Canad. Math. Bull. Vol. 16 (2), 1973.

ON CHARACTERIZATIONS OF CONDITIONAL EXPECTATION

BY
A. N. AL-HUSSAINI[1]

In the following (Ω, α, μ) is a totally σ-finite measure space except where noted. For a sub-σ-algebra $\beta \subset \alpha$, the conditional expectation $E\{f \mid \beta\}$ of f given β is a function measurable relative to β, such that

$$\int_B E\{f \mid \beta\}\, d\mu = \int_B f\, d\mu, \quad \text{all } B \in \beta.$$

In [5] R. G. Douglas proved, among other things the following, in the finite case:

THEOREM 1. *Suppose $\mu(\Omega)=1$. Then a linear operator T on $L_1(\Omega, \alpha, \mu)$ is a conditional expection if and only if*

(1.1) $$\|T\| \le 1$$

(1.2) $$T^2 = T$$

(1.3) $$T1 = 1.$$

The point of this note is to characterize conditional expectation in the σ-finite case (Theorems 2, 3).

As will be shown below we reduce Theorem 2 to Theorem 1. However to prove Theorem 3 we make use of the identification of the limit of the Chacon-Ornstein ergodic theorem ([1], [2], [4], [6]). Finally we prove Theorem 1 as a Corollary to Theorem 3.

THEOREM 2. *A linear operator T on $L_1(\Omega, \alpha, \mu)$ is a conditional expectation relative to some σ-finite sub-σ-algebra $\beta \subset \alpha$ if and only if,*

(2.1) $\|T\| \le 1$

(2.2) $T^2 = T$

(2.3) $T \ge 0$ *i.e.* $Tf \ge 0$ *if* $f \ge 0$

(2.4) *There is $g \in L_1(\Omega, \alpha, \mu)$ such that $Tg > 0$ almost everywhere and $T(Tg \cdot Tg) = Tg \cdot Tg$.*

Received by the editors May 18, 1971 and, in revised form, September 23, 1971.
(1) Supported by National Research Council Grant A-7253.

THEOREM 3. *A linear operator T on $L_1(\Omega, \alpha, \mu)$ is a conditional expectation relative to some σ-finite sub-σ-algebra $\beta \subset \alpha$ if and only if:*

(3.1) $\|T\| \leq 1$

(3.2) $T^2 = T$

(3.3) $T \geq 0$ *i.e.* $Tf \geq 0$ *if* $f \geq 0$

(3.4) *There is $g \in L_1(\Omega, \alpha, \mu)$ such that $Tg > 0$ almost everywhere and $T^*Tg = Tg$.*

Here T^* denotes the adjoint of T i.e.

$$\int Tf \cdot h\, d\mu = \int f \cdot T^*h\, d\mu, \qquad f \in L_1(\Omega, \alpha, \mu), \qquad h \in L_\infty(\Omega, \alpha, \mu).$$

We give the if parts of the proofs only. The only if parts are rather trivial.

Proof of Theorem 2. Let $\nu(A) = \int_A Tg \cdot d\mu$. For $f \in L_1(\Omega, \alpha, \nu)$ we define

$$Pf = \frac{T(f \cdot Tg)}{Tg}.$$

Clearly

(1) $$\|P\| \leq 1$$

(2) $$P^2 = P$$

(3) $$P1 = 1$$

Thus by Theorem 1, there exists β,

$$Pf = E\{f \mid \beta\} \cdots ($$

Now let $f \in L_1(\Omega, \alpha, \mu)$, then by substitution $(f/Tg) \in L_1(\Omega, \alpha, \nu)$. Hence

$$P\left(\frac{f}{Tg}\right) = E\left\{\frac{f}{Tg} \,\middle|\, \beta\right\} \cdots (\nu)$$

and consequently

$$\frac{Tf}{Tg} = E\left\{\frac{f}{Tg} \,\middle|\, \beta\right\} \cdots (\nu)$$

But Tg is a β-measurable function by virtue of assumption (2.4) of the hypothesis. Therefore Tf is β-measurable. This together with the above equation, and definition of ν implies

$$Tf = E\{f \mid \beta\} \cdots (\mu).$$

Finally σ-finiteness of β follows from the positivity of Tg.

Proof of Theorem 3. By B([1], [2]), ([4, pp. 26–29] and [6, pp. 194–211]) T is a conservative operator, and satisfies:

$$\lim_{N\to\infty} \frac{\sum_0^N T^n f}{\sum_0^N T^{n+1} g} = \frac{Tf}{Tg} = \frac{E\{f \mid \beta\}}{E\{Tg \mid \beta\}}.$$

Where β is the σ-finite, σ-algebra of sets invariant under T^*. Since Tg is β-measurable by assumption (3.4) of the hypothesis, $Tf = E\{f \mid \beta\}$.

Proof of Theorem 1. Trivially $\|T\|=1$, $T^*1=1$, and $T1$ satisfies assumption (3.4) of the hypothesis of Theorem 3. We shall show that $T \geq 0$.

By [3], it is easy to show that there is a linear operator $|T|$ the modulus of T satisfying:

(a) $\| \, |T| \, \| \leq 1$

(b) $|Tf| \leq |T| \, |f| \, f \in L_1(\Omega, \alpha, \mu)$

(c) $|T| f(\omega) = \sup_{|g| \leq f} |Tg| (\omega)$, where $f \geq 0$,

$|T| 1 = 1$ follows from (a) and (c), which together with (b) imply that $|T| = T$ i.e. $T \geq 0$. Theorem 3 completes the proof.

References

1. R. V. Chacon, *Identification of the limit of operator averages*, J. Math. and Mech. (1962), 961–968.

2. R. V. Chacon and D. Ornstein, *A general ergodic theorem*, Illinois, J. Math. (1960), 153–160.

3. R. V. Chacon and U. Krengel, *Linear modulus of linear operators*, Proc. Amer. Math. Soc. (1964), 553–559.

4. S. R. Foguel, *The ergodic theory of Markov processses*, Van Nostrand, Princeton, N.J., 1969.

5. R. G. Douglas, *Contractive projections on an L_1 space*, Pacific J. Math. (1965), 443–462.

6. J. Neveu, *Mathematical foundations of the calculus of probability*, Holden-Day, San Francisco, Calif., 1965.

University of Alberta,
Edmonton, Alberta