II.

Re-discussion, from the Wiewpoint of Aerodynamics.

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Remarks on Steady Perfect Fluid Flow with Spherical Symmetry.

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1. – Introduction.

As discussed by DEUTSCH, the following gas-dynamical problem arises in the study of a stellar atmosphere: find a steady outward flow, originating from a star, which is compatible with the interstellar medium. In what follows, spherical symmetry will be assumed; moreover, the gas will be considered as a perfect gas with constant specific heats.

The notation is as follows:

r, distance from the center of the star;
ϱ, density;
τ, specific volume;
T, absolute temperature;
C_p, specific heat per unit mass at constant pressure;
γ, adiabatic index $(1<\gamma\leqslant\frac{5}{3})$;
GM/r, gravitational potential of the star;
τ_∞, T_∞, values of τ and T in the interstellar medium which are assumed to be known.

2. – Equations.

This flow is ruled by the usual conservation laws: mass, momentum and energy. The continuity equation gives:

$$\varrho u r^2 = m \qquad \text{or} \qquad u = m\tau r^{-2}\,, \tag{1}$$

m, the rate of mass flux, is a constant.

https://doi.org/10.1017/S0074180900104498 Published online by Cambridge University Press

The energy equation says that the specific entropy is a constant; as a result the momentum equation may be replaced by the Bernoulli theorem

$$\frac{u^2}{2} + C_p T - \frac{GM}{r} = B\,, \tag{2}$$

while the equation of state shows that

$$T\tau^{\gamma-1} = \text{const}\,. \tag{3}$$

The physical problem is a study of mass-loss from the star; thus $B > 0$. For large values of r, as τ tends towards τ_∞, u tends to zero, and (2) shows that $B = C_p T_\infty$. Thus, B is a known positive constant. It is very easy to study the variations of u, τ, T as functions of r. For instance $\tau(r)$ is implicitly given by

$$\left(B + \frac{GM}{r} - \frac{m^2\tau^2}{2r^2}\right)\tau^{\gamma-1} = \text{const}\,. \tag{4}$$

It is convenient to introduce non-dimensional variables.

If r_L (length) and τ_L (specific volume) are defined by

$$GM = Br_L\,, \qquad m^2\tau_L^2 = r_L^4 B \tag{5}$$

and if one introduces

$$x = \frac{r}{r_L}\,, \qquad y = \frac{\tau}{\tau_L}\,, \tag{6}$$

(4) may be written

$$F(x, y) = \left(1 + \frac{1}{x} - \frac{y^2}{2x^4}\right) y^{\gamma-1} = C\,, \tag{7}$$

where C is a constant ($C > 0$).

One must investigate the shape of the curves (ζ) defined by (7) in the domain $x > 0$, $y > 0$.

3. – Pattern of the curves (ζ).

First of all, one has

$$\frac{\partial F}{\partial x} = \frac{y^{\gamma-1}}{x^5}\,(-x^3 + 2y^2)\,, \qquad \frac{\partial F}{\partial y} = \frac{y^{\gamma-2}}{x^4}\left[(\gamma-1)(x^3 + x^4) - \frac{\gamma+1}{2}\,y^2\right]. \tag{8}$$

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a) The curve $(\bar{\Gamma})$ defined by

$$2y^2 = x^3$$

is the locus of points of (ζ) where the tangent is parallel to the x axis.

For a given y, C is an increasing function of x for $x^3 < 2y^2$ and a decreasing function of x for $x^3 > 2y^2$. Every curve (ζ) is cut in at most two points by a parallel to the x axis.

b) The curve (Γ_*) defined by

$$y^2 = \frac{2(\gamma - 1)}{\gamma + 1}(x^3 + x^4), \tag{10}$$

is the locus of points of (ζ) where the tangent is parallel to the y axis. As

$$C_p T = \frac{c^2}{\gamma - 1},$$

with c the speed of sound, it is easily checked with (2) that (Γ_*) is the locus of points which correspond to a Mach number equal to 1. This curve (Γ_*) divides the domain into a subsonic and a supersonic region. That is, for a given x, a value of y exceeding that from eq. (10) requires supersonic flow; less, subsonic.

For a given x, C is an increasing function of y for

$$y^2 < \frac{2(\gamma - 1)}{\gamma + 1}(x^3 + x^4),$$

and a decreasing function of y for

$$y^2 > \frac{2(\gamma - 1)}{\gamma + 1}(x^3 + x^4).$$

Every curve (ζ) is cut in two points by a parallel to the axis.

c) When $1 < \gamma < \frac{5}{3}$:

$(\bar{\Gamma})$ and (Γ_*) have one point of intersection A, apart from the origin

$$x_A = \frac{5 - 3\gamma}{4(\gamma - 1)}, \qquad y_A^2 = \frac{1}{2^7}\left[\frac{5 - 3\gamma}{\gamma - 1}\right]^3. \tag{11}$$

Obviously, A is a saddle point for the family of the ζ-curves. The particular

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curve which goes through A—say (ζ_*), and C_* the corresponding value of C—admits the point A as a double point [1].

In what follows the two branches for $x > x_L$ will be noted $(\zeta_*)_s$ and $(\zeta_*)_S$ respectively, $(\zeta_*)_s$ lying below $(\zeta_*)_S$.

The previous results allow one to sketch the pattern of the (ζ) curves (Fig. 3).

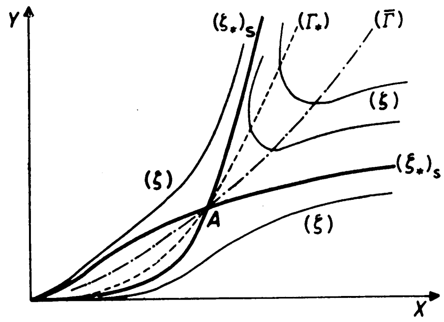

Fig. 3. – Sketch of the pattern of the (ζ) curve.

The following information may be added

d) When $x \to \infty$, either y remains finite ($y^{\gamma-1} = C$), or y may increase indefinitely ($y^2 \sim 2x^4$).

e) When $x \to 0$, $y \to 0$; but one has either

$$y^{\gamma-1} \sim Cx$$

or

$$y^2 \sim 2x^3(1 - \beta x^{(5-3\gamma)/2}) , \qquad \beta > 0 .$$

4. – Shocks.

At a point distant r from the origin, the critical sound speed is given by

$$c_*^2 = \frac{2(\gamma-1)}{\gamma+1}\left(B + \frac{GM}{r}\right) = \frac{2(\gamma-1)}{\gamma+1} B \left(1 + \frac{1}{x}\right) . \tag{12}$$

If one has a shock at this point, according to Prandtl's relation (cf., *e.g.* *Gas Dynamics*, by K. Oswatitsch, p. 33, (New York, 1956))

$$u_1 u_2 = c_*^2 , \qquad u_1 > u_2 . \tag{13}$$

Where u_1 and u_2 are the velocities for $r-0$ and $r+0$ respectively.

Thus.

$$\frac{m^2 \tau_1 \tau_2}{r^4} = c_*^2 , \qquad \tau_1 > \tau_2 ,$$

or

$$y_1 y_2 = \frac{2(\gamma-1)}{\gamma+1}(x^3 + x^4) , \qquad y_1 > y_2 . \tag{14}$$

[1] For $\gamma = \frac{3}{2}$, $y^2 = 2x^4$ is one of the branches of (ζ_*).

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Notice that (10)—the equation of (Γ_*)—appears as a special case of (14).

In the x, y plane, when a shock is present the image point of the flow jumps from a curve (ζ_1) to another (ζ_2). It is inportant to compare the values of the related constants C_1 and C_2. If one notes that the constant in the right-hand side of (3) is proportional to

$$\exp\left[\frac{s}{C_v}\right],$$

where s is the specific entropy and C_v the specific heat at constant volume, it is clear that C_2/C_1 is proportional to

$$\exp\left[\frac{s_2 - s_1}{C_v}\right],$$

and, as $s_2 > s_1$, one concludes that

$$C_2 > C_1 . \tag{15}$$

5. – Discussion of uniqueness.

Let us recall that it is always assumed that τ_∞ and T_∞ are given. Thus the constant $B = C_p T_\infty$ is known. The following assumptions are also considered

a) m is given;

b) the radius of the star is very small in comparison with r_L; and in its neighborhood, the flow is subsonic. This excludes for the moment the special limiting case $\gamma = \frac{5}{3}$.

This last assumption means that in the x, y plane, the image of the flow is to be defined for $0 < x < \infty$, and that the curve must reach the origin in a path lying in the subsonic region in the neighborhood of this point. Of course, it may be a serious limitation, which has to be discussed with figures coming from the observations, but without such an assumption uniqueness cannot be proved.

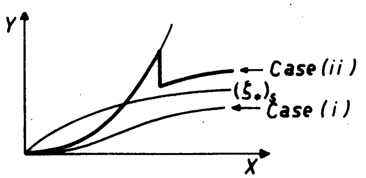

Fig. 4.

With these assumptions, the flow is uniquely defined.

First of all, r_L and τ_L may be computed by (5). Thus y_∞ is known and defines uniquely the (ζ) curve for large x. Let us call (ζ_∞) this arc. Two cases are possible (Fig. 4).

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i) This arc (ζ_∞) lies below the arc $(\zeta_*)_s$ which goes through A. The corresponding (ζ_∞) curve goes through the origin and checks the assumption b). Thus, it is a solution. No other solution is possible because according to (15), such an arc (ζ_∞) would be the image of a flow lying downwards of a shock, if the image of the upward flow were a (ζ) curve lying above the two branches of the critical curve (ζ_*) and thus condition b) would not be satisfied. (ii) Or this arc (ζ_∞) lies above $(\zeta_*)_s$. A shock must be present. For the reason given in i), the only possibility to check assumption b) is to have as image of the upward part of the flow an arc of curve $(\zeta_*)_s$ which corresponds to a flow starting as a subsonic flow for $x=0$ and which is accelerated smoothly through the trans-sonic regime (saddle point A).

The position of the shock is defined as the intersection of the (ζ_∞) curve with curve (ζ), which represents the locus of flows lying just behind a shock when the above state lies on the $(\zeta_*)_s$ branch. It may be checked that these two curves have one and only one point of intersection.

The special case $\gamma=\frac{5}{3}$ is a limiting case whose discussion is left to the reader. We want to emphasize that assumption b) is necessary to guarantee the uniqueness property.

6. – Signification of the results.

The values of τ_∞, T_∞, B are given by the data of the interstellar medium. But the value of m depends on data on the state observed on the surface of the star. Let us note with a subscript the corresponding values of various quantities $(r_0, \tau_0, T_0 \ldots)$

$$m = u_0 r_0^2 \tau_0^{-1}\,.$$

In order to compute m, u_0 and τ_0 (or T_0 and τ_0 according to the Bernoulli equation) must be known. To compare the theory with experimental data, one must check that the flow on the star is subsonic and that the flow at r_0 may be compatible with the data at infinity thanks to the uniquely defined flow in **5**.

Another way to test the validity of such a theory is to notice that data for various stars must be fitted as explained above with the same interstellar medium.

Discussion:

— A. J. DEUTSCH:

I should like to clarify several points. First, I think it is not quite correct to say that there is a controversy between PARKER and myself. As I understand the situation, there is not necessarily any controversy. We depart from different sets of observations, with different physical problems in view, and