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DEMICONTINUITY OF NEMITSKY OPERATORS ON ORLICZ–SOBOLEV SPACES

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We give an extension, to Orlicz-Sobolev spaces, of a theorem of Marcus and Mizel on the demicontinuity of Nemitsky operators on Sobolev spaces.

1. INTRODUCTION

In our paper [3], we gave an extension to Orlicz–Sobolev spaces of a theorem of Marcus and Mizel (see [6]) on the mapping of Sobolev spaces by Nemitsky operators (defined in Section 4 of this paper.) However we gave no extension of their theorem (Theorem 4.1 in [6]) on the demicontinuity (that is,"strong $\rightarrow$ weak"continuity) of Nemitsky operators. In this paper, we give a theorem of this type.

We shall find that, by replacing Lebesgue L_p spaces by suitable Orlicz spaces, and using the appropriate Orlicz space analogues of theorems on Lebesgue and Sobolev spaces (contained in references [1], [3], [4] and [5] and Section 3 of this paper), the proof of Marcus and Mizel for Lebesgue spaces can be modified so as to apply to Orlicz spaces.

2. PRELIMINARIES

ORLICZ SPACES

We shall use the definitions and properties of N-functions and Orlicz spaces as given in Krasnosel'skiĭ and Rutickiĭ [5]. For ease of reference, and because our notation occasionally differs from that in [5], we shall quote a number of results.

Throughout this section, M denotes an N-function; that is, a real valued, continuous, convex, even function on $\mathbb{R}$ such that $M(u)/u \rightarrow 0$ or ∞ as $u \rightarrow 0$ or ∞ respectively, and Ω denotes a bounded domain in $\mathbb{R}^n$.

(i) We say that M satisfies the Δ_2-*condition* if there exist constants $K > 0$, $u_0 \geqslant 0$ such that

$$M(2u) \leqslant KM(u) \quad \text{for all } u \geqslant u_0. \tag{2.1}$$

We shall also need the equivalent criteria given in (ii).

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(ii) M satisfies the Δ_2-condition if and only if either

(*) there exists a constant $u_0 \geqslant 0$ and a real valued function K_M (which may be taken to have the form $K_M(\ell) = C\ell^\mu$, where $C \geqslant 1$ and $\mu > 1$) such that

$$M(\ell u) \leqslant K_M(\ell)M(u) \quad \text{for } \ell > 1 \text{ and } u \geqslant u_0; \tag{2.2}$$

or

(**) there exist constants $u_0 \geqslant 0$ and $p > 1$ such that

$$uM'_+(u)/M(u) \leq p \quad \text{for all } u \geqslant u_0 \tag{2.3}$$

(M'_+ denotes the right derivative of M).

(iii) Let $\tilde{M}(v) = \sup_{u \geqslant 0}[u|v| - M(u)]$ so that $\tilde{M}$ is the N - function complementary to M; then the following inequalities hold:

$$u \leqslant M^{-1}(u)\tilde{M}^{-1}(u) \leqslant 2u, \qquad u \geqslant 0. \tag{2.4}$$

(iv) The *Orlicz class* $L^*_M(\Omega)$ is the set of all functions u, measurable on Ω, such that $\int_\Omega M \circ u < \infty$, and the *Orlicz space* $L_M(\Omega)$ is the set of all functions u, measurable on Ω, such that either

* there exists $k > 0$ such that $\int_\Omega M[ku(x)]dx < \infty$, or

** for all $v \in L^*_{\tilde{M}}(\Omega)$, $|\int_\Omega uv| < \infty$.

(v) The *Luxemburg norm* $\|u\|_{M,\Omega} = \inf\{\lambda > 0 \colon \int_\Omega M(u/\lambda) \leq 1\}$ is monotone in the sense that if $|u(x)| \leqslant |v(x)|$ for almost all $x \in \Omega$, then $\|u\|_{M,\Omega} \leqslant \|v\|_{M,\Omega}$.

ORLICZ-SOBOLEV SPACES

We shall use the definitions and properties of Orlicz-Sobolev spaces as given in Donaldson and Trudinger [**1**]. We shall only need to consider these spaces defined on bounded domains $\Omega \subset \mathbf{R}^n$ satisfying the cone condition.

(vi) Let Q be an N-function. Then the *Sobolev conjugate* N-function Q^* of Q is defined by

$$(Q^*)^{-1}(s) = \int_0^{|s|} Q^{-1}(t) \cdot t^{-1-1/n}\,dt \tag{2.5}$$

where it is assumed that, if necessary, $Q(t)$ is redefined for small values of t (obtaining an equivalent N-function) so that

$$\int_0^1 Q^{-1}(t) \cdot t^{-1-1/n}\,dt < \infty. \tag{2.6}$$

Q^* has the property that $Q \prec Q^*$, that is, that there exist constants u_o and k such that $Q(u) \leqslant Q^*(ku)$ for all $u \geqslant u_o$. This implies that there exists a constant C such that, for all $u \in L_{Q^*}(\Omega)$,

$$\|u\|_{Q,\Omega} \leqslant C\|u\|_{Q^*,\Omega}. \tag{2.7}$$

(vii) The *Orlicz-Sobolev space* $W^1L_Q(\Omega)$ is defined as the set of all functions u in $L_Q(\Omega)$ whose distributional derivatives $\partial_{x_i}u$ also belong to $L_Q(\Omega)$. A norm $\|u\|_{1,Q} = \|u\|_{1,Q,\Omega}$ may be defined on $W^1L_Q(\Omega)$ by

$$\|u\|_{1,Q} = \max\{\|u\|_Q, \|\partial_{x_1}u\|_Q, \dots, \|\partial_{x_n}u\|_Q\}.$$

In the case that $\lim_{s\to\infty}(Q^*)^{-1}(s) = \infty$, there exists a constant C such that

$$\|u\|_{Q^*,\Omega} \leqslant C\|u\|_{1,Q,\Omega}. \tag{2.8}$$

(viii) Let $Q_1, \dots, Q_m$ be N-functions. We shall denote the set of vector-valued functions $u = (u_1, \dots, u_m)$, for which $u_k \in W^1L_{Q_k}(\Omega), k = 1, \dots, m$, by $W^1L_{\overline{Q}}(\Omega)$, and use the norm

$$\|u\|_{1,\overline{Q},\Omega} = \max_{1\leqslant k\leqslant m} \|u_k\|_{1,Q_k,\Omega}$$

on this space.

We shall also need the following chain rule (see [**3**]):

(ix) *Let $g\colon \mathbf{R} \to \mathbf{R}$ be a locally absolutely continuous function and let Ω be a bounded domain in $\mathbf{R}^n$ having the cone property. Suppose $u \in W_{1,1}(\Omega)$, and let $v = g \circ u$. Then $v \in W^1L_P(\Omega)$ if and only if*

$$v_i = [g' \circ u]\,\partial_{x_i}u \in L_P(\Omega), \quad i = 1, \dots, n, \tag{2.9}$$

the product being interpreted as zero wherever $\partial_{x_i}u = 0$. Moreover, if (2.9) holds, $v_i = \partial_{x_i}v$ almost everywhere in $\Omega, i = 1, \dots, n$.

3. Mappings Between Orlicz Spaces

We shall need an extension to Orlicz spaces of a theorem of Halmos (Theorem 1 in Halmos [**2**]) on the mapping of Lebesgue spaces by Borel measurable functions. For convenience, we shall first quote the following lemma - essentially a particular case of Lemma 3.2 in [**4**].

LEMMA 3.1. *Let P and Q be N-functions, and let Ω be a bounded domain in $\mathbb{R}^n$. Then*

(i) $Q^{-1} \circ P \circ u \in L_Q(\Omega)$ *for all* $u \in L_P(\Omega)$ *if and only if* $Q^{-1} \circ P$ *satisfies the* $\triangle_2$*-condition,*

(ii) *in the case that* $Q^{-1} \circ P$ *satisfies the* $\triangle_2$*-condition, there exists a constant* C *such that*

$$\|Q^{-1} \circ P \circ u\|_{Q,\Omega} \leqslant C\left[1 + K_{Q^{-1}\circ P}(\|u\|_{P,\Omega})\right], \tag{3.1}$$

where $K_{Q^{-1}\circ P}$ *denotes the function which occurs in the alternative criterion 2 (ii*) for the* $\triangle_2$*-condition.*

NOTE: We do not assume that $Q^{-1} \circ P$ is an N-function, however the definition of the $\triangle_2$-condition still makes sense in the present context.

Our version of Halmos' theorem is then:

THEOREM 3.2. *Let f be a real valued Borel measurable function on $\mathbb{R}^m$, and Ω be a bounded domain in $\mathbb{R}^n$. Let $P_1, \dots P_m$ and Q be N-functions.*

(a) *If* $f(u_1, \dots, u_m) \in L_Q(\Omega)$ *for all* $u_k \in L^*_{P_k}(\Omega), k = 1, \dots, m$, *then there exists a constant* C *such that, for all points* $(\sigma_1, \dots, \sigma_m)$ *in* $\mathbb{R}^m$, *the inequality*

$$|f(\sigma_1, \dots, \sigma_m)| \leqslant C \sum_{k=1}^{m} Q^{-1} \circ P_k(1 + |\sigma_k|) \tag{3.2}$$

is satisfied.

(b) *If* f *satisfies an inequality of the form (3.2) for all* $(\sigma_1, \dots, \sigma_m) \in \mathbb{R}^m$, *and if each function* $Q^{-1} \circ P_k, k = 1, \dots, m$, *satisfies the* $\triangle_2$*-condition, then* $f(u_1, \dots, u_m) \in L_Q(\Omega)$ *whenever* $u_k \in L_{P_k}(\Omega), k = 1, \dots, m$.

PROOF: (a) Suppose (3.2) is false. Then there exists a sequence $\{(y_1^\nu, \dots, y_m^\nu)\}$ of points in $\mathbb{R}^m$ such that, for $\nu = 1, 2, \dots$,

$$|f(y_1^\nu, \dots, y_m^\nu)| > 2^\nu \sum_{k=1}^{m} Q^{-1} \circ P(1 + |y_k^\nu|). \tag{3.3}$$

Let $\Omega_\nu, \nu = 1, 2, \dots$, be pairwise disjoint measurable subsets of Ω such that

$$|\Omega_\nu| = |\Omega| \left[\sum_{k=1}^{m} P_k(1)\right] \Big/ \left[2^\nu \sum_{k=1}^{m} P_k(1 + |y_k^\nu|)\right], \tag{3.4}$$

where $|\cdot|$ denotes Lebesgue n-dimensional measure.

We define functions $u_k, k = 1, \ldots, m$, by

$$u_k(t) = \begin{cases} y_k^\nu & \text{for } t \in \Omega_\nu, \\ 0 & \text{otherwise in } \Omega. \end{cases}$$

Then, for $k = 1, \ldots, m$, using (3.4), we have

$$\int_\Omega P_k \circ u_k = \sum_{\nu=1}^{\infty} P_k(y_k^\nu)\,|\Omega_\nu| < \infty,$$

so that each $u_k \in L^*_{P_k}(\Omega)$.

We define the function v by

$$v(t) = \begin{cases} \sum\limits_{k=1}^{m} (1/m)\tilde{Q}^{-1} \circ P_k(1 + |y_k^\nu|) & \text{for } t \in \Omega_\nu, \\ 0 & \text{otherwise in } \Omega. \end{cases}$$

Using the convexity of $\tilde{Q}$ and (3.4), we have

$$\begin{aligned} \int_\Omega \tilde{Q} \circ v &= \sum_{\nu=1}^{\infty} \tilde{Q}\left[\sum_{k=1}^{m} (1/m)\tilde{Q}^{-1} \circ P_k(1 + |y_k^\nu|)\right] |\Omega_\nu| \\ &\leqslant \sum_{\nu=1}^{\infty} (1/m) \sum_{k=1}^{m} \tilde{Q} \circ \tilde{Q}^{-1} \circ P_k(1 + |y_k^\nu|)\,|\Omega_\nu| \\ &= \sum_{\nu=1}^{\infty} |\Omega|(1/m) \left[\sum_{k=1}^{m} P_k(1)\right] /2^\nu < \infty, \end{aligned}$$

so that $v \in L^*_{\tilde{Q}}(\Omega)$.

However, using (2.4), we have

$$\begin{aligned} &\int_\Omega f(u_1, \ldots, u_m)v \\ &> \sum_{\nu=1}^{\infty} 2^\nu \left[\sum_{k=1}^{m} Q^{-1} \circ P_k(1 + |y_k^\nu|)\right] \left[\sum_{k=1}^{m} (1/m)\tilde{Q}^{-1} \circ P_k(1 + |y_k^\nu|)\right] |\Omega_\nu| \\ &\geqslant (1/m) \sum_{\nu=1}^{\infty} 2^\nu \sum_{k=1}^{m} \left[Q^{-1} \circ P_k(1 + |y_k^\nu|)\right] \left[\tilde{Q}^{-1} \circ P_k(1 + |y_k^\nu|)\right] |\Omega_\nu| \\ &\geqslant (1/m) \sum_{\nu=1}^{\infty} 2^\nu \left[\sum_{k=1}^{m} P_k(1 + |y_k^\nu|)\right] |\Omega_\nu| \\ &= \infty, \quad \text{using (3.4).} \end{aligned}$$

Therefore, using the criterion (**) from 2(iv), we see that $f(u_1,\ldots,u_m) \notin L_Q(\Omega)$. This contradiction shows that the inequality (3.2) holds. The proof of (b) follows immediately from Lemma 3.1.

REMARK: Halmos also considers the case in which the domain has infinite measure - Theorem 2 in [2]. While this case is not needed for this paper, for completeness we observe that Theorem 3.2 remains valid if we make the following changes:

(i) "Ω is a bounded domain in $\mathbb{R}^n$" is replaced by "Ω is a domain in $\mathbb{R}^n$ with $|\Omega| = \infty$" and "Δ_2-condition" is replaced by "global Δ_2-condition", that is, 2(i) with $u_0 = 0$.

(ii) The inequality (3.2) is replaced by

$$|f(\sigma_1,\ldots,\sigma_m)| \leqslant C\sum_{k=1}^{m} Q^{-1}\circ P_k(\sigma_k). \tag{3.5}$$

We give the proof only in outline, as it is almost the same as that of Theorem 3.2.

Suppose (3.5) is false. Then there exists a sequence $\{(y_1^\nu,\ldots,y_m^\nu)\}$ such that

$$|f(y_1^\nu,\ldots,y_m^\nu)| > 2^\nu \sum_{k=1}^{m} Q^{-1}\circ P_k(y_k^\nu), \quad \nu = 1,2,\ldots. \tag{3.6}$$

Because a function which is identically zero on Ω belongs to each $L^*_{P_k}(\Omega)$, and a constant function can belong to $L_Q(\Omega)$ with $|\Omega| = \infty$ only if it is identically zero, we must have $f(0,\ldots,0) = 0$, and then (3.6) implies that $(y_1^\nu,\ldots,y_m^\nu) \neq (0,\ldots,0)$. Therefore we may choose a pairwest disjoint sequence $\{\Omega_\nu\}$ of measurable subsets of Ω such that

$$|\Omega_\nu| = 1/2^\nu \sum_{k=1}^{m} P_k(y_k^\nu).$$

We define functions $u_1,\ldots,u_m$ and v by

$$u_k(t) = \begin{cases} y_k^\nu & \text{for } t \in \Omega_\nu, \\ 0 & \text{otherwise in } \Omega, \end{cases}$$

$$v(t) = \begin{cases} \sum_{k=1}^{m} (1/m)\tilde{Q}^{-1}\circ P_k(y_k^\nu) & \text{for } t \in \Omega_\nu, \\ 0 & \text{otherwise in } \Omega, \end{cases}$$

and then, as before, we find that $u_k \in L^*_{P_k}(\Omega), v \in L^*_{\tilde{Q}}(\Omega)$, and $\int_\Omega f(u_1,\ldots,u_m)v = \infty$, that is, $f(u_1,\ldots,u_m) \notin L_Q(\Omega)$. This contradiction shows that the estimate (3.5) must hold.

Since Lemma 3.1 remains valid if we make the changes (i) in its hypotheses (this follows by routine modifications of the arguments used to prove Lemma 3.2 in [4]), the modified version of (b) follows immediately.

4. Demicontinuity of Nemitsky Operators

Definitions and notation

Let $A(\Omega)$ denote the class of real measurable functions u on Ω such that, for almost every line τ parallel to any co-ordinate axis, u is locally absolutely continuous on $\tau \cap \Omega$. $A'(\Omega)$ denotes the class of functions u such that u coincides almost everywhere in Ω with a function $\tilde{u}$ in $A(\Omega)$. For $u \in A'(\Omega)$, the symbol $\partial'_{x_j} u$ denotes any member of the equivalence class of functions measurable on Ω which contains the classical partial derivative $\partial \tilde{u}/\partial x_j$.

A function $g\colon \Omega \times \mathbf{R}^m \to \mathbf{R}$ is said to be a *locally absolutely continuous Caratheodory function* if

(i) there exists a null subset N_g of Ω such that if $x \in \Omega - N_g$,
  (a) $g(x, \cdot)$ is continuous in $\mathbf{R}^m$,
  (b) for every line τ parallel to one of the axes in $\mathbf{R}^m$, the function $g(x, \cdot)$ restricted to this line is locally absolutely continuous;

(ii) for every fixed $t \in \mathbf{R}^m$, we have $g(\cdot, t) \in A'(\Omega)$.

An operator G on vector valued functions $u = (u_1, \ldots, u_m)$, measurable on Ω, defined by

$$Gu(x) = g(x, u(x))$$

is called a *Nemitsky operator*.

We can now give our theorem on demicontinuity, an extension to Orlicz-Sobolev spaces of Marcus and Mizel's Theorem 4.1.

Theorem 4.1. *Let Ω be a bounded domain in $\mathbf{R}^n$ having the cone property, and let g be a locally absolutely continuous Caratheodory function on $\Omega \times \mathbf{R}^m$. Let P, Q_k, and $Q_k^{\dagger}$ for $k = 1, \ldots, m$, be N-functions having the following properties:*

(i) *$P, \tilde{P}$, and Q_k satisfy the $\triangle_2$-condition;*

(ii) *$P \prec Q_k$;*

(iii) *there exist complementary N-functions R_k and $\tilde{R}_k$, and constants α_k, β_k and γ_k for $k = 1, \ldots, m$, such that the inequalities*

$$R_k(u) \leqslant P^{-1}\left[Q_k(\alpha_k u)\right]$$

and

$$\tilde{R}_k(u) \leqslant P^{-1}\left[Q_k^{\dagger}(\beta_k u)\right]$$

are satisfied for $u \geqslant \gamma_k$;

(iv) *$(Q_k^{\dagger})^{-1} \circ Q_j^*$ for $k, j = 1, \ldots, m$, and $P^{-1} \circ Q_j^*$ for $j = 1, \ldots, m$, satisfy the $\triangle_2$-condition;*

(v) *$(Q_k^*)^{-1}(v) \to \infty$ as $v \to \infty$.*

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Suppose a, b, a_k *and* $b_{k,j}$ *are functions such that*

I *For every fixed* $t \in \mathbf{R}^m$,

$$|\partial'_{x_i} g(x,t)| \leqslant a(x) + b(t) \text{ almost everywhere in } \Omega, \text{for } i = 1, \dots, n.$$

II *The inequality*

$$|\partial g(x,t)/\partial t_k| \leqslant a_k(x) + \sum_{j=1}^{m} b_{k,j}(t_j), \text{for } k = 1, \dots, m$$

holds at every point $(x,t) \in (\Omega - N_g) \times \mathbf{R}^m$ *at which the derivative exists in the classical sense.*

Furthermore, a, b, a_k *and* $b_{k,j}$ *have the properties (vi) to (xiii) given below:*

(vi) $o \leqslant a \in L_p(\Omega)$;

(vii) b *is non-negative and continuous in* $\mathbf{R}^m$;

(viii) $o \leqslant a_k \in L_{Q_k^\dagger}(\Omega), k = 1, \dots, m,$ *where* a_k *is everywhere finite;*

(ix) $o \leqslant b_{k,j}$ *is an everywhere finite Borel function on* $\mathbf{R}$, $k, j = 1, \dots, m$;

(x) $b_{k,k}$ *is continuous,* $k = 1, \dots, m$;

(xi) b *defines, by composition, a mapping from* $L_{Q_1^*}(\Omega) \times \dots \times L_{Q_m^*}(\Omega)$ *to* $L_P(\Omega)$;

(xii) $b_{k,j}$ *defines, by composition, a mapping from* $L_{Q_j^*}(\Omega)$ *to* $L_{Q_k^\dagger}(\Omega)$, *for* $k, j = 1, \dots, m$;

(xiii) *the mappings in (xi) and, for* $j = k$, *in (xii), are continuous.*

Then G *maps* $W^1 L_{\overline{Q}}(\Omega)$ *into* $W^1 L_P(\Omega)$ *and is demicontinuous and bounded. Moreover* G *is continuous as a mapping from* $W^1 L_{\overline{Q}}(\Omega)$ *to* $L_P(\Omega)$.

We follow the method of Marcus and Mizel; however, as the details are somewhat different, we shall still briefly give the proof for the convenience of the reader.

PROOF: From (xi), (xii) and Theorem 3.2, for $\sigma \in \mathbf{R}$ and $(\sigma_1, \dots, \sigma_m) \in \mathbf{R}^m$ we have

$$(4.1) \qquad |b_{k,j}(\sigma)| \leqslant \text{ const. } (Q_k^\dagger)^{-1}\left[Q_j^*(1 + |\sigma|)\right], \quad \text{for } k, j = 1, \dots, m,$$

and

$$(4.2) \qquad |b(\sigma_1, \dots, \sigma_m)| \leqslant \text{ const. } \left[P^{-1} \circ Q_1^*(1 + |\sigma_1|) + \dots + P^{-1} \circ Q_m^*(1 + |\sigma_m|)\right].$$

Further, (iii) implies that, for $u \in L_{Q_k^\dagger}(\Omega)$ and $v \in L_{Q_k}(\Omega)$, we have

$$(4.3) \qquad \|uv\|_{P,\Omega} \leqslant \text{ const. } \|u\|_{Q_k^\dagger,\Omega} \|v\|_{Q_k,\Omega}, \quad k = 1, \dots, m.$$

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(See [5], p.223).

Let $u^\nu = (u_1^\nu, \dots, u_m^\nu)$ for $\nu \geqslant 1$, converge to $u^0 = (u_1^0, \dots, u_m^0)$ in $W^1L_{\overline{Q}}(\Omega)$. Using (2.8), we find that $u_k^\nu \to u_k^0$ in $L_{Q_k^*}(\Omega)$, $k = 1, \dots, m$, and further using 2(ix) and (xii), we have

$$\partial_{x_i}(\beta_k \circ u_k^\nu) = (b_{k,k} \circ u_k^\nu)\partial_{x_i} u_k^\nu, \quad \text{for } \nu = 0, 1, 2, \dots, \tag{4.4}$$

where $\beta_k(\sigma)$ denotes $\displaystyle\int_0^\sigma b_{k,k}(\rho)\,d\rho$.

From (4.3),

$$\begin{aligned} &\|(b_{k,k} \circ u_k^\nu - b_{k,k} \circ u_k^0)\partial_{x_i} u_k^\nu\|_{P,\Omega} \\ &\qquad \leqslant \text{const. } \|b_{k,k} \circ u_k^\nu - b_{k,k} \circ u_k^0\|_{Q_k^\dagger,\Omega} \|\partial_{x_i} u_k^\nu\|_{Q_k,\Omega} \\ &\qquad \to 0, \text{using (xiii); and} \\ &\|b_{k,k} \circ u_k^0(\partial_{x_i} u_k^\nu - \partial_{x_i} u_k^0)\|_{P,\Omega} \\ &\qquad \leqslant \text{const. } \|b_{k,k} \circ u_k^0\|_{Q_k^\dagger,\Omega} \|\partial_{x_i} u_k^\nu - \partial_{x_i} u_k^0\|_{Q_k,\Omega} \to 0, \end{aligned} \tag{4.5}$$

and so, from (4.4),

$$\|\partial_{x_i}(\beta_k \circ u_k^\nu) - \partial_{x_i}(\beta_k \circ u_k^0)\|_{P,\Omega} \to 0, \quad \text{for } k = 1, \dots, m. \tag{4.6}$$

From the definition of β_k and (4.1), we have

$$|\beta_k(\sigma'') - \beta_k(\sigma')| \leqslant \text{const. } \left[(Q_k^\dagger)^{-1} \circ Q_k^*(1 + |\sigma''|) + (Q_k^\dagger)^{-1} \circ Q_k^*(1 + |\sigma'|)\right] |\sigma'' - \sigma'|$$

and then using (4.3), the triangle inequality (see [5], p.79), Lemma 3.1, (iv), and denoting $(Q_k^\dagger)^{-1} \circ Q_k^*$ by Ψ, we obtain

$$\begin{aligned} &\|\beta_k \circ u_k^\nu - \beta_k \circ u_k^0\|_{P,\Omega} \\ &\qquad \leqslant \text{const. } \left\|(Q_k^\dagger)^{-1} \circ Q_k^*(1 + |u_k^\nu|) + (Q_k^\dagger)^{-1} \circ Q_k^*(1 + |u_k^0|)\right\|_{Q_k^\dagger,\Omega} \\ &\qquad\quad \times \|u_k^\nu - u_k^0\|_{Q_k,\Omega} \\ &\qquad \leqslant \text{const. } \left[1 + K_\Psi\left(\|1 + |u_k^\nu|\,\|_{Q_k^*,\Omega}\right) + K_\Psi\left(\|1 + |u_k^0|\,\|_{Q_k^*,\Omega}\right)\right] \\ &\qquad\quad \times \|u_k^\nu - u_k^0\|_{Q_k,\Omega} \to 0. \end{aligned} \tag{4.7}$$

Following Marcus and Mizel, we denote Gu^ν by v^ν for $v = 0, 1, 2, \dots$, $\displaystyle\sum_{j=1}^{k-1} b_{k,j}(t_j)$ by $c_k^{(1)}(t)$, and $\displaystyle\sum_{j=k+1}^{m} b_{k,j}(t_j)$ by $c_k^{(2)}(t)$ for $k = 1, \dots, m$.

Then (see [3]) their arguments can be modified (by replacing a number of results for L_p spaces by their analogues for Orlicz spaces) to show that the following analogue of their inequality (4.7) holds:

$$\text{(4.8)}\quad \|v^\nu - v^0\|_{P,\Omega} \leqslant \sum_{k=1}^{m} \text{const.}\ \left[\|a_k + c_k^{(1)} \circ u^\nu + c_k^{(2)} \circ u^0\|_{Q_k^\dagger,\Omega}\|u_k^\nu - u_k^0\|_{Q_k,\Omega}\right] + \sum_{k=1}^{m} \|\beta_k \circ u_k^\nu - \beta_k \circ u_k^0\|_{P,\Omega}.$$

Using (xii) and (4.7), (4.8) shows that $\|v^\nu - v^0\|_{P,\Omega} \to 0$.

Similarly, we have the following analogue of (4.8) in [6]:

(4.9)

$$\begin{aligned}\|\partial_{x_i} v^\nu\|_{P,\Omega} \leqslant& \|a\|_{P,\Omega} + \|b \circ u^\nu\|_{P,\Omega} \\ &+ \sum_{k=1}^{m} \text{const.}\ \left[\|a_k\|_{Q_k^\dagger,\Omega} + \|c_k \circ u^\nu\|_{Q_k^\dagger,\Omega}\right] \|\partial_{x_i} u_k^\nu\|_{Q_k,\Omega} \\ &+ \sum_{k=1}^{m} \|\partial_{x_i}(\beta_k \circ v_k^\nu)\|_{P,\Omega} \quad \text{for } i = 1, \ldots, n,\end{aligned}$$

where $c_k(t) = c_k^{(1)}(t) + c_k^{(2)}(t)$.

(It appears to the author that, in [6], Ω should be substituted for Ω' in (4.8), and the words "by Lemma 1.6" in the following paragraph should be deleted.)

Using Lemma 3.1 with the estimates (4.1) and (4.2), together with (4.6) and the convergence of each sequence u_k^ν in $L_{Q_k^*}(\Omega)$, we can show that the right-hand side of (4.9) is uniformly bounded in ν, and so, for each i, $i = 1, \ldots, n$, the functions $\partial_{x_i} v^\nu$ form a bounded set in $L_P(\Omega)$. From this, together with the facts that $L_P(\Omega)$ is reflexive (because we assume that both P and $\bar{P}$ satisfy the $\triangle_2$-condition) and that $v^\nu \to v^0$ in $L_P(\Omega)$, we can show that $\partial_{x_i} v^\nu \to \partial_{x_i} v^0$ weakly in $L_P(\Omega)$. Thus $v^\nu \to v^0$ weakly in $W^1 L_P(\Omega)$, and so G is demicontinuous as a map from $W^1 L_{\overline{Q}}(\Omega)$ to $W^1 L_P(\Omega)$. ∎

Notes on the Hypotheses

Note 1. We can slightly simplify the hypotheses (at the expense of the generality) of Theorem 4.1 if we assume that each function Q_k^*, $k = 1, \ldots, m$, satisfies the $\triangle_2$-condition. Then if K is the constant in the definition 2(i) (for Q_k^*), on using the concavity of P^{-1} and the fact that necessarily $K > 2$, we find that, for large enough v,

$$P^{-1}(Kv) \leqslant K P^{-1}(v)$$

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and so, for large enough u,

$$P^{-1}\left[Q_k^*(2u)\right] \leqslant P^{-1}\left[KQ_k^*(u)\right] \leqslant KP^{-1}\left[Q_k^*(u)\right]$$

and $P^{-1}\circ Q_k^*$ satisfies the $\triangle_2$-condition. Since the same comment applies to $(Q_k^\dagger)^{-1}\circ Q_j^*$, the hypothesis (iv) may be replaced by

(iv$'$) *Q_k^* for $k=1,\ldots,m$ satisfies the $\triangle_2$-condition.*

It is evident that, in this case, to apply the theorem, we shall need to identify classes of N-functions Q_k such that both Q_k and Q_k^* satisfy the $\triangle_2$-condition, and $(Q_k^*)^{-1}(v)\to\infty$ as $v\to\infty$. A simple means of doing this is provided in Proposition 4.2.

PROPOSITION 4.2. *Let Q be an N-function, and let Q^* be defined (in terms of the positive integer n) as in (2.5). Suppose that there exist constants U and p, where $U\geqslant 0$ and $1<p<n$, such that*

$$uQ'_+(u)/Q(u)\leqslant p,\quad u\geqslant U. \tag{4.10}$$

Then

(i) *Q satisfies the $\triangle_2$-condition;*
(ii) *Q^* satisfies the $\triangle_2$-condition;*
(iii) *$(Q^*)^{-1}(v)\to\infty$ as $v\to\infty$.*

PROOF: (i) follows immediately from (4.10) and (2.3).

From (4.10) there exist constants $C>0$ and $T>0$ such that

$$Q^{-1}(t)/t(Q^{-1})'_+(t)\leqslant p \tag{4.11}$$

and (see [**5**], p.25)

$$Q^{-1}(t)\geqslant Ct^{1/p} \tag{4.12}$$

for all $t\geqslant T$. Because of (2.6), we shall assume that $T\geqslant 1$. Then, for $v>T$, we have, using (4.11) and then integrating by parts,

$$\begin{aligned}\int_T^v Q^{-1}(t).t^{-1-1/n}\,dt &\leqslant p\int_T^v (Q^{-1})'_+(t).t^{-1/n}\,dt\\ &\leqslant pQ^{-1}(v).v^{-1/n}-pQ^{-1}(T).T^{-1/n}\\ &\quad+(p/n)\int_T^v Q^{-1}(t).t^{-1-1/n}\,dt,\end{aligned}$$

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that is (using (2.5))

$$(4.13)\qquad \left[(Q^*)^{-1}(v)-(Q^*)^{-1}(T)\right][1-(p/n)]\leqslant pQ^{-1}(v).v^{-1/n}-pQ^{-1}(T).T^{-1/n}.$$

Let $p^*=(np)/(n-p)$. Since $v\left[(Q^*)^{-1}\right]'(v)=Q^{-1}(v).v^{-1/n}$, we may write (4.13) in the form

$$(Q^*)^{-1}(v)/v\left[(Q^*)^{-1}\right]'(v)\leqslant p^*+p^*\left[(Q^*)^{-1}(T)-Q^{-1}(T).T^{-1/n}\right]v^{1/n}/Q^{-1}(v).$$

From (4.12) and the assumption that $1/n<1/p$, we find that $v^{1/n}/Q^{-1}(v)\to 0$ as $v\to\infty$, and so

$$(4.14)\qquad \lim_{v\to\infty}\sup\left[(Q^*)^{-1}\right](v)/v\left[(Q^*)^{-1}\right]'(v)\leqslant p^*.$$

As (4.14) is equivalent ot 2(ii)(**), this proves (ii).

(iii) follows because, from (4.12),

$$\int_T^v Q^{-1}(t).t^{-1-1/n}\,dt\geqslant C\int_T^v t^{1/p-1-1/n}\,dt\to\infty\quad\text{as } v\to\infty.$$

∎

Note 2. We can construct families of N-functions satisfying the conditions (i), (ii), and (iii) of Theorem 4.1, from (suitable) known N-functions, such as can be found in [**5**], using the following proposition:

PROPOSITION 4.3. *Let P and R be N-functions satisfying the $\triangle_2$-condition, and let $Q=P\circ R$, $Q^\dagger=R\circ\tilde{R}$. Then Q and $Q^\dagger$ are N-functions having the properties:*

(i) *Q satisfies the $\triangle_2$-condition;*
(ii) *$P<Q$;*
(iii) *$R=P^{-1}\circ Q$ and $\tilde{R}=P^{-1}\circ Q^\dagger$.*

PROOF: By their definition (see [**5**], p.10) Q and $Q^\dagger$ are N-functions. For $l>1$, and large enough u, using the notation in 2(ii*), we have

$$Q(lu)=P\circ R(lu)\leqslant P\left[K_R(l)R(u)\right]\leqslant K_P\left[K_R(l)\right]R(u)$$

which proves (i).

Because R is an N-function, $R(v)/v\to\infty$ as $v\to\infty$, from which we obtain, for any $\kappa>0$

$$u\leqslant R(\kappa u)$$

and further

$$P(u) \leqslant P \circ R(\kappa u) = Q(\kappa u)$$

for large enough u. This proves (ii).

(iii) is immediate. ∎

We shall give two concrete examples. The first shows that our Theorem 4.1 contains Marcus and Mizel's Theorem 4.1 (for $p < q_i < n$ only) as a particular case, and the second shows that our theorem 4.1 contains results not included in Marcus and Mizel's Theorem 4.1.

EXAMPLE 4.4: Let n be a positive integer $\geqslant 2$, and let $1 < p < q_k < n$, $(1/q_k') + (1/q_k) = 1/p$, and $q_k^* = nq_k/(n-q_k)$. Let $P(u) = |u|^p$ and $R_k(u) = (p/q_k)|u|^{q_k/p}$, so that

$$\tilde{R}_k(u) = (p/q_k')|u|^{q_k'/p}.$$

Then P and $\tilde{P}$ satisfy the $\triangle_2$-condition,

$$Q_k = P \circ R_k = (p/q_k)^p |u|^{q_k}$$

and

$$Q_k^\dagger = P \circ \tilde{R}_k = (p/q_k')^p |u|^{q_k'}.$$

It is evident that conditions (i), (ii), and (iii) of Theorem 4.1 are satisfied. Moreover, since $Q_k^*(u) = \text{const.}\ |u|^{q_k^*}$, it follows (either directly, or using Proposition 4.2 and the fact that $\lim\limits_{u\to\infty} [uQ_k'(u)/Q_k(u)] = q_k < n$) that conditions (iv) and (v) of Theorem 4.1 are also satisfied.

EXAMPLE 4.5: Let P be as in Example 4.4; now let $R_k(u) = (1+|u|)ln(1+|u|) - |u|$. Then (see [5]) R_k satisfies the $\triangle_2$-condition, and $\tilde{R}_k = e^{|u|} - |u| - 1$. Using Proposition 4.3, it is immediate that P, $\tilde{P}$, $Q_k = P \circ R_k$ and $Q_k^\dagger = P \circ \tilde{R}_k$ satisfy conditions (i), (ii) and (iii) of Theorem 4.1. Since $\lim\limits_{u\to\infty} [uQ_k'(u)/Q_k(u)] = p < n$, Proposition 4.2 shows that conditions (iv) and (v) are also satisfied.

Note 3. Marcus and Mizel do not assume the continuity of the composition operators associated with b and $b_{k,k}$ (our hypothesis (xiii)) because in the case of Lebesgue spaces, this is an immediate consequence of the Lebesgue - space analogues of our hypotheses (xi) and (xii), that is, conditions (4.1) of Section 4 of [**6**]. For references to the proof of this, see [**6**]. In our case, continuity does not follow from our hypotheses. We remark that the required continuity would follow if, say, we assume that, in addition to the hypotheses of our Theorem 4.1, (except for (xiii)) that

(α) Each function Q_k^* for $k = 1, \dots, m$, satisfies the $\triangle_2$-condition;

(β) Each function $Q_k^\dagger$ for $k = 1, \dots, m$, satisfies the $\triangle_2$-condition.

(α) and (β), together with the assumption already made, that P satisfies the $\triangle_2$-condition, would then give the required continuity, so that the hypothesis (xiii) could be dispensed with. This follows from suitable modifications of Theorem 17.3 in [**5**], using the fact that $E_M(\Omega) = L_M(\Omega) = L^*_M(\Omega)$ if M satisfies the $\triangle_2$-condition.

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