathcal{A}_\lambda(\mathbf{d})}$ and $c_\lambda := c_{\lambda,\mathcal{A}_\lambda(\mathbf{d})}$ for $\lambda \in \mathbb{X}$. Then $c_\lambda \in \mathrm{SI}[\mathbf{\Delta}, \mathbf{d}]_\theta$ for each $\lambda \in \mathbb{X}$. Observe that Lemma 4.2(ii) implies that $c_\lambda = c^{R^{(r_\lambda)}_{\lambda,i}}$ for any $\lambda \in \mathbb{X}$ and $i \in \mathcal{A}_\lambda(\mathbf{d})$. More generally, $c^n_\lambda = c^{R^{(nr_\lambda)}_{\lambda,i}}$ for any $n \in \mathbb{N}_+$, $\lambda \in \mathbb{X}$ and $i \in \mathcal{A}_\lambda(\mathbf{d})$.

We have the following information about $\Lambda_\theta(\mathbf{d})$.

Proposition 4.5 *The algebra $\Lambda_\theta(\mathbf{d})$ is generated by the functions c_λ, $\lambda \in \mathbb{X}$.*

Proof First we show that if $\lambda \in \mathbb{X}$, $i \in \mathbb{Z}$, $n \in \mathbb{N}_+$, $\theta^{R^{(n)}_{\lambda,i}}(\mathbf{d}) = 0$, and $c^{R^{(n)}_{\lambda,i}} \neq 0$, then $p^{\mathbf{d}}_{\lambda,i \bmod r_\lambda} = p^{\mathbf{d}}_{\lambda,(i-n) \bmod r_\lambda}$ and $p^{\mathbf{d}}_{\lambda,j \bmod r_\lambda} \geq p^{\mathbf{d}}_{\lambda,i \bmod r_\lambda}$ for each $j \in [i-n+1, i-1]$. Indeed, the former condition follows from the equality $\langle \mathbf{e}^{(n)}_{\lambda,i}, \mathbf{d} \rangle_{\mathbf{\Delta}} = -\theta^{R^{(n)}_{\lambda,i}}(\mathbf{d}) = 0$. Moreover, if there exists $j \in [i-n+1, i-1]$ such that $p^{\mathbf{d}}_{\lambda,j \bmod r_\lambda} < p^{\mathbf{d}}_{\lambda,i \bmod r_\lambda}$, then $\operatorname{Hom}_{\mathbf{\Delta}}(R^{(n)}_{\lambda,i}, R) \neq 0$ for each $R \in \mathcal{R}(\mathbf{d})$, hence $c^{R^{(n)}_{\lambda,i}} = 0$.

We have the following important consequence of the above observation. Assume that $\lambda \in \mathbb{X}$, $i \in [0, r_\lambda - 1]$, $n \in \mathbb{N}_+$ and $c^{R_{\lambda,i}^{(nr_\lambda)}} \neq 0$. Then $p_{\lambda,j}^{\mathbf{d}} \geq p_{\lambda,i}^{\mathbf{d}}$ for each $j \in [0, r_\lambda - 1]$, hence $i \in \mathcal{A}_\lambda(\mathbf{d})$. In particular, $c^{R_{\lambda,i}^{(nr_\lambda)}} = c_\lambda^n$.

Now assume that $R \in \operatorname{rep} \mathbf{\Delta}$, $\theta^R = n\theta$ for some $n \in \mathbb{N}$ and $c^R \neq 0$. We know that $R \in \operatorname{add} \mathcal{R}$ and $\mathbf{dim}\, R = n\mathbf{h}$. If $R = \bigoplus_{\lambda \in \mathbb{X}} R_\lambda$ for $R_\lambda \in \mathcal{R}_\lambda$, $\lambda \in \mathbb{X}$, then $\mathbf{dim}\, R_\lambda = p_\lambda^R \mathbf{h}$ for each $\lambda \in \mathbb{X}$. We show that $c^{R_\lambda} = c_\lambda^{p_\lambda^R}$ for each $\lambda \in \mathbb{X}$, hence the claim will follow from Lemma 4.2.

Fix $\lambda \in \mathbb{X}$ and write $R_\lambda = \bigoplus_{j \in [1,m]} R_{\lambda,i_j}^{(n_j)}$ for $m \in \mathbb{N}_+$, $i_1, \dots, i_m \in \mathbb{Z}$ and $n_1, \dots, n_m \in \mathbb{N}_+$. If $n_j \equiv 0 \pmod{r_\lambda}$ for each $j \in [1, m]$, then the claim follows. Thus assume $n_1 \not\equiv 0 \pmod{r_\lambda}$. Since $\mathbf{dim}\, R_\lambda = p_\lambda^R \mathbf{h}$, we may assume that $i_2 = i_1 - n_1$. Then we have an exact sequence $0 \to R_{\lambda,i_2}^{(n_2)} \to R_{\lambda,i_1}^{(n_1+n_2)} \to R_{\lambda,i_1}^{(n_1)} \to 0$, hence Lemma 4.2(ii) implies that $c^R = c^{R'}$, where

$$R' := R_{\lambda,i_1}^{(n_1+n_2)} \oplus \bigoplus_{j \in [3,n]} R_{\lambda,i_j}^{(n_j)}.$$

Now the claim follows by induction. ∎

As a consequence we get the following.

Corollary 4.6 *If $M, N \in \mathcal{R}(\mathbf{d})$, then M and N are S-equivalent if and only if there exists $\mu \in k$ such that $c_\lambda(M) = \mu c_\lambda(N)$ for each $\lambda \in \mathbb{X}$.*

Proof Follows immediately form Propositions 4.1(ii) and 4.5. ∎

We list some consequences of the description of the maximal orbits in $\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$ given in [10, Proposition 5] (see also [35, Theorem 3.5]). Recall that $M \in \mathcal{R}(\mathbf{d})$ for each maximal orbit $\mathcal{O}(M)$ in $\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$. Next, if $\mathcal{O}(M)$ is maximal in $\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$, then $\dim \mathcal{O}(M) = a_{\mathbf{\Delta}}(\mathbf{d}) - p^{\mathbf{d}}$. In particular, the maximal orbits in $\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$ coincide with the orbits of maximal dimension. Moreover, if $\lambda \in \mathbb{X}$, then there exists at most one $i \in \mathcal{A}_\lambda(\mathbf{d})$ such that $c_{\lambda,i}(M) = 0$. We put

$$\hat{\mathbb{X}}(M) := \{(\lambda, i) : \lambda \in \mathbb{X}, i \in \mathcal{A}_\lambda(\mathbf{d}) \text{ and } c_{\lambda,i}(M) = 0\}$$

and denote by $\mathbb{X}(M)$ the image of $\hat{\mathbb{X}}(M)$ under the projection on the first coordinate. If $\lambda \in \mathbb{X}$, then $\lambda \in \mathbb{X}(M)$ if and only if $p_\lambda^M \neq 0$. In particular, $|\mathbb{X}(M)| \leq p^{\mathbf{d}}$. Finally, if $M, N \in \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$ are S-equivalent, the orbits $\mathcal{O}(M)$ and $\mathcal{O}(N)$ are maximal, and $\hat{\mathbb{X}}(M) \subseteq \hat{\mathbb{X}}(N)$, then $\mathcal{O}(M) = \mathcal{O}(N)$.

For a representation V of $\mathbf{\Delta}$ such that $\theta^V(\mathbf{d}) = 0$ we denote by $\mathcal{H}^V(\mathbf{d})$ the zero set of c^V, *i.e.*, $\mathcal{H}^V(\mathbf{d}) := \{M \in \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d}) : \operatorname{Hom}_{\mathbf{\Delta}}(V, M) \neq 0\}$. Moreover, we say that an exact sequence $0 \to M \to N \to L \to 0$ is V-exact if the induced sequence

$$0 \to \operatorname{Hom}_{\mathbf{\Delta}}(V, M) \to \operatorname{Hom}_{\mathbf{\Delta}}(V, N) \to \operatorname{Hom}_{\mathbf{\Delta}}(V, L) \to 0$$

is exact. We need the following version of [34, Corollary 7.4].

Proposition 4.7 *Let V be a representation of $\mathbf{\Delta}$ such that $\theta^V(\mathbf{d}) = 0$.*

(i) *If* $M \in \mathcal{H}^V(\mathbf{d})$ *and* $\dim_k \operatorname{Hom}_{\mathbf{\Delta}}(V, M) = 1$, *then*

$$\operatorname{Ker} \partial c^V(M) = \{Z \in T_M \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d}) : \pi_M(Z) \text{ is } V\text{-exact}\}.$$

(ii) *If* $M \in \mathcal{H}^V(\mathbf{d})$ *and* $\dim_k \operatorname{Hom}_{\mathbf{\Delta}}(V, M) \geq 2$, *then* $\operatorname{Ker} \partial c^V(M) = T_M \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$.

5 Auxiliary Lemmas

Throughout this section we fix a tame bound quiver $\mathbf{\Delta}$ and a sincere separating exact subcategory $\mathcal{R}$ of ind $\mathbf{\Delta}$. We use freely notation introduced in Section 2. We also fix $\mathbf{d} \in \mathbf{R}$ such that $p := p^{\mathbf{d}} > 0$.

Lemma 5.1 *If* $\lambda_0, \dots, \lambda_p \in \mathbb{X}$ *are pairwise different, then*

$$\bigcap_{l \in [0,p]} \mathcal{H}^{V_{\lambda_l}}(\mathbf{d}) = \bigcap_{\lambda \in \mathbb{X}} \mathcal{H}^{V_\lambda}(\mathbf{d}) = \bigcup_{\substack{\mathbf{d}' \in \mathbf{P}+\mathbf{R},\ \mathbf{d}'' \in \mathbf{Q} \\ \mathbf{d}'+\mathbf{d}''=\mathbf{d},\ \mathbf{d}'' \neq 0}} (\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'').$$

Proof Obviously, $\bigcap_{l \in [0,p]} \mathcal{H}^{V_{\lambda_l}}(\mathbf{d}) \supseteq \bigcap_{\lambda \in \mathbb{X}} \mathcal{H}^{V_\lambda}(\mathbf{d})$.

Now fix $\lambda \in \mathbb{X}$, $\mathbf{d}' \in \mathbf{P} + \mathbf{R}$ and $\mathbf{d}'' \in \mathbf{Q}$ such that $\mathbf{d}' + \mathbf{d}'' = \mathbf{d}$ and $\mathbf{d}'' \neq 0$. If $P \in (\mathcal{P} \cup \mathcal{R})(\mathbf{d}')$ and $Q \in \mathcal{Q}(\mathbf{d}'')$, then Proposition 2.1(iii) implies that

$$\dim_k \operatorname{Hom}_{\mathbf{\Delta}}(V_\lambda, P \oplus Q) \geq \dim_k \operatorname{Hom}_{\mathbf{\Delta}}(V_\lambda, Q) = \langle \mathbf{h}, \mathbf{d}'' \rangle_{\mathbf{\Delta}} > 0,$$

hence $(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'') \subseteq \mathcal{H}^{V_\lambda}(\mathbf{d})$.

Finally, assume that $R \in \mathcal{R}(\mathbf{d}) \cap \bigcap_{l \in [0,p]} \mathcal{H}^{V_{\lambda_l}}(\mathbf{d})$. Then $p^R_{\lambda_l} > 0$ for each $l \in [0, p]$. Consequently, $p^R \geq \sum_{l \in [0,p]} p^R_{\lambda_l} > p$, a contradiction. ■

Corollary 5.2 *Let* $\lambda_0, \dots, \lambda_p \in \mathbb{X}$ *be pairwise different. If* $\mathcal{C}$ *is an irreducible component of* $\bigcap_{l \in [0,p]} \mathcal{H}^{V_{\lambda_l}}(\mathbf{d})$, *then* $\dim \mathcal{C} = a_{\mathbf{\Delta}}(\mathbf{d}) - p - 1$ *and there exist* $\mathbf{d}' \in \mathbf{P} + \mathbf{R}$ *and* $\mathbf{d}'' \in \mathbf{Q}$ *such that* $\mathbf{d}' + \mathbf{d}'' = \mathbf{d}$, $\mathbf{d}'' \neq 0$ *and* $\mathcal{C} = \overline{(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')}$.

Proof It follows from Lemma 5.1 that $\mathcal{C}$ is an irreducible component of

$$\overline{(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')}$$

for some $\mathbf{d}' \in \mathbf{P} + \mathbf{R}$ and $\mathbf{d}'' \in \mathbf{Q}$ such that $\mathbf{d}' + \mathbf{d}'' = \mathbf{d}$ and $\mathbf{d}'' \neq 0$. Since $(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')$ is irreducible by Proposition 3.3, $\mathcal{C} = \overline{(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')}$.

We know from Proposition 3.2(i) that $\dim \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d}) = a_{\mathbf{\Delta}}(\mathbf{d})$, hence Krull's Principal Ideal Theorem [30, Section V.3] implies that $\dim \mathcal{C} \geq a_{\mathbf{\Delta}}(\mathbf{d}) - p - 1$. On the other hand, $\dim \mathcal{C} = \dim\big((\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')\big) \leq a_{\mathbf{\Delta}}(\mathbf{d}) - p - 1$ by Corollary 3.4, and the claim follows. ■

Lemma 5.3 *Let* $\lambda_0, \dots, \lambda_p \in \mathbb{X}$ *be pairwise different and* $J_l \subseteq \mathcal{A}_{\lambda_l}(\mathbf{d})$, $l \in [0, p]$. *If* $\mathcal{C}$ *is an irreducible component of* $\bigcap_{l \in [0,p]} \mathcal{H}^{V_{\lambda_l, J_l}}(\mathbf{d})$, *then* $\mathcal{C}$ *is an irreducible component of* $\bigcap_{l \in [0,p]} \mathcal{H}^{V_{\lambda_l}}(\mathbf{d})$.

Proof Similar to the proof of Corollary 5.2, we show that $\dim \mathcal{C} \geq a_{\boldsymbol{\Delta}}(\mathbf{d}) - p - 1$. On the other hand, $\mathcal{C} \subseteq \bigcap_{l\in[0,p]} \mathcal{H}^{V_{\lambda_l}}(\mathbf{d})$, hence there exists an irreducible component $\mathcal{C}'$ of $\bigcap_{l\in[0,p]} \mathcal{H}^{V_{\lambda_l}}(\mathbf{d})$ such that $\mathcal{C} \subseteq \mathcal{C}'$. Corollary 5.2 says that $\dim \mathcal{C}' = a_{\boldsymbol{\Delta}}(\mathbf{d}) - p - 1$, hence $\mathcal{C} = \mathcal{C}'$. ∎

Corollary 5.4 *Let $\lambda_0, \dots, \lambda_p \in \mathbb{X}$ be pairwise different and $J_l \subseteq \mathcal{A}_{\lambda_l}(\mathbf{d})$, $l \in [0, p]$. If $\mathcal{C}$ is an irreducible component of $\bigcap_{l\in[0,p]} \mathcal{H}^{V_{\lambda_l, J_l}}(\mathbf{d})$, then $\dim \mathcal{C} = a_{\boldsymbol{\Delta}}(\mathbf{d}) - p - 1$ and there exist $\mathbf{d}' \in \mathbf{P} + \mathbf{R}$ and $\mathbf{d}'' \in \mathbf{Q}$ such that $\mathbf{d}' + \mathbf{d}'' = \mathbf{d}$, $\mathbf{d}'' \neq 0$ and $\mathcal{C} = \overline{(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')}$.*

Proof Immediate from Lemma 5.3 and Corollary 5.2. ∎

Lemma 5.5 *If $\mathbf{d}' \in \mathbf{P} + \mathbf{R}$, $\mathbf{d}'' \in \mathbf{Q}$, $\mathbf{d}' + \mathbf{d}'' = \mathbf{d}$ and $\langle \mathbf{d}'', \mathbf{d}' \rangle_{\boldsymbol{\Delta}} = -p - 1$, then there exists $M \in (\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')$ such that $\operatorname{Ext}^2_{\boldsymbol{\Delta}}(M, M) = 0$.*

Proof Before we start the proof we recall that the functions

$$\operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d}) \to \mathbb{Z}, M \mapsto \dim_k \operatorname{Ext}^2_{\boldsymbol{\Delta}}(M, M),$$

and

$$\operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d}') \times \operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d}'') \to \mathbb{Z}, (P, Q) \mapsto \dim_k \operatorname{Ext}^1_{\boldsymbol{\Delta}}(Q, P),$$

are upper semi-continuous, see [5, 3.4] and [22, Lemma 4.3], respectively.

Proposition 3.3 implies that $\overline{(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')}$ is an irreducible set of dimension $a_{\boldsymbol{\Delta}}(\mathbf{d}) - p - 1$, hence $\overline{(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')}$ is an irreducible component of $\bigcap_{\lambda\in\mathbb{X}} \mathcal{H}^{\lambda}(\mathbf{d})$ by Lemma 5.1 and Corollary 5.2. Let $M \in (\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')$ be such that $\mathcal{O}(M)$ is maximal in $\bigcap_{\lambda\in\mathbb{X}} \mathcal{H}^{\lambda}(\mathbf{d})$ and

$$\dim_k \operatorname{Ext}^2_{\boldsymbol{\Delta}}(M, M) = \min\{\dim_k \operatorname{Ext}^2_{\boldsymbol{\Delta}}(N, N) : N \in (\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')\}.$$

Write $M = P \oplus Q$ for $P \in (\mathcal{P} \cup \mathcal{R})(\mathbf{d}')$ and $Q \in \mathcal{Q}(\mathbf{d}'')$.

There exists an exact sequence $\xi\colon 0 \to P \to R \to Q \to 0$ with $R \in \mathcal{R}(\mathbf{d})$. Indeed, since $M \notin \operatorname{add} \mathcal{R}$, $\mathcal{O}(M)$ is not maximal in $\operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d})$, hence there exist indecomposable direct summands X and Y of M such that $\operatorname{Ext}^1_{\boldsymbol{\Delta}}(Y, X) \neq 0$. Since $\mathcal{O}(M)$ is maximal in $\bigcap_{\lambda\in\mathbb{X}} \mathcal{H}^{\lambda}(\mathbf{d})$, it follows that $\mathcal{O}(P)$ and $\mathcal{O}(Q)$ are maximal in $(\mathcal{P} \cup \mathcal{R})(\mathbf{d}')$ and $\mathcal{Q}(\mathbf{d}'')$, respectively, hence either X is a direct summand of P and Y is a direct summand of Q or Y is a direct summand of P and X is a direct summand of Q. However, the latter case in not possible, since $\operatorname{Ext}^1_{\boldsymbol{\Delta}}(P, Q) = 0$. Consequently, we obtain an exact sequence of the above form. Finally, since $\mathcal{O}(M)$ is maximal in $\bigcap_{\lambda\in\mathbb{X}} \mathcal{H}^{\lambda}(\mathbf{d})$, it follows that $R \notin \bigcap_{\lambda\in\mathbb{X}} \mathcal{H}^{\lambda}(\mathbf{d})$, hence $R \in \mathcal{R}(\mathbf{d})$ by Lemma 5.1. Note that $\operatorname{pdim}_{\boldsymbol{\Delta}} R \leq 1$, hence the map $\Phi\colon \operatorname{Ext}^1_{\boldsymbol{\Delta}}(P, P) \to \operatorname{Ext}^2_{\boldsymbol{\Delta}}(Q, P)$, $\xi' \mapsto \xi' \circ \xi$, is an epimorphism.

Minimality of

$$\dim_k \operatorname{Ext}^2_{\boldsymbol{\Delta}}(M, M) = \dim_k \operatorname{Ext}^2_{\boldsymbol{\Delta}}(Q, P) = \dim_k \operatorname{Ext}^1_{\boldsymbol{\Delta}}(Q, P) + \langle \mathbf{d}'', \mathbf{d}' \rangle_{\boldsymbol{\Delta}}$$

implies that the set

$$\mathcal{E} := \{(U, V) \in (\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \times \mathcal{Q}(\mathbf{d}'') : \dim_k \operatorname{Ext}^1_{\boldsymbol{\Delta}}(V, U) = \dim_k \operatorname{Ext}^1_{\boldsymbol{\Delta}}(Q, P)\}$$

is an open subset of $(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \times \mathcal{Q}(\mathbf{d}'')$. However, $(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \times \mathcal{Q}(\mathbf{d}'')$ is an open subset of $\operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d}') \times \operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d}'')$ by [5, Lemmas 3.7 and 3.8] (these lemmas are formulated in the case of canonical algebras, but the proofs transfer without any changes to arbitrary concealed-canonical algebras). Consequently, $\mathcal{E}$ is an open subset of $\operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d}') \times \operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d}'')$, hence $T_{P,Q}\mathcal{E} = T_P \operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d}') \times T_Q \operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d}'')$. On the other hand, [6, Proposition 3.3] implies that

$$T_{P,Q}\mathcal{E} \subseteq \{(Z', Z'') \in T_P \operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d}') \times T_Q \operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d}'') : \pi_P(Z') \circ \xi + \xi \circ \pi_Q(Z'') = 0\}.$$

Since π_P is an epimorphism (as $\operatorname{pdim}_{\boldsymbol{\Delta}} P \leq 1$), this implies that $\Phi = 0$. Since Φ is an epimorphism, $\operatorname{Ext}^2_{\boldsymbol{\Delta}}(Q, P) = 0$, and the claim follows. ■

Proposition 5.6 *Let $\lambda_0, \dots, \lambda_p \in \mathbb{X}$ be pairwise different and $J_l \subseteq \mathcal{A}_{\lambda_l}(\mathbf{d})$, $l \in [0, p]$. If $\mathbf{d}' \in \mathbf{P} + \mathbf{R}$, $\mathbf{d}'' \in \mathbf{Q}$ and $\overline{(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')}$ is an irreducible component of $\bigcap_{l \in [0,p]} \mathcal{H}^{V_{\lambda_l, J_l}}(\mathbf{d})$, then $\langle \mathbf{dim}\, V_{\lambda_l, J_l}, \mathbf{d}'' \rangle_{\boldsymbol{\Delta}} > 0$ for each $l \in [0, p]$. Moreover, if $\langle \mathbf{dim}\, V_{\lambda_l, J_l}, \mathbf{d}'' \rangle_{\boldsymbol{\Delta}} = 1$ for each $l \in [0, p]$, then there exists $M \in (\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')$ such that $\operatorname{Ext}^2_{\boldsymbol{\Delta}}(M, M) = 0$ and $\partial c_{\lambda_0, J_0}(M), \dots, \partial c_{\lambda_p, J_p}(M)$ are linearly independent.*

Proof We know from Lemma 5.3 that $\overline{(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')}$ is an irreducible component of $\bigcap_{l \in [0,p]} \mathcal{H}^{V_{\lambda_l}}(\mathbf{d})$. In particular, $\dim\big((\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')\big) = a_{\boldsymbol{\Delta}}(\mathbf{d}) - p - 1$ by Corollary 5.2. Consequently, Proposition 3.3 implies that $\langle \mathbf{d}'', \mathbf{d}' \rangle_{\boldsymbol{\Delta}} = -p - 1$. Using Lemma 5.5 we may choose $M \in (\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')$ such that $\mathcal{O}(M)$ is maximal in $\bigcap_{l \in [0,p]} \mathcal{H}^{V_{\lambda_l}}(\mathbf{d})$ and $\operatorname{Ext}^2_{\boldsymbol{\Delta}}(M, M) = 0$. Write $M = P \oplus Q$ for $P \in (\mathcal{P} \cup \mathcal{R})(\mathbf{d}')$ and $Q \in \mathcal{Q}(\mathbf{d}'')$.

First we prove that $\operatorname{Hom}_{\boldsymbol{\Delta}}(V_{\lambda_l, J_l}, P) = 0$ for each $l \in [0, p]$. This will imply in particular that

$$\langle \mathbf{dim}\, V_{\lambda_l, J_l}, \mathbf{d}'' \rangle_{\boldsymbol{\Delta}} = \dim_k \operatorname{Hom}_{\boldsymbol{\Delta}}(V_{\lambda_l, J_l}, Q) = \dim_k \operatorname{Hom}_{\boldsymbol{\Delta}}(V_{\lambda_l, J_l}, M) > 0$$

for each $l \in [0, p]$. Write $P = P' \oplus R$ for $P' \in \operatorname{add} \mathcal{P}$ and $R \in \operatorname{add} R$, and assume $\operatorname{Hom}_{\boldsymbol{\Delta}}(V_{\lambda_l, i}, R) \neq 0$ for some $l \in [0, p]$ and $i \in J_l$. Then $q^R_{\lambda_l, i} > 0$. If $p^R > 0$, then $\langle \mathbf{dim}\, Q, \mathbf{dim}\, R \rangle_{\boldsymbol{\Delta}} \leq \langle \mathbf{d}'', \mathbf{h} \rangle_{\boldsymbol{\Delta}} < 0$ by Proposition 2.1(iii) (recall that $\mathbf{d}'' \neq 0$ by Corollary 5.4). Otherwise, we fix $n \in \mathbb{N}$ such that $q^R_{\lambda, (i+n) \bmod r_\lambda} = 0$ and $q^R_{\lambda, (i+j) \bmod r_\lambda} > 0$ for each $j \in [1, n-1]$. Then

$$\begin{aligned} \langle \mathbf{d}'', \mathbf{e}^n_{\lambda_l, i+n-1} \rangle_{\boldsymbol{\Delta}} &= \langle \mathbf{d} - \mathbf{dim}\, P' - \mathbf{dim}\, R, \mathbf{e}^n_{\lambda_l, i+n-1} \rangle_{\boldsymbol{\Delta}} \\ &\leq -p^{\mathbf{d}}_{\lambda_l, (i+n) \bmod r_\lambda} - \langle \mathbf{dim}\, P', \mathbf{e}^n_{\lambda, i+n-1} \rangle_{\boldsymbol{\Delta}} - q^R_{\lambda_l, i} < 0. \end{aligned}$$

This again implies that $\langle \mathbf{dim}\, Q, \mathbf{dim}\, R \rangle_{\boldsymbol{\Delta}} < 0$, hence $\operatorname{Ext}^1_{\boldsymbol{\Delta}}(Q, R) \neq 0$. If $0 \to R \to Q' \to Q \to 0$ is a non-split exact sequence, then $P' \oplus Q' \in \bigcap_{l \in [0,p]} \mathcal{H}^{V_{\lambda_l}}(\mathbf{d})$, since $\dim_k \underline{\operatorname{Hom}_{\boldsymbol{\Delta}}}(V_\lambda, Q') \geq \langle \mathbf{h}, \mathbf{dim}\, Q' \rangle_{\boldsymbol{\Delta}} = \langle \mathbf{h}, \mathbf{d}'' \rangle_{\boldsymbol{\Delta}} > 0$ for each $\lambda \in \mathbb{X}$. Moreover, $M \in \overline{\mathcal{O}(P' \oplus Q')}$ and $M \not\simeq P' \oplus Q'$, which contradicts the maximality of $\mathcal{O}(M)$.

Now we assume that $\langle \mathbf{dim}\, V_{\lambda_l, J_l}, \mathbf{d}'' \rangle_{\boldsymbol{\Delta}} = 1$ for each $l \in [0, p]$ and we prove that under this assumption $\partial c_{\lambda_0, J_0}(M), \dots, \partial c_{\lambda_p, J_p}(M)$ are linearly independent. Our assumption implies that

$$\dim_k \operatorname{Hom}_{\boldsymbol{\Delta}}(V_{\lambda_l, J_l}, M) = \dim_k \operatorname{Hom}_{\boldsymbol{\Delta}}(V_{\lambda_l, J_l}, Q) = 1$$

for each $l \in [0,p]$. Let $K := \bigcap_{l\in[0,p]} \operatorname{Ker} \partial c^{V_{\lambda_l,J_l}}(M) \subseteq T_M \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$. We have the canonical inclusion $\operatorname{Ext}^1_{\mathbf{\Delta}}(Q,P) \hookrightarrow \operatorname{Ext}^1_{\mathbf{\Delta}}(M,M)$, which sends an exact sequence $\xi\colon 0 \to P \to N \to Q \to 0$ to the sequence

$$\xi'\colon 0 \to M \to N \oplus M \to M \to 0.$$

Using Proposition 4.7(i) we obtain that $\xi' \in \pi_M(K)$ if and only if

$$\dim_k \operatorname{Hom}_{\mathbf{\Delta}}(V_{\lambda_l,J_l}, N) = 1 \quad \text{for each} \quad l \in [0,p].$$

In particular, this implies that $N \in \bigcap_{l\in[0,p]} \mathcal{H}^{V_{\lambda_l,J_l}}(\mathbf{d})$. By the maximality of $\mathcal{O}(M)$, $N \simeq M$, *i.e.*, ξ splits, thus Proposition 3.2(iv) implies that

$$\operatorname{codim}_{T_M \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})} K \geq \dim_k \operatorname{Ext}^1_{\mathbf{\Delta}}(Q,P) \geq -\langle \mathbf{d}'', \mathbf{d}' \rangle_{\mathbf{\Delta}} = p+1,$$

and this finishes the proof. ■

Let $\lambda_0,\dots,\lambda_p \in \mathbb{X}$ be pairwise different and $J_l \subseteq \mathcal{A}_{\lambda_l}(\mathbf{d})$, $l \in [0,p]$. Assume that $\overline{(\mathcal{P}\cup\mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')}$ is an irreducible component of $\bigcap_{l\in[0,p]} \mathcal{H}^{V_{\lambda_l,J_l}}(\mathbf{d})$ for $\mathbf{d}' \in \mathbf{P}+\mathbf{R}$ and $\mathbf{d}'' \in \mathbf{Q}$. We know from Corollary 5.4 that $\dim\big((\mathcal{P}\cup\mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')\big) = a_{\mathbf{\Delta}}(\mathbf{d}) - p - 1$. Consequently, either $q_{\mathbf{\Delta}}(\mathbf{d}'') = 0$ or $q_{\mathbf{\Delta}}(\mathbf{d}'') = 1$ by Corollary 3.4. We prove that in the latter case there is always $M \in (\mathcal{P}\cup\mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')$ such that $\operatorname{Ext}^2_{\mathbf{\Delta}}(M,M) = 0$ and $\partial c_{\lambda_0,J_0}(M),\dots,\partial c_{\lambda_p,J_p}(M)$ are linearly independent.

Corollary 5.7 *Let $\lambda_0,\dots,\lambda_p \in \mathbb{X}$ be pairwise different and $J_l \subseteq \mathcal{A}_{\lambda_l}(\mathbf{d})$, $l \in [0,p]$. If $\mathbf{d}' \in \mathbf{P}+\mathbf{R}$, $\mathbf{d}'' \in \mathbf{Q}$, $\overline{(\mathcal{P}\cup\mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')}$ is an irreducible component of $\bigcap_{l\in[0,p]} \mathcal{H}^{V_{\lambda_l,J_l}}(\mathbf{d})$ and $q_{\mathbf{\Delta}}(\mathbf{d}'') = 1$, then there exists $M \in (\mathcal{P}\cup\mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')$ such that $\operatorname{Ext}^2_{\mathbf{\Delta}}(M,M) = 0$ and $\partial c_{\lambda_0,J_0}(M),\dots,\partial c_{\lambda_p,J_p}(M)$ are linearly independent.*

Proof From the previous proposition we know that $\langle \mathbf{dim}\, V_{\lambda_l,J_l}, \mathbf{d}'' \rangle_{\mathbf{\Delta}} > 0$ for each $l \in [0,p]$. On the other hand, Corollary 3.4 implies that

$$\langle \mathbf{dim}\, V_{\lambda_l,J_l}, \mathbf{d}'' \rangle_{\mathbf{\Delta}} \leq \langle \mathbf{h}, \mathbf{d}'' \rangle_{\mathbf{\Delta}} = 1$$

for each $l \in [0,p]$. Consequently, $\langle \mathbf{dim}\, V_{\lambda_l,J_l}, \mathbf{d}'' \rangle_{\mathbf{\Delta}} = 1$ for each $l \in [0,p]$, and the claim follows from the previous proposition. ■

6 Non-singular Dimension Vectors

Throughout this section we fix a sincere separating exact subcategory $\mathcal{R}$ of $\operatorname{ind}\mathbf{\Delta}$ for a tame bound quiver $\mathbf{\Delta}$ and use freely notation introduced in Section 2. We also fix $\mathbf{d} \in \mathbf{R}$ such that $p := p^{\mathbf{d}} > 0$. Finally, we assume that $\mathbf{d}$ is not singular. This assumption implies, according to Proposition 2.3(ii) and Corollary 3.4, that $q_{\mathbf{\Delta}}(\mathbf{d}'') = 1$ for any $\mathbf{d}' \in \mathbf{P}+\mathbf{R}$ and $\mathbf{d}'' \in \mathbf{Q}$ such that $\mathbf{d}'+\mathbf{d}'' = \mathbf{d}$ and $\dim\big((\mathcal{P}\cup\mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')\big) = a_{\mathbf{\Delta}}(\mathbf{d}) - p - 1$. Consequently, we have the following.

Lemma 6.1 *Let $\lambda_0, \dots, \lambda_p \in \mathbb{X}$ be pairwise different and $J_l \subseteq \mathcal{A}_{\lambda_l}(\mathbf{d})$, $l \in [0, p]$. If $\mathcal{C}$ is an irreducible component of $\bigcap_{l \in [0,p]} \mathcal{H}^{V_{\lambda_l, J_l}}(\mathbf{d})$, then there exists $M \in \mathcal{C}$ such that $\mathrm{Ext}^2_{\boldsymbol{\Delta}}(M, M) = 0$ and $\partial c_{\lambda_0, J_0}(M), \dots, \partial c_{\lambda_p, J_p}(M)$ are linearly independent.*

Proof We know from Corollary 5.4 that $\dim \mathcal{C} = a_{\boldsymbol{\Delta}}(\mathbf{d}) - p - 1$ and

$$\mathcal{C} = \overline{(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')}$$

for $\mathbf{d}' \in \mathbf{P} + \mathbf{R}$ and $\mathbf{d}'' \in \mathbf{Q}$. Since $q_{\boldsymbol{\Delta}}(\mathbf{d}'') = 1$, the claim follows from Corollary 5.7. ∎

Corollary 6.2 *If $\lambda_0, \dots, \lambda_p \in \mathbb{X}$ are pairwise different, then*

$$\Big\{ f \in k[\mathrm{rep}_{\boldsymbol{\Delta}}(\mathbf{d})] : f(M) = 0 \text{ for each } M \in \bigcap_{l \in [0,p]} \mathcal{H}^{V_{\lambda_l}}(\mathbf{d}) \Big\} = (c_{\lambda_0}, \dots, c_{\lambda_p}).$$

Proof We know from Proposition 3.2(i) that $\mathrm{rep}_{\boldsymbol{\Delta}}(\mathbf{d})$ is a complete intersection. Moreover, the previous lemma implies that for each irreducible component $\mathcal{C}$ of $\bigcap_{l \in [0,p]} \mathcal{H}^{V_{\lambda_l}}(\mathbf{d})$ there exists $M \in \mathcal{C}$ such that $\partial c_{\lambda_0}(M), \dots, \partial c_{\lambda_p}(M)$ are linearly independent and $\dim_k \mathrm{Ext}^2_{\boldsymbol{\Delta}}(M, M) = 0$. Then $\dim_k T_M \mathrm{rep}_{\boldsymbol{\Delta}}(\mathbf{d}) = \dim \mathrm{rep}_{\boldsymbol{\Delta}}(\mathbf{d})$ by Proposition 3.2(v), hence the claim follows from Proposition 3.1(ii). ∎

Proposition 6.3 *Let $\lambda_0, \dots, \lambda_p \in \mathbb{X}$ be pairwise different. If $M, N \in \mathcal{R}(\mathbf{d})$ and there exists $\mu \in k$ such that $c_{\lambda_l}(M) = \mu c_{\lambda_l}(N)$ for each $l \in [0, p]$, then M and N are S-equivalent.*

Proof Lemma 5.1 implies that $c_{\lambda_l}(M) \neq 0$ for some $l \in [0, p]$. Without loss of generality, we may assume that $c_{\lambda_0}(M) \neq 0$. Then $\mu \neq 0$ and $c_{\lambda_0}(N) \neq 0$. For $l \in [0, p]$ we put $\mu_l := \frac{c_{\lambda_l}(M)}{c_{\lambda_0}(M)}$. Observe that $c_{\lambda_l}(N) = \mu_l c_{\lambda_0}(N)$ for each $l \in [0, p]$.

Fix $\lambda \in \mathbb{X}$, and put $\mu' := \frac{c_\lambda(M)}{c_{\lambda_0}(M)}$ and $\mu'' := \frac{c_\lambda(N)}{c_{\lambda_0}(N)}$. We know from Lemma 5.1 that $c_\lambda(V) = 0$ for each $V \in \bigcap_{l \in [0,p]} \mathcal{H}^{V_{\lambda_l}}(\mathbf{d})$, hence Corollary 6.2 implies that there exist $f_0, \dots, f_p \in k[\mathrm{rep}_{\boldsymbol{\Delta}}(\mathbf{d})]$ such that $c_\lambda = \sum_{l \in [0,p]} f_l c_{\lambda_l}$. Put $f := \sum_{l \in [0,p]} \mu_l f_l$. Then

$$\begin{aligned} c_\lambda(g \cdot M) &= \sum_{l \in [0,p]} f_l(g \cdot M) c_{\lambda_l}(g \cdot M) \\ &= \sum_{l \in [0,p]} \mu_l f_l(g \cdot M) c_{\lambda_0}(g \cdot M) = f(g \cdot M) c_{\lambda_0}(g \cdot M) \end{aligned}$$

for each $g \in \mathrm{GL}(\mathbf{d})$. Recall that c_λ and c_{λ_0} are semi-invariants of the same weight, hence $f(g \cdot M) = \frac{c_\lambda(M)}{c_{\lambda_0}(M)} = \mu'$ for each $g \in \mathrm{GL}(\mathbf{d})$. Similarly, $f(g \cdot N) = \mu''$ for each $g \in \mathrm{GL}(\mathbf{d})$. Since $\overline{\mathcal{O}(M)} \cap \overline{\mathcal{O}(N)} \neq \varnothing$ ($S^{\mathbf{d}} \in \overline{\mathcal{O}(M)} \cap \overline{\mathcal{O}(N)}$), $\mu' = \mu''$. Consequently,

$$c_\lambda(M) = \mu' c_{\lambda_0}(M) = \mu'' \mu c_{\lambda_0}(N) = \mu c_\lambda(N),$$

and the claim follows from Corollary 4.6. ∎

Proposition 6.4 *If $\mathcal{O}(M) \subseteq \operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d})$ is maximal, then there exist $\lambda_0, \dots, \lambda_p \in \mathbb{X}$, $i_0 \in \mathcal{A}_{\lambda_0}(\mathbf{d}), \dots, i_p \in \mathcal{A}_{\lambda_p}(\mathbf{d})$, and $\mu_1, \dots, \mu_p \in k$, such that*

$$\{f \in k[\operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d})] : f(N) = 0 \text{ for each } N \in \overline{\mathcal{O}(M)}\} \\ = (c_{\lambda_1,i_1} - \mu_1 c_{\lambda_0,i_0}, \dots, c_{\lambda_p,i_p} - \mu_p c_{\lambda_0,i_0}).$$

In particular, $\overline{\mathcal{O}(M)}$ is a complete intersection of dimension $a_{\boldsymbol{\Delta}}(\mathbf{d}) - p$.

Proof First, let $(\lambda_1, i_1), \dots, (\lambda_q, i_q)$ be the pairwise different elements of $\hat{\mathbb{X}}(M)$. We put $\mu_l := 0$ for $l \in [1, q]$. Next, we choose pairwise different $\lambda_0, \lambda_{q+1}, \dots, \lambda_p \in \mathbb{X} \setminus \big(\mathbb{X}_0 \cup \mathbb{X}(M)\big)$. Finally, we put $i_0 := 0$, and $i_l := 0$ and $\mu_l := \frac{c_{\lambda_l}(M)}{c_{\lambda_0}(M)}$ for $l \in [q+1, p]$.

Let

$$\mathcal{V} := \{N \in \operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d}) : c_{\lambda_l,i_l}(N) - \mu_l c_{\lambda_0,i_0}(N) = 0 \text{ for each } l \in [1, p]\}$$

and $\mathcal{V}' := \bigcap_{l \in [0,p]} \mathcal{H}^{V_{\lambda_l,i_l}}(\mathbf{d})$. Obviously $\mathcal{V}' \subseteq \mathcal{V}$. Moreover, every irreducible component of $\mathcal{V}'$ has dimension $a_{\boldsymbol{\Delta}}(\mathbf{d}) - p - 1$ by Corollary 5.4, hence Krull's Principal Ideal Theorem implies that every irreducible component of $\mathcal{V}$ has dimension $a_{\boldsymbol{\Delta}}(\mathbf{d}) - p$. In particular, Corollary 3.4 implies that $\mathcal{R}(\mathbf{d}) \cap \mathcal{C}$ is a non-empty open subset of $\mathcal{C}$ for each irreducible component $\mathcal{C}$ of $\mathcal{V}$. Note that $c_{\lambda_l}(R) = \frac{c_{\lambda_0}(R)}{c_{\lambda_0}(M)} c_{\lambda_l}(M)$ for any $l \in [0, p]$ and $R \in \mathcal{R}(\mathbf{d}) \cap \mathcal{V}$, thus Proposition 6.3 implies that R is S-equivalent to M for each $R \in \mathcal{V} \cap \mathcal{R}(\mathbf{d})$. Consequently, there are only finitely many orbits in $\mathcal{R}(\mathbf{d}) \cap \mathcal{V}$ by Proposition 4.4. This implies that every irreducible component of $\mathcal{V}$ is of the form $\overline{\mathcal{O}(R)}$ for some $R \in \mathcal{R}(\mathbf{d})$. Fix $R \in \mathcal{R}(\mathbf{d})$ such that $\overline{\mathcal{O}(R)}$ is an irreducible component of $\mathcal{V}$. Since $\dim \mathcal{O}(R) = a_{\boldsymbol{\Delta}}(\mathbf{d}) - p$, $\mathcal{O}(R)$ is maximal in $\operatorname{rep}_{\boldsymbol{\Delta}}(R)$. Moreover, R and M are S-equivalent and $\hat{\mathbb{X}}(M) \subseteq \hat{\mathbb{X}}(R)$, hence $\mathcal{O}(R) = \mathcal{O}(M)$. Consequently, $\mathcal{V} = \overline{\mathcal{O}(M)}$.

Lemma 6.1 implies that there exists $N \in \mathcal{V}$ such that $\operatorname{Ext}^2_{\boldsymbol{\Delta}}(N, N) = 0$ and $\partial c_{\lambda_0,i_0}(N), \dots, \partial c_{\lambda_p,i_p}(N)$ are linearly independent. Then obviously

$$\partial c_{\lambda_1,i_1}(N) - \mu_1 \partial c_{\lambda_0,i_0}(N), \dots, \partial c_{\lambda_p,i_p}(N) - \mu_p \partial c_{\lambda_0,i_0}(N)$$

are linearly independent as well. Moreover, $\dim T_N \operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d}) = \dim \operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d})$ by Proposition 3.2(v). Since $\operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d})$ is a complete intersection by Proposition 3.2(i), the claim follows from Proposition 3.1(ii). ∎

Proposition 6.5 *If $\mathcal{O}(M) \subseteq \operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d})$ is maximal, then the variety $\overline{\mathcal{O}(M)}$ is normal.*

Proof We know from Proposition 6.4 that there exist

$$\lambda_0, \dots, \lambda_p \in \mathbb{X}, \quad i_0 \in \mathcal{A}_{\lambda_0}(\mathbf{d}), \dots, i_p \in \mathcal{A}_{\lambda_p}(\mathbf{d}), \quad \text{and} \quad \mu_1, \dots, \mu_p \in k,$$

such that

$$\{f \in k[\operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d})] : f(N) = 0 \text{ for each } N \in \overline{\mathcal{O}(M)}\} = (c_{\lambda_l,i_l} - \mu_l c_{\lambda_0,i_0} : l \in [1, p]).$$

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Let

$$\mathcal{U} := \{N \in \overline{\mathcal{O}(M)} : \dim_k T_N\overline{\mathcal{O}(M)} = \dim \mathcal{O}(M)\}.$$

Proposition 3.2(v) implies that $\mathcal{U}$ is the set of all $N \in \overline{\mathcal{O}(M)}$ such that $\operatorname{Ext}^2_{\mathbf{\Delta}}(N, N) = 0$ and $\partial c_{\lambda_1,i_1}(N) - \mu_1\partial c_{\lambda_0,i_0}(N), \dots, \partial c_{\lambda_p,i_p}(N) - \mu_p\partial c_{\lambda_0,i_0}(N)$ are linearly independent.

By general theory, $\mathcal{O}(M) \subseteq \mathcal{U}$, hence $\overline{\mathcal{O}(M)} \setminus \mathcal{U} \subseteq \mathcal{V}' \cup \mathcal{V}''$, where $\mathcal{V}' := \bigcap_{l\in[0,p]} \mathcal{H}^{V_{\lambda_l,i_l}}(\mathbf{d})$ and $\mathcal{V}'' := \big(\overline{\mathcal{O}(M)} \setminus \mathcal{O}(M)\big) \cap \mathcal{R}(\mathbf{d})$. Lemma 6.1 says that $\mathcal{U} \cap \mathcal{C} \neq \varnothing$ for each each irreducible component $\mathcal{C}$ of $\mathcal{V}'$, thus $\dim(\mathcal{V}' \setminus \mathcal{U}) < \dim \mathcal{V}' = a_{\mathbf{\Delta}}(\mathbf{d}) - p - 1 = \dim \mathcal{O}(M) - 1$. On the other hand, if $R \in \mathcal{V}''$, then R is S-equivalent to M by Proposition 6.3, hence $\mathcal{V}''$ is a union of finitely many orbits according to Proposition 4.4. Moreover, [48, Theorem 1.1] implies that $R \in \mathcal{U}$ for each $R \in \mathcal{V}''$ such that $\dim \mathcal{O}(R) = \dim \mathcal{O}(M) - 1$. Concluding, we obtain that $\dim\big(\overline{\mathcal{O}(M)} \setminus \mathcal{U}\big) < \dim \mathcal{O}(M) - 1$. Since $\overline{\mathcal{O}(M)}$ is a complete intersection by Proposition 6.4, the claim follows from Proposition 3.1(i). ■

7 Singular Dimension Vectors

Throughout this section we fix a sincere separating exact subcategory $\mathcal{R}$ of ind $\mathbf{\Delta}$ for a tame bound quiver $\mathbf{\Delta}$ and use freely notation introduced in Section 2. We also fix singular $\mathbf{d} \in \mathbf{R}$. Proposition 2.3(i) implies that $\mathbf{d} = \mathbf{h}$ and $\mathbf{\Delta}$ is of type $(2, 2, 2, 2)$. Let $\mathcal{O}(M) \subseteq \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{h})$ be maximal. It follows from [10, Proposition 5] that $M \simeq R_{\lambda,i}^{(r_\lambda)}$ for some $\lambda \in \mathbb{X}$ and $i \in [0, r_\lambda - 1]$. We prove that $\overline{\mathcal{O}(M)}$ is normal if and only if $r_\lambda = 2$. Note that $\hat{\mathbb{X}}(M) = \{(\lambda, j)\}$, where $j := (i - 1) \bmod r_\lambda$. Moreover, $V_{\lambda,j} = R_{\lambda,j}$.

Proposition 7.1 *We have*

$$\{f \in k[\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{h})] : f(N) = 0 \textit{ for each } N \in \overline{\mathcal{O}(M)}\} = (c_{\lambda,j}).$$

In particular, $\overline{\mathcal{O}(M)}$ is a complete intersection of dimension $a_{\mathbf{\Delta}}(\mathbf{h}) - 1$.

Proof We know from Proposition 3.2(i) that $\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{h})$ is an irreducible variety of dimension $a_{\mathbf{\Delta}}(\mathbf{h})$, hence Krull's Principal Ideal Theorem implies that every irreducible component of $\mathcal{H}^{V_{\lambda,j}}(\mathbf{h})$ has dimension $a_{\mathbf{\Delta}}(\mathbf{h}) - 1$. Observe that $\mathcal{R}(\mathbf{h}) \cap \mathcal{H}^{V_{\lambda,j}}(\mathbf{h})$ is a union of finitely many orbits. Since $\dim\big(\mathcal{H}^{V_{\lambda,j}}(\mathbf{h}) \setminus \mathcal{R}(\mathbf{h})\big) \leq a_{\mathbf{\Delta}}(\mathbf{h}) - 2$ by Corollary 3.4, this implies that every irreducible component of $\mathcal{V}$ is of the form $\overline{\mathcal{O}(R)}$ for a maximal orbit $\mathcal{O}(R)$ in $\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{h})$. However, [10, Proposition 5] implies that $\mathcal{O}(M)$ is a unique maximal orbit in $\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{h})$ which is contained in $\mathcal{H}^{V_{\lambda,j}}(\mathbf{h})$, hence $\mathcal{H}^{V_{\lambda,j}}(\mathbf{h}) = \overline{\mathcal{O}(M)}$.

We know that $\dim_k \operatorname{Ext}^1_{\mathbf{\Delta}}(M, M) = 1$ and the non-split exact sequences in $\operatorname{Ext}^1_{\mathbf{\Delta}}(M, M)$ are of the form $\xi\colon 0 \to M \to N \to M \to 0$ with $N \simeq R_{\lambda,i}^{(2r_\lambda)}$. In particular, $\dim_k \operatorname{Hom}_{\mathbf{\Delta}}(V_{\lambda,j}, N) = 1$. Consequently, the sequence $0 \to M \to M \oplus M \to M \to 0$ is the only $V_{\lambda,j}$-exact sequence in $\operatorname{Ext}^1_{\mathbf{\Delta}}(M, M)$. Propositions 4.7(i) and 3.2(iv) imply that $\partial c^{V_{\lambda,j}}(M)$ is non-zero. Since $\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{h})$ is a complete intersection by Proposition 3.2(i) and $\dim_k T_M \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d}) = \dim \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$ by Proposition 3.2(v) (note that $\operatorname{pdim}_{\mathbf{\Delta}} M \leq 1$, since $M \in \mathcal{R}$), the claim follows from Proposition 3.1(ii). ■

Proposition 7.2 *Let*

$$\mathcal{U} := \{N \in \overline{\mathcal{O}(M)} : \dim_k T_N \overline{\mathcal{O}(M)} = \dim \mathcal{O}(M)\}.$$

(i) *If* $r_\lambda = 1$, *then* $\dim\big(\overline{\mathcal{O}(M)} \setminus \mathcal{U}\big) = \dim \mathcal{O}(M) - 1$. *In particular,* $\overline{\mathcal{O}(M)}$ *is not normal.*

(ii) *If* $r_\lambda = 2$, *then* $\dim\big(\overline{\mathcal{O}(M)} \setminus \mathcal{U}\big) < \dim \mathcal{O}(M) - 1$. *In particular,* $\overline{\mathcal{O}(M)}$ *is normal.*

Proof Fix $\lambda' \in \mathbb{X} \setminus (\mathbb{X}_0 \cup \{\lambda\})$. Lemma 5.1 implies that $\operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{h}) \setminus \mathcal{R}(\mathbf{h}) = \mathcal{H}^{V_\lambda}(\mathbf{h}) \cap \mathcal{H}^{V_{\lambda'}}(\mathbf{h})$. By general theory $\mathcal{O}(M) \subseteq \mathcal{U}$, hence $\overline{\mathcal{O}(M)} \setminus \mathcal{U} \subseteq \mathcal{V}' \cup \mathcal{V}''$, where $\mathcal{V}' := \big(\overline{\mathcal{O}(M)} \setminus \mathcal{O}(M)\big) \cap \mathcal{R}(\mathbf{h})$ and $\mathcal{V}'' := \mathcal{H}^{V_{\lambda,j}}(\mathbf{h}) \cap \mathcal{H}^{V_{\lambda'}}(\mathbf{h})$. We know that $\mathcal{V}'$ is a union of finitely many orbits. Moreover, [48, Theorem 1.1] implies that $R \in \mathcal{U}$ for each $R \in \mathcal{V}'$ such that $\dim \mathcal{O}(R) = \dim \mathcal{O}(M) - 1$. Consequently, $\dim(\mathcal{V}' \setminus \mathcal{U}) < \dim \mathcal{V}' \leq \dim \mathcal{O}(M) - 1$.

Now let $\mathcal{C}$ be an irreducible component of $\mathcal{V}''$. Corollary 5.4 implies that $\dim \mathcal{C} = a_{\boldsymbol{\Delta}}(\mathbf{h}) - 2$ and there exist $\mathbf{d}' \in \mathbf{P}+\mathbf{R}$ and $\mathbf{d}'' \in \mathbf{Q}$ such that $\mathcal{C} = \overline{(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')}$. Moreover, Corollary 3.4 implies that either $q_{\boldsymbol{\Delta}}(\mathbf{d}'') = 1$ or $q_{\boldsymbol{\Delta}}(\mathbf{d}'') = 0$. If $q_{\boldsymbol{\Delta}}(\mathbf{d}'') = 1$, then Corollary 5.7 implies that $\mathcal{U} \cap \mathcal{C} \neq \varnothing$.

Assume that $q_{\boldsymbol{\Delta}}(\mathbf{d}'') = 0$ (according to Proposition 2.3(ii), this case appears, since $\mathbf{d}''$ is singular). Then $\langle \mathbf{h}, \mathbf{d}'' \rangle_{\boldsymbol{\Delta}} = 2$ by Corollary 3.4. If $r_\lambda = 2$, then $\langle \operatorname{\mathbf{dim}} V_{\lambda,j}, \mathbf{d}'' \rangle_{\boldsymbol{\Delta}} = 1$. Indeed, we know from Proposition 5.6 that $\langle \operatorname{\mathbf{dim}} V_{\lambda,j}, \mathbf{d}'' \rangle_{\boldsymbol{\Delta}} > 0$. On the other hand, Proposition 2.1(iv) implies that

$$\langle \operatorname{\mathbf{dim}} V_{\lambda,j}, \mathbf{d}'' \rangle_{\boldsymbol{\Delta}} = \langle \mathbf{h}, \mathbf{d}'' \rangle_{\boldsymbol{\Delta}} - \langle \mathbf{e}_{\lambda,i}, \mathbf{d}'' \rangle_{\boldsymbol{\Delta}} \leq 2 - 1 = 1.$$

Consequently, Proposition 5.6 implies that also in this case $\mathcal{U} \cap \mathcal{C} \neq \varnothing$. On the other hand, if $r_\lambda = 1$, then $\dim_k \operatorname{Hom}_{\boldsymbol{\Delta}}(V_{\lambda,j}, N) \geq \langle \mathbf{h}, \mathbf{d}'' \rangle_{\boldsymbol{\Delta}} = 2$ for each $N \in \mathcal{C}$. Thus in this case $\mathcal{U} \cap \mathcal{C} = \varnothing$ by Proposition 4.7(ii).

Concluding, $\dim\big(\overline{\mathcal{O}(M)} \setminus \mathcal{U}\big) < \dim \mathcal{O}(M) - 1$ if and only if $r_\lambda = 2$. Since we know from Proposition 7.1 that $\overline{\mathcal{O}(M)}$ is a complete intersection, the claims about (non-)normality of $\overline{\mathcal{O}(M)}$ follow immediately from Proposition 3.1(i). ■

We finish this section with a remark about the relationship between the degenerations and the hom-order. Let $\boldsymbol{\Delta}'$ be a bound quiver and $\mathbf{d}_0$ a dimension vector. If $U, V \in \operatorname{rep}_{\boldsymbol{\Delta}'}(\mathbf{d}_0)$, then we say that V is a degeneration of U (and write $U \leq_{\deg} V$) if $\mathcal{O}(V) \subseteq \overline{\mathcal{O}(U)}$. Similarly, we write $U \leq_{\hom} V$ if $\dim_k \operatorname{Hom}_{\boldsymbol{\Delta}'}(X, U) \leq \dim_k \operatorname{Hom}_{\boldsymbol{\Delta}'}(X, V)$ for each $X \in \operatorname{rep} \boldsymbol{\Delta}'$ (equivalently, $\dim_k \operatorname{Hom}_{\boldsymbol{\Delta}'}(U, X) \leq \dim_k \operatorname{Hom}_{\boldsymbol{\Delta}'}(V, X)$ for each $X \in \operatorname{rep} \boldsymbol{\Delta}'$). Both $\leq_{\deg}$ and $\leq_{\hom}$ induce partial orders in the set of the isomorphism classes of the representations of $\boldsymbol{\Delta}'$. It is also known that $\leq_{\deg}$ implies $\leq_{\hom}$. The reverse implication is not true in general, however $\leq_{\hom}$ implies $\leq_{\deg}$ if either $\boldsymbol{\Delta}'$ is of finite representation type [44] or $\operatorname{gl.dim} \boldsymbol{\Delta}' = 1$ and $\boldsymbol{\Delta}'$ is of tame representation type [16] (*i.e.*, $R = \varnothing$ and Δ' is an Euclidean quiver). We present an example showing that $\leq_{\hom}$ does not imply $\leq_{\deg}$ for the tame concealed-canonical algebras in general.

We return to the setup of this section and assume that $r_\lambda = 2$. Let $R := R_{\lambda,0} \oplus R_{\lambda,1}$. Moreover, we fix $\mathbf{d}'' \in \mathbf{Q}$ such that $q_{\boldsymbol{\Delta}}(\mathbf{d}'') = 0$, $\langle \mathbf{h}, \mathbf{d}'' \rangle_{\boldsymbol{\Delta}} = 2$ and $\mathbf{d}' \in \mathbf{P}$, where $\mathbf{d}' := \mathbf{h} - \mathbf{d}''$. If $N \in \mathcal{P}(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')$, then

$$\dim_k \operatorname{Hom}_{\boldsymbol{\Delta}}(R_{\lambda',i'}, R) \leq 1 \leq \dim_k \operatorname{Hom}_{\boldsymbol{\Delta}}(R_{\lambda',i'}, N)$$

for any $\lambda' \in \mathbb{X}$ and $i' \in [0, r_{\lambda'} - 1]$. By adapting [18, Corollary 4.2] to the considered situation we get that $R \leq_{\text{hom}} N$ for each $N \in \mathcal{P}(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')$. On the other hand,

$$\dim \mathcal{O}(R) = a_{\boldsymbol{\Delta}}(\mathbf{d}) - 2 = \dim\big(\mathcal{P}(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')\big),$$

hence $\mathcal{P}(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'') \not\subseteq \overline{\mathcal{O}(R)}$, *i.e.*, there exists $N \in \mathcal{P}(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')$ such that $R \not\leq_{\text{deg}} N$.

8 Proof of Theorem 4

Let M be a periodic representation of a tame concealed-canonical quiver $\boldsymbol{\Delta}$ such that $\mathcal{O}(M)$ is maximal.

If $\operatorname{Ext}^1_{\boldsymbol{\Delta}}(M, M) = 0$, then $\overline{\mathcal{O}(M)} = \operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d})$ by Proposition 3.2(ii). Consequently, $\overline{\mathcal{O}(M)}$ is a normal complete intersection by Proposition 3.2(i). Observe, that $\mathbf{dim}\, M$ is not singular in this case.

Now assume $\operatorname{Ext}^1_{\boldsymbol{\Delta}}(M, M) \neq 0$. Using Proposition 3.2(iii) we may assume that $M \in \operatorname{add} \mathcal{R}$ for a sincere separating exact subcategory $\mathcal{R}$ of $\operatorname{ind} \boldsymbol{\Delta}$. Proposition 3.2(iv) implies that $\overline{\mathcal{O}(M)} \neq \operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d})$. Consequently, $p^M \neq 0$ (since $\dim \mathcal{O}(M) = \dim \operatorname{rep}_{\boldsymbol{\Delta}}(\mathbf{d}) - p^M$) and the claim follows from Propositions 6.4 and 6.5.

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