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On Classes $Q_p^\#$ for Hyperbolic Riemann Surfaces

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Abstract. The Q_p spaces of holomorphic functions on the disk, hyperbolic Riemann surfaces or complex unit ball have been studied deeply. Meanwhile, there are a lot of papers devoted to the $Q_p^\#$ classes of meromorphic functions on the disk or hyperbolic Riemann surfaces. In this paper, we prove the nesting property (inclusion relations) of $Q_p^\#$ classes on hyperbolic Riemann surfaces. The same property for Q_p spaces was also established systematically and precisely in earlier work by the authors of this paper.

1 Introduction

Let R be a hyperbolic Riemann surface, $a \in R$ and let $g(z,a)$ be Green's function of R with logarithmic singularity at a. Let $M(R)$ denote the collection of all functions meromorphic on R. For $f \in M(R)$, we consider the second order differential $f^\#(z)^2\,dxdy$, where

$$f^\#(z) = \frac{|f'(z)|}{1+|f(z)|^2}$$

is the spherical derivative of f with respect to the local parameter $z = x + iy$, and define

$$D^\#(f) = \frac{1}{\pi}\iint_R f^\#(z)^2\,dxdy,$$

and

$$B_p^\#(f) = \sup_{a\in R}\frac{1}{\pi}\iint_R f^\#(z)^2 g^p(z,a)\,dxdy \text{ for } p \ge 0.$$

Note that $\pi D^\#(f)$ is the spherical area of $f(R)$ as a covering surface. By $Q_p^\#(R)$ and $Q_{p,0}^\#(R)$, we denote the classes of functions $f \in M$ such that $B_p^\#(f) < \infty$ and

$$\lim_{a\to\partial R}\iint_R f^\#(z)^2 g^p(z,a)\,dxdy = 0,$$

respectively (*cf.* [5–7]). The class $Q_{p,0}^\#(R)$ is defined for $p > 0$ only. For the special case $p = 1$ these classes have been defined and studied by S. Yamashita; that is, $Q_1^\#(R) = UBC(R)$ (meromorphic functions of uniformly bounded characteristic) and $Q_{1,0}^\#(R) = UBC_0(R)$ (*cf.* [15]). We have $B_0^\#(f) = D^\#(f)$, and $Q_0^\#(R)$ is the spherical Dirichlet class usually denoted by $AD^\#(R)$.

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Recall that the class $\mathcal{N}(R)$ of normal functions (see [2, 11]) and class $\mathcal{N}_0(R)$ of strongly normal functions are defined as follows:

$$\mathcal{N}(R) = \Big\{ f \in M(R) : \sup_{a \in R} \frac{f^{\#}(a)}{\lambda(a)} < \infty \Big\},$$

$$\mathcal{N}_0(R) = \Big\{ f \in M(R) : \lim_{a \to \partial R} \frac{f^{\#}(a)}{\lambda(a)} = 0 \Big\},$$

where $\lambda(z)|dz|$ is the Poincaré metric of R. We also define other kinds of normal functions:

$$\mathcal{CN}(R) = \Big\{ f \in M(R) : \sup_{a \in R} \frac{f^{\#}(a)}{c(a)} < \infty \Big\},$$

$$\mathcal{CN}_0(R) = \Big\{ f \in M(R) : \lim_{a \to \partial R} \frac{f^{\#}(a)}{c(a)} = 0 \Big\},$$

where $c(a) = e^{-\gamma(a)}$ and

$$\gamma(a) = \lim_{z \to a} \Big(g(z,a) - \log \frac{1}{|z-a|} \Big)$$

is the Robin constant under the same local parameter z. By using the universal covering and the Schwarz lemma, we may show that $c(a) \leq \lambda(a)$ so that $\mathcal{CN}(R) \subset \mathcal{N}(R)$ and $\mathcal{CN}_0(R) \subset \mathcal{N}_0(R)$. It is obvious that $\mathcal{CN}(\Delta) = \mathcal{N}(\Delta)$ and $\mathcal{CN}_0(\Delta) = \mathcal{N}_0(\Delta)$ for the unit disk Δ.

If we consider analytic functions and their derivatives on the unit disk or a hyperbolic Riemann surface or the unit ball in the above definitions, we obtain Q_p, $Q_{p,0}$ spaces, Bloch spaces, and the Dirichlet space. These spaces have been studied extensively and deeply. There are a lot of references dedicated to this subject (see [3–7, 9, 10, 12–14], in particular, [14]). The nesting property (inclusion relations) of Q_p spaces on hyperbolic Riemann surfaces was established systematically and precisely in [4]. In this paper, we investigate the same problem for $Q_p^{\#}$ classes. Our results are as follows:

(a) For $f \in M(R)$, let

$$I_p(f) = \frac{1}{\pi} \int\!\!\int_R f^{\#}(z)^2 g^p(z,a)\, dx\, dy.$$

If $D^{\#} = D^{\#}(f) < 1$, then

$$\Big(\frac{f^{\#}(a)}{c(a)} \Big)^2 \leq \frac{I_p(f)}{\lambda_p} \leq \frac{I_q(f)}{\lambda_q} \leq \frac{D^{\#}}{1 - D^{\#}} \quad \text{for } p > q > 0,$$

where

$$\lambda_p = 2\pi (1 - D^{\#})^2 \int_0^\infty \frac{t^p e^{2t}\, dt}{(D^{\#} + (1 - D^{\#}) e^{2t})^2} \quad \text{for } p > 0.$$

Further, the estimates are precise, and one of the equalities holds if and only if R is obtained from a hyperbolic surface R_0 by deleting a set of capacity 0 and f is extended to a conformal mapping of R_0 onto a spherical disk such that $f(a)$ is the spherical center of the disk.

(b) If the assumption $D^\# < 1$ is omitted, then

$$\frac{f^\#(a)}{c(a)} \le b_{p,I_p(f)}, \quad I_p(f) \le a_{p,q,I_q(f)}, \quad \text{for } p > q > 0.$$

(c) The following inclusion relations hold:

$$\mathcal{CN}(R) \supset Q_p^\#(R) \supset Q_q^\#(R) \supset AD^\#(R),$$
$$\mathcal{CN}_0(R) \supset Q_{p,0}^\#(R) \supset Q_{q,0}^\#(R), \quad \text{for } p > q > 0.$$

2 A Spherical Area Inequality

First we formulate the following elementary spherical isoperimetric inequality on the sphere (see the proof of [8, Lemma II]).

Lemma 2.1 *Let γ be a piecewise smooth simple closed curve on the Riemann sphere that has spherical length l and bounds two domains of spherical areas S and $\pi - S$. Then*

$$S(\pi - S) \le \frac{l^2}{4}.$$

Lemma 2.2 *Let $S = S_1 + \cdots + S_n$ and $-\pi < S_j < \pi$ for $j = 1, \ldots, n$. If $|S| < \pi$, then*

$$|S|(\pi - |S|) \le \sum_{j=1}^{n} |S_j|(\pi - |S_j|).$$

Proof We prove the lemma by induction to n. The lemma is obviously true for $n = 1$. Assume that the lemma is true for $n = k$. Denote $S' = S_2 + \cdots + S_{k+1}$ and $S = S_1 + S'$. Without loss of generality, assume that $S \ge 0$ and $S_1 \ge 0$. Then we have $-S_1 \le S' \le S$ and

$$|S'|(\pi - |S'|) \le \sum_{j=2}^{k+1} |S_j|(\pi - |S_j|)$$

by the induction assumption. Thus

$$\begin{aligned} |S|(\pi - |S|) &= \pi S_1 - S_1^2 + \pi S' - S'^2 - 2S_1 S' \\ &\le \sum_{j=1}^{k+1} |S_j|(\pi - |S_j|) - \big(\pi(|S'| - S') + 2S_1 S'\big). \end{aligned}$$

Note that $\pi(|S'| - S') + 2S_1S' = 2S_1S' \ge 0$ if $S' \ge 0$ and $\pi(|S'| - S') + 2S_1S' = 2|S'|(\pi - S_1) \ge 0$ if $S' \le 0$. This shows that the lemma is also true for $n = k + 1$. The lemma is proved. ∎

Lemma 2.3 *Let γ be a piecewise smooth closed curve (or a finite sum of such curves) on the complex w-plane of spherical length l, and let*

$$S = \frac{i}{2} \int_\gamma \frac{w\,\overline{dw}}{1 + |w|^2}.$$

If $|S| < \pi$, then $|S|(\pi - |S|) \le \frac{l^2}{4}$.

Proof By an approximation, we may assume that γ is a polygonal closed curve. It is obvious that such a γ may be written as a finite sum of polygonal Jordan closed curves γ_j, where γ_j has spherical lengths l_j and bounds a domain Ω_j, for $j = 1, \dots, n$. Then, S is a sum of S_j, where S_j is the same integral taken over l_j, for $j = 1, \dots, n$. By Green's formula, for $j = 1, \dots, n$, we have

$$S_j = \pm\frac{i}{2}\int\!\!\int_{\Omega_j} d\Big(\frac{w\,d\overline{w}}{1+|w|^2}\Big) = \pm\frac{i}{2}\int\!\!\int_{\Omega_j} d\left(\frac{w}{1+|w|^2}\right)\overline{dw}$$
$$= \pm\frac{i}{2}\int\!\!\int_{\Omega_j}\frac{dw\,d\overline{w}}{(1+|w|^2)^2} = \pm\int\!\!\int_{\Omega_j}\frac{du dv}{(1+|w|^2)^2}$$

according to whether γ_j has an anti-clockwise or clockwise direction. The last integral in the above equality is the spherical area of Ω_j.

Since $|S| < \pi$ and $|S_j| < \pi$ for $j = 1, \dots, n$, we may use Lemmas 2.1 and 2.2, and obtain

$$|S|(\pi - |S|) \le \sum_{j=1}^{n} |S_j|(\pi - |S_j|) \le \sum_{j=1}^{n}\frac{l_j^2}{4} \le \frac{l^2}{4}. \qquad \blacksquare$$

Theorem 2.4 (Spherical isoperimetric inequality of meromorphic functions on Riemann surfaces) *Let Ω be a relatively compact domain on a Riemann surface with a piecewise smooth boundary $\Gamma = \partial\Omega$, let f be a non-constant function meromorphic on $\overline{\Omega}$, and let S and l be the spherical area of $f(\Omega)$ as a covering surface and the spherical length of $\gamma = f(\Gamma)$, respectively. If $S < \pi$, then $S(\pi - S) \le l^2/4$.*

Proof Assume that Γ is positively oriented. First, we assume that f is holomorphic on $\overline{\Omega}$. Then, as the proof of Lemma 2.3, using Green's formula, we have

$$S = \int\!\!\int_{\Omega} f^{\#}(z)^2\,dxdy = \frac{i}{2}\int\!\!\int_{\Omega}\frac{df(z)\,d\overline{f(z)}}{(1+|f(z)|^2)^2}$$
$$= \frac{i}{2}\int_{\Gamma}\frac{f(z)\,d\overline{f(z)}}{1+|f(z)|^2} = \frac{i}{2}\int_{\gamma}\frac{w\overline{dw}}{1+|w|^2}.$$

Using Lemma 2.3 we obtain the conclusion of the lemma. In the case where f has finitely many poles on $\overline{\Omega}$, the lemma can be proved by considering $\overline{\Omega}'$ obtained from $\overline{\Omega}$ by deleting some parameter disks (or half disks) around the poles, and letting these disks shrink to points. The proof is complete. $\blacksquare$

Theorem 2.5 *Let R be a hyperbolic Riemann surface, let $g(z,a)$ be a Green's function of R with logarithmic singularity at $a \in R$ and for $t \ge 0$, let $R_t = \{z \in R : g(z,a) > t\}$. For $f \in M(R)$ and $t \ge 0$, let*

$$\psi(t) = \int\!\!\int_{R_t} f^{\#}(z)^2\,dxdy,$$

which denotes the spherical area of $f(R_t)$ as a covering surface. If $\psi(t) > 0$ for $t > 0$, then ψ is a continuous decreasing function and

$$\int\!\!\int_{R_t} f^{\#}(z)^2 g^p(z,a)\,dxdy = -\int_t^{\infty}\sigma^p\,d\psi(\sigma), \quad \text{for } p \ge 0,\ t \ge 0, \tag{2.1}$$

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where the integral at the right side is understood as a Stieltjes integral and defined by the limit as $t \to 0$ if $t = 0$. If $\psi(t_0) < \pi$ for some $t_0 \geq 0$, then

$$\psi(\sigma) \leq \frac{\pi\psi(t)}{\psi(t) + (\pi - \psi(t))e^{2(\sigma-t)}} \quad \textit{for } \sigma \geq t \geq t_0. \tag{2.2}$$

Proof First we assume that R is a finite surface and f is meromorphic on $\overline{R}$. Let $g^*(z,a)$ be the harmonic conjugate of $g(z,a)$ and let $G(z) = g(z,a) + ig^*(z,a)$. For $t \geq 0$, let

$$\Gamma_t = \{z \in \overline{R} : g(z,a) = t\}, \quad \psi_0(t) = \int_{\Gamma_t} \frac{f^\#(z)^2}{|G'(z)|^2} \frac{\partial g}{\partial n}\, ds.$$

By substituting $dgdg^* = |G'(z)|^2\, dxdy$, we have

$$\begin{aligned}
\iint_{R_t} f^\#(z)^2 g^p(z,a)\, dxdy &= \iint_{R_t} \frac{f^\#(z)^2}{|G'(z)|^2} g^p(z,a)|G'(z)|^2\, dxdy \\
&= \iint_{R_t} \frac{f^\#(z)^2}{|G'(z)|^2} g^p(z,a)\, dgdg^* \\
&= \int_t^\infty \int_{\Gamma_\sigma} \frac{f^\#(z)^2}{|G'(z)|^2} g^p(z,a)\Big(\frac{\partial g}{\partial n}\Big)^2 dsdn \\
&= \int_t^\infty \int_{\Gamma_\sigma} \frac{f^\#(z)^2}{|G'(z)|^2} g^p(z,a)\frac{\partial g}{\partial n}\, ds\, d\sigma \\
&= \int_t^\infty \sigma^p \psi_0(\sigma)\, d\sigma,
\end{aligned} \tag{2.3}$$

where we have set $\sigma = g(z,a)$ and hence $d\sigma = (\partial g/\partial n)\, dn$ along Γ_σ. In particular,

$$\psi(t) = \int_t^\infty \psi_0(\sigma)\, d\sigma. \tag{2.4}$$

Combining (2.3) and (2.4) gives (2.1).

For $t \geq t_0$, by Schwarz's inequality, we have

$$\begin{aligned}
\psi_0(t) &= \frac{1}{2\pi} \int_{\Gamma_t} \frac{f^\#(z)^2}{|G'(z)|^2} \frac{\partial g}{\partial n}\, ds \int_{\Gamma_t} \frac{\partial g}{\partial n}\, ds \\
&\geq \frac{1}{2\pi}\Big(\int_{\Gamma_t} \frac{f^\#(z)}{|G'(z)|} \frac{\partial g}{\partial n}\, ds\Big)^2 = \frac{1}{2\pi}\Big(\int_{\Gamma_t} f^\#(z)\, ds\Big)^2.
\end{aligned}$$

Thus, using Lemma 3.1, the spherical isoperimetric inequality for the function f on the domain R_t, we have

$$\psi(t)(\pi - \psi(t)) \leq \frac{1}{4}\Big(\int_{\Gamma_t} f^\#(z)\, ds\Big)^2 \leq \frac{\pi}{2}\psi_0(t). \tag{2.5}$$

Since $\psi(t) \leq \psi(t_0) < \pi$, it follows from (2.4) and (2.5) that

$$-\frac{\psi'(t)}{\psi(t)(\pi - \psi(t))} \geq \frac{2}{\pi}.$$

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Integrating the above inequality from t_0 to t gives

$$\frac{\pi-\psi(t)}{\psi(t)} \geq \frac{\pi-\psi(t_0)}{\psi(t_0)} e^{2(t-t_0)}$$

from which (2.2) follows.

Now, assume that R is a general hyperbolic Riemann surface. Let R^n be a regular exhaustion of R, and let $\psi_n(t)$ be defined for f, R^n and a, as in the theorem. By what we have proved for finite surfaces, for $n = 1, 2, \ldots$, we have

$$\psi_n(\sigma) \leq \frac{\pi\psi_n(t)}{\psi_n(t) + (\pi - \psi_n(t))e^{2(\sigma-t)}} \quad \text{for } \sigma \geq t \geq t_0, \tag{2.6}$$

$$\iint_{R_t^n} f^\#(z)^2 g_n^p(z,a)\,dx\,dy = -\int_t^\infty \sigma^p\,d\psi_n(\sigma), \quad \text{for } p \geq 0,\ \ t \geq 0, \tag{2.7}$$

where $g_n(z, a)$ is a Green's function of R^n. It is obvious that $\psi_n(t) \to \psi(t)$ for every fixed $t \geq 0$, so letting $n \to \infty$ in (2.6) we obtain (2.2). Using (2.7) and integrating by parts, it follows that

$$\iint_{R_t^n} f^\#(z)^2 g_n^p(z,a)\,dx\,dy = t^p\psi_n(t) + p\int_t^\infty \sigma^{p-1}\psi_n(\sigma)\,d\sigma, \tag{2.8}$$

for $p \geq 0$, $t \geq 0$. Letting $n \to \infty$ in (2.8) and integrating by parts again gives

$$\iint_{R_t} f^\#(z)^2 g^p(z,a)\,dx\,dy = t^p\psi(t) + p\int_t^\infty \sigma^{p-1}\psi(\sigma)\,d\sigma = -\int_t^\infty \sigma^p\,d\psi(\sigma).$$

This shows (2.1). Taking the limit as $t \to 0$ completes the proof. ■

We call (2.2) the spherical area inequality.

3 A Lemma from Calculus

All of our main results are obtained by using a lemma that belongs entirely to real analysis. We formulate and prove it in this section.

Lemma 3.1 *Let $\psi(t)$, $t \geq 0$, be a positive continuous and decreasing function ($\psi(0)$ is allowed to assume ∞). For $p \geq 0$ and $t \geq 0$, define*

$$h_p(t) = -\int_t^\infty \sigma^p\,d\psi(\sigma),$$

where the integral is understood as in Theorem 2.4. If $\psi(t_0) < \pi$ for some t_0 and (2.2) holds for $\sigma > t \geq t_0$, then for $p > q > 0$,

$$\frac{1}{\pi}\lim_{t\to\infty} e^{2t}\psi(t) \leq \frac{h_p(t_0)}{\lambda_p} \leq \frac{h_q(t_0)}{\lambda_q} \leq \frac{h_0(t_0)}{\lambda_0} = \frac{e^{2t_0}\psi(t_0)}{\pi - \psi(t_0)}, \tag{3.1}$$

where

$$\lambda_p = 2\pi e^{-4t_0}(\pi - \psi(t_0))^2 \int_{t_0}^\infty \frac{t^p e^{2t}\,dt}{(\psi(t_0) + (\pi - \psi(t_0))e^{2(t-t_0)})^2} \quad \textit{for } p \geq 0.$$

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Proof Let $p > q > 0$. Note that (2.2) implies that $e^{2t}\psi(t)/(\pi-\psi(t))$ is decreasing and, consequently, $d(e^{2t}\psi(t)/(\pi-\psi(t))) \le 0$ and

$$dh_q(t) = t^q d\psi(t) \le -\frac{2}{\pi} t^q \psi(t)(\pi-\psi(t))\,dt \quad \text{for } t \ge t_0.$$

On the other hand, using (2.2) and integrating by parts twice, we have

$$\begin{aligned} h_q(t) &= t^q\psi(t) + q\int_t^\infty \sigma^{q-1}\psi(\sigma)\,d\sigma \\ &\le t^q\psi(t) + q\pi\psi(t)\int_t^\infty \frac{\sigma^{q-1}d\sigma}{\psi(t)+(\pi-\psi(t))e^{2(\sigma-t)}} \\ &= 2\pi e^{-2t}\psi(t)(\pi-\psi(t))\int_t^\infty \frac{e^{2\sigma}\sigma^q d\sigma}{(\psi(t)+(\pi-\psi(t))e^{2(\sigma-t)})^2}. \end{aligned}$$

Thus,

$$\frac{dh_q(t)}{h_q(t)} \le -\frac{t^q e^{2t}dt}{\pi^2\int_t^\infty e^{2\sigma}\sigma^q(\psi(t)+(\pi-\psi(t))e^{2(\sigma-t)})^{-2}d\sigma}. \tag{3.2}$$

Since, by (2.2),

$$\psi(t) \le \frac{\pi\psi(t_0)}{\psi(t_0)+(\pi-\psi(t_0))e^{2(t-t_0)}} \quad \text{for } t \ge t_0, \tag{3.3}$$

we may replace $\psi(t)$ in (3.2) by the right side of (3.3) and obtain

$$\frac{dh_q(t)}{h_q(t)} \le -\frac{t^q e^{2t}(\psi(t_0)+(\pi-\psi(t_0))e^{2(t-t_0)})^{-2}dt}{\int_t^\infty e^{2\sigma}\sigma^q(\psi(t_0)+(\pi-\psi(t_0))e^{2(\sigma-t_0)})^{-2}d\sigma} \quad \text{for } t \ge t_0.$$

Consequently, integrating this from t_0 to t gives

$$h_q(t) \le h_q(t_0)\cdot\frac{\int_t^\infty e^{2\sigma}\sigma^q(\psi(t_0)+(\pi-\psi(t_0))e^{2(\sigma-t_0)})^{-2}d\sigma}{\int_{t_0}^\infty e^{2\sigma}\sigma^q(\psi(t_0)+(\pi-\psi(t_0))e^{2(\sigma-t_0)})^{-2}d\sigma} \quad \text{for } t \ge t_0. \tag{3.4}$$

Finally integrating by parts, using (3.4), and exchanging the order of integral, we have

$$\begin{aligned} h_p(t_0) &= t_0^{p-q}h_q(t_0) + (p-q)\int_{t_0}^\infty t^{p-q-1}h_q(t)\,dt \\ &\le t_0^{p-q}h_q(t_0) + \frac{(p-q)h_q(t_0)}{\lambda_q'}\int_{t_0}^\infty t^{p-q-1}dt \\ &\quad \cdot\int_t^\infty \frac{\sigma^q e^{2\sigma}d\sigma}{(\psi(t_0)+(\pi-\psi(t_0))e^{2(\sigma-t_0)})^2} \\ &= t_0^{p-q}h_q(t_0) + \frac{(p-q)h_q(t_0)}{\lambda_q'} \\ &\quad \cdot\int_{t_0}^\infty \frac{\sigma^q e^{2\sigma}d\sigma}{(\psi(t_0)+(\pi-\psi(t_0))e^{2(\sigma-t_0)})^2}\int_{t_0}^\sigma t^{p-q-1}dt \\ &= h_q(t_0)\cdot\frac{\int_{t_0}^\infty \sigma^p e^{2\sigma}(\psi(t_0)+(\pi-\psi(t_0))e^{2(\sigma-t_0)})^{-2}d\sigma}{\int_{t_0}^\infty \sigma^q e^{2\sigma}(\psi(t_0)+(\pi-\psi(t_0))e^{2(\sigma-t_0)})^{-2}d\sigma}, \end{aligned}$$

where

$$\lambda_q' = \int_{t_0}^{\infty} \frac{t^q e^{2t} dt}{(\psi(t_0) + (\pi - \psi(t_0))e^{2(t-t_0)})^2}.$$

The second inequality of (3.1) is proved.

The third inequality of (3.1) follows from

$$\lim_{q\to 0} \frac{h_q(t_0)}{\lambda_q} = \frac{h_0(t_0)}{\lambda_0} = \frac{e^{2t_0}\psi(t_0)}{\pi - \psi(t_0)}.$$

To show the first one of (3.1), it suffices to prove that

$$\lim_{p\to\infty} \frac{h_p(t_0)}{\lambda_p} = \frac{1}{\pi} \lim_{t\to\infty} e^{2t}\psi(t). \tag{3.5}$$

Integrating by parts, we have

$$h_p(t_0) = t_0^p \psi(t_0) + p \int_{t_0}^{\infty} t^{p-1}\psi(t)\, dt,$$

$$\lambda_p = e^{-2t_0} t_0^p (\pi - \psi(t_0)) + p\pi e^{-2t_0}(\pi - \psi(t_0)) \int_{t_0}^{\infty} \frac{t^{p-1} dt}{\psi(t_0) + (\pi - \psi(t_0))e^{2(t-t_0)}}.$$

Denote the last integral by $\omega(p)$. It is easy to see that

$$\omega(p) \geq \frac{1}{\pi} \int_{t_0}^{\infty} t^{p-1} e^{-2t} dt \geq \frac{e^{-2t_0}\Gamma(p)}{2^p \pi}$$

and $t_0^p = o(\omega(p))$ as $p \to \infty$. So, we only need to calculate the limit of $B(p) = \omega(p)^{-1} \int_{t_0}^{\infty} t^{p-1}\psi(t)\, dt$.

Letting $k(t) = (\psi(t_0) + (\pi - \psi(t_0))e^{2(t-t_0)})\psi(t)$ and integrating by parts, we obtain

$$\begin{aligned} B(p) &= \frac{1}{\omega(p)} \int_{t_0}^{\infty} \frac{t^{p-1} k(t) dt}{\psi(t_0) + (\pi - \psi(t_0))e^{2(t-t_0)}} \\ &= e^{-2t_0}(\pi - \psi(t_0)) \lim_{p\to\infty} e^{2t}\psi(t) \\ &\quad - \frac{1}{\omega(p)} \int_{t_0}^{\infty} dk(t) \int_{t_0}^{t} \frac{\sigma^{p-1} d\sigma}{\psi(t_0) + (\pi - \psi(t_0))e^{2(\sigma-t_0)}}. \end{aligned}$$

The last term I is estimated as a sum of two terms I' and I'', which correspond to the integrals from T to ∞ and from t_0 to T, respectively. Suppose T is sufficiently large. Let $\tau_1(t) = e^{2t}\psi(t)/(\pi - \psi(t))$, which is decreasing as mentioned before, let $\tau_2(t) = \psi(t_0)e^{-2t} + (\pi - \psi(t_0))e^{-2t_0}$, which is also decreasing, and let $\mu(t) = \tau_1(t)\tau_2(t)$. Then

$$\begin{aligned} dk(t) &= d\{\mu(t)(\pi - \psi(t))\} = (\pi - \psi(t))d\mu(t) - \mu(t)d\psi(t), \\ |I'| &\leq \frac{1}{\omega(p)} \int_{T}^{\infty} |dk(t)| \int_{t_0}^{t} \frac{\sigma^{p-1} d\sigma}{\psi(t_0) + (\pi - \psi(t_0))e^{2(\sigma-t_0)}} \leq \int_{T}^{\infty} |dk(t)| \\ &\leq -\int_{T}^{\infty} (\pi d\mu(t) + \mu(t_0)d\psi(t)) \to 0 \quad \text{as } T \to \infty, \end{aligned}$$

since μ and ψ are both positive and decreasing. This shows that $I' \to 0$ as $T \to \infty$ uniformly for $p > 0$. On the other hand, for a given $T > t_0$,

$$|I''| \le \frac{1}{\omega(p)} \int_{t_0}^{T} |dk(t)| \int_{t_0}^{t} \frac{\sigma^{p-1} d\sigma}{\psi(t_0) + (\pi - \psi(t_0)) e^{2(\sigma - t_0)}}$$
$$\le \frac{T^p}{p\pi\omega(p)} \int_{t_0}^{T} |dk(t)| \to 0 \quad \text{as } p \to \infty.$$

Thus, $I \to 0$ as $p \to \infty$. This shows that $B(p) \to e^{-2t_0}(\pi - \psi(t_0)) \lim_{t\to\infty} e^{2t}\psi(t)$. So (3.5) and the first inequality of (3.1) are proved. The proof of the lemma is complete. ∎

4 The Case $D^\#(f) < 1$

Different from the holomorphic case (see [4]), precise estimates can be obtained under the assumption $D^\#(f) < 1$ only.

Theorem 4.1 *Let R be a hyperbolic Riemann surface, $a \in R$, and let $f \in M(R)$ be a non-constant function such that*

$$D^\# = D^\#(f) = \frac{1}{\pi} \iint_R f^\#(z)^2 \, dx dy < 1.$$

For $p > 0$, denote

$$I_p(f) = \iint_R f^\#(z)^2 g^p(z,a) \, dx dy, \quad \lambda_p = 2\pi(1 - D^\#)^2 \int_0^\infty \frac{t^p e^{2t} dt}{(D^\# + (1 - D^\#)e^{2t})^2}.$$

Then, for $p > q > 0$, we have

$$\left(\frac{f^\#(a)}{c(a)}\right)^2 \le \frac{I_p(f)}{\lambda_p} \le \frac{I_q(f)}{\lambda_q} \le \frac{D^\#}{1 - D^\#}. \tag{4.1}$$

Proof Let $\psi(t)$ and h_p be the functions defined in Theorem 2.4 for R, a, f and in Lemma 6.1 for $\psi(t)$ and $t_0 = 0$, respectively. We have $I_p(f) = h_p(0)$ for $p \ge 0$ by (2.1). The λ_p is the same in Lemma 6.1. Using (3.1), we obtain the second and third inequalities of (4.1). It remains to prove that

$$\lim_{t\to\infty} e^{2t}\psi(t) = \pi\left(\frac{f^\#(a)}{c(a)}\right)^2. \tag{4.2}$$

To show this, we take $\zeta = \xi + i\eta = \exp\{-g(z,a) - ig^*(z,a)\}$ as a local parameter around a. Then, for sufficient large t,

$$e^{2t}\psi(t) = e^{2t} \iint_{|\zeta|<e^{-t}} f^\#(\zeta)^2 d\xi d\eta,$$

which tends to $\pi f^\#(a)^2$ obviously as $t \to \infty$. Note that $\gamma(a) = 0$ and $c(a) = 1$ under this parameter. The theorem is proved. ∎

Example 4.2 Consider the function $f(z) = z$ on the unit disk Δ and let $a = 0$. We have

$$\psi(t) = \frac{\pi}{1+e^{2t}}, \quad D^{\#}(f) = \frac{1}{2}, \quad \frac{f^{\#}(a)}{c(a)} = 1,$$

$$I_p(f) = 2\pi \int_0^{\infty} \frac{t^p e^{2t} dt}{(1+e^{2t})^2} \quad \text{for } p > 0.$$

So, the equalities in (2.2) and (4.1) hold for $\sigma > t \geq 0$ and $p > q > 0$, respectively. This shows that (2.2) and (4.1) are all sharp.

The conditions $\psi(t_0) < \pi$ for (2.2) and $D^{\#}(f) < 1$ for (4.1) are essential in their proofs. In fact, the following examples show that it is impossible to bound $I_p(f)$ or $f^{\#}(a)/c(a)$ in terms of $D^{\#}(f)$ without the condition $D^{\#}(f) < 1$.

Example 4.3 For $\delta > 0$, let $f_\delta(z) = \delta(z + 1/z)$ for $z \in \Delta$. We have $D^{\#}(f_\delta) = 1$ for $\delta > 0$, since f_δ maps Δ univalently onto the extended complex plane with a slit $[-2\delta, 2\delta]$. Let $\delta_n = (n+1/n)^{-1}$, then $f_{\delta_n}(1/n) = 1$ and f_{δ_n} maps the disk $\{z : |z| < 1/n\}$ onto a domain that covers the exterior of the unit disk and has a spherical area bigger than $\pi/2$. Thus,

$$I_1(f_{\delta_n}) \geq \iint_{|z|<1/n} f_{\delta_n}^{\#}(z)^2 \log\frac{1}{z} dxdy \geq \log n \iint_{|z|<1/n} f_{\delta_n}^{\#}(z)^2 dxdy \geq \frac{\pi}{2}\log n,$$

which tends to the infinity as $n \to \infty$. Meanwhile, $c(0) = 1$ and $f_{\delta_n}^{\#}(0) = 1/\delta_n$.

Example 4.4 Let $f_n(z) = nz$ for $z \in \Delta$ and $n = 1, 2, \ldots$. Then $D^{\#}(f_n) \to 1$ as $n \to \infty$. However, $c(0) = 1$ and $f^{\#}(0) = n \to \infty$ as $n \to \infty$.

5 Nesting Property of Classes $\mathcal{C}N(R)$ and $Q_p^{\#}(R)$ for $p > 0$

Although the condition $D^{\#}(f) < 1$ is necessary for (4.1), yet it is really possible to bound $I_p(f)$ by $I_q(f)$ for $p > q > 0$ and to bound $f^{\#}(a)/c(a)$ by $I_p(f)$ for $p > 0$ without any additional condition about $D^{\#}(f)$. We will formulate and prove these results in this section, from which the nesting property of classes $\mathcal{C}N(R)$ and $Q_p^{\#}(R)$ for $p > 0$ follows.

Theorem 5.1 *Let R be a hyperbolic Riemann surface, $a \in R$ and let $f \in M(R)$ be not a constant. Let $p > q > 0$, and let $I_p(f)$, $I_q(f)$ be defined in Theorem 2.5. If $I_q(f) < \infty$, then*

$$I_p(f) \leq a_{p,q,I_q(f)}, \quad \left(\frac{f^{\#}(a)}{c(a)}\right)^2 \leq b_{q,I_q(f)}. \tag{5.1}$$

Proof Let $\psi(t)$ be the function defined in Theorem 2.4 for f. Assume that $I_q(f) < \infty$. Then $0 < \psi(t) < \infty$ for $t > 0$. Take $t_0 = (2I_q(f)/\pi)^{1/q}$. Then $\psi(t_0) \leq \pi/2$, since, by Theorem 2.4,

$$I_q(f) = -\int_0^{\infty} t^q d\psi(t) \geq -\int_{t_0}^{\infty} t^q d\psi(t) \geq t_0^q \psi(t_0).$$

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Using Theorems 2.4 and 2.5, the second inequality of (3.1) with $t_0 = (2I_q(f)/\pi)^{1/q}$, we have

$$\begin{aligned} I_p(f) &= -\int_0^\infty t^p \, d\psi(t) = -\int_0^{t_0} t^p \, d\psi(t) - \int_{t_0}^\infty t^p \, d\psi(t) \\ &\le -t_0^{p-q} \int_0^{t_0} t^q \, d\psi(t) - \frac{4\int_{t_0}^\infty t^p e^{2t} e^{-4(t-t_0)}\,dt}{\int_{t_0}^\infty t^q e^{2t}(1+2e^{2(t-t_0)})^{-2}\,dt} \int_{t_0}^\infty t^q \, d\psi(t) \le a_{p,q,t_0}. \end{aligned}$$

This shows the first one of (5.1).

Using the first one of (3.1) (p is replaced by q) with $t_0 = (2I_q(f)/\pi)^{1/q}$ and (4.2), we obtain

$$\Big(\frac{f^\#(a)}{c(a)}\Big)^2 \le \frac{h_q(t_0)}{\lambda_q} \le \frac{I_q(f)}{\lambda_q}.$$

Note that

$$\begin{aligned} \lambda_q &= 2\pi e^{-4t_0}\big(\pi - \psi(t_0)\big)^2 \int_{t_0}^\infty \frac{t^q e^{2t}\,dt}{(\psi(t_0) + (\pi - \psi(t_0))e^{2(t-t_0)})^2} \\ &\ge 2\pi e^{-4t_0} \int_{t_0}^\infty \frac{t^q e^{2t}\,dt}{(1+e^{2(t-t_0)})^2} = b_{q,t_0}. \end{aligned}$$

This shows the second one of (5.1). The theorem is proved. ∎

The nesting property of $Q_p(R)$ and $\mathcal{CN}(R)$ is formulated in the following theorem, which are consequences of Theorem 4.1.

Theorem 5.2 *For any hyperbolic Riemann surface R, we have*

$$Q_q^\#(R) \subset Q_p^\#(R) \subset \mathcal{CN}(R) \quad \textit{and} \quad Q_{q,0}^\#(R) \subset Q_{p,0}^\#(R) \subset \mathcal{CN}_0(R) \quad \textit{for } \ p > q > 0.$$

6 Spherical Dirichlet Class

As indicated in Section 4, $I_p(f)$ and $f^\#(a)/c(a)$ cannot be bounded in terms of $D^\#(f)$ without an additional condition. However, it is still true that $AD^\#(R) \subset Q_p^\#(R)$ for $p > 0$ and, consequently, $AD^\#(R) \subset \mathcal{CN}(R)$.

Lemma 6.1 *Let R be a hyperbolic Riemann surface, $a \in R$, and let $f \in M(R)$ be not a constant. If $D^\#(f) < \infty$ and*

$$D_{t_0}^\#(f) = \frac{1}{\pi}\iint_{g(z,a)>t_0} f^\#(z)^2\,dx\,dy = k < 1$$

for some $t_0 > 0$, then

$$I_p(f) \le \pi t_0^p D^\#(f) + \frac{2\pi k}{1-k}\int_{t_0}^\infty t^p e^{-2(t-t_0)}\,dt \quad \textit{for } \ p > 0, \tag{6.1}$$

where $I_p(f)$ is defined in Theorem 2.5.

Proof Let $p > 0$ be given. Let $\psi(t)$ be the function defined in Theorem 2.4. We have

$$-\int_0^{t_0} t^p\,d\psi(t) \le -t_0^p \int_0^{t_0} d\psi(t) \le t_0^p \psi(0) = \pi t_0^p D^\#(f)$$

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and, using the third inequality of (3.1) to $\psi(t)$,

$$
\begin{aligned}
-\int_{t_0}^{\infty} t^p d\psi(t) = h_p(t_0) &\le \lambda_p \cdot \frac{e^{2t_0}\psi(t_0)}{\pi - \psi(t_0)} \\
&= 2\pi e^{-2t_0}\psi(t_0)(\pi - \psi(t_0)) \int_{t_0}^{\infty} \frac{t^p e^{2t} dt}{(\psi(t_0) + (\pi - \psi(t_0))e^{2(t-t_0)})^2} \\
&\le \frac{2\pi k}{1-k} \int_{t_0}^{\infty} t^p e^{-2(t-t_0)} dt.
\end{aligned}
$$

Thus, (6.1) follows, since $I_p(f) = -\int_0^{\infty} t^p d\psi(t)$ by Theorem 2.4. The lemma is proved. ∎

Theorem 6.2 *If R is a hyperbolic Riemann surface, then*

$$AD^{\#}(R) \subset Q_p^{\#}(R) \subset \mathcal{CN}(R) \quad for \;\; p > 0.$$

Proof Let $p > 0$ and $f \in AD^{\#}(R)$, *i.e.*, $D^{\#}(f) < \infty$. Then there exists a compact set $E_1 \subset R$ such that

$$\frac{1}{\pi} \iint_{R \setminus E_1} f^{\#}(z)^2 \, dx dy < \frac{1}{2}.$$

Since

$$\limsup_{a \to \partial R} \max_{z \in E_1} g(z, a) = M < \infty,$$

we have a compact set $E_2 \subset R$ such that $g(z, a) < t_0 = M + 1$ for $a \in R \setminus E_2$ and $z \in E_1$. Let $a \in R \setminus E_2$ and let $D_{t_0}^{\#}(f)$ be defined in Lemma 7.2 for a. Since

$$D_{t_0}^{\#}(f) \le \frac{1}{\pi} \iint_{R \setminus E_1} f^{\#}(z)^2 \, dx dy < \frac{1}{2},$$

we may use Lemma 7.2 and obtain

$$\frac{1}{\pi} \iint_R f^{\#}(z)^2 g(z, a)^p \, dx dy \le t_0^p D^{\#}(f) + 2 \int_{t_0}^{\infty} t^p e^{-2(t-t_0)} dt = A_1 \quad \text{for } a \in R \setminus E_2.$$

On the other hand, the integral on the left side of the above inequality is continuous with respect to a. Letting

$$A_2 = \max_{a \in E_2} \frac{1}{\pi} \iint_R f^{\#}(z)^2 g(z, a)^p \, dx dy < \infty,$$

we have $B_p^{\#}(f) \le \max\{A_1, A_2\}$. This shows that $f \in Q_p^{\#}(R)$, and the theorem is proved. ∎

As a special case $p = 1$, since $Q_1^{\#}(R) = UBC$ (*cf.* [15]), the first inclusion in Theorem 5.2 gives an affirmative solution to Yamashita's question [16].

A hyperbolic Riemann surface is called a regular surface if $\max\{g(z, a) : z \in E\} \to 0$ as $a \to \partial R$ for any compact set $E \subset R$. For regular surfaces the previous theorem can be strengthened.

Theorem 6.3 *If R is a regular hyperbolic Riemann surface, then*

$$AD^\#(R) \subset Q_{p,0}^\#(R) \subset \mathcal{CN}_0(R) \quad \textit{for } p > 0.$$

Proof The proof is similar to that of the above theorem. This time, instead of the number $1/2$, we take a number ϵ, which may be arbitrarily small. Now, $M = 0$ since R is regular. So, we may take t_0 arbitrarily small. Then, for $a \in R \setminus E_2$,

$$\frac{1}{\pi}\int\int_R f^\#(z)g(z,a)^p\,dxdy \le t_0^p D^\#(f) + \frac{2\epsilon}{1-\epsilon}\int_{t_0}^\infty t^p e^{-2(t-t_0)}dt = A_1.$$

The theorem is proved, since $A_1 \to 0$ as $\epsilon, t_0 \to 0$. ∎

7 The Equality Condition of (3.1) and (4.1)

It is easy to verify that all equalities in (4.1) hold for $R = \Delta$, $a = 0$ and the function $f(z) = z$. More generally, by considering a rotation of the Riemann sphere, we may conclude that all equalities in (4.1) hold if R is a simply-connected hyperbolic Riemann surface, $a \in R$ and f is a conformal mapping of R onto a spherical disk such that $f(a)$ is the spherical center of the spherical disk. In this section we want to prove the following theorem.

Theorem 7.1 *Let R be a hyperbolic Riemann surface, $a \in R$, and let $f \in M(R)$ be not a constant. If R is obtained from a simply-connected hyperbolic Riemann surface R' by deleting at most a set of capacity zero and f is extended to a conformal mapping of R' onto a spherical disk such that $f(a)$ is the spherical center of the disk, then all equalities in* (4.1) *(for any $p > q > 0$) hold. Conversely, if $0 < D^\#(f) < 1$ and the equality in* (4.1) *holds for some $p > q > 0$, then the above condition, denoted by condition $(*)$, is satisfied.*

The notion of capacity is defined in terms of the logarithmic potential or transfinite diameter (*cf.* [1]). Let E be a compact set in the complex plane, let Ω be the complement of E that is connected, and let $g(z)$ be a Green's function of Ω whose asymptotic behavior at the infinity is of the form

$$g(z) = \log|z| + \gamma + \epsilon(z),$$

where γ is a constant and $\epsilon(z) \to 0$ as $z \to \infty$. It was proved that the capacity of E, denoted by $\operatorname{cap} E$, is equal to $e^{-\gamma}$, which assumes 0 if Ω possesses no Green's function. Also, it is known that if $\operatorname{cap} E = 0$, then E is totally disconnected and of Lebesque measure zero (dimension 2), and the complement of E is connected.

For a relatively closed set E in the unit disk Δ, we say that E is of capacity zero if any compact subset of E is of capacity zero. This definition is extended naturally to a hyperbolic Riemann surface R. We say that a closed set $E \subset R$ has capacity zero if for any point in R there exists a parameter disk $D \in R$ around this point such that $D \bigcap E$ is of capacity zero. The following sufficient and necessary condition for a set to be of capacity zero was shown by the authors in [4] for the case of the unit disk, and, by considering a universal covering, it can be proved for Riemann surfaces.

Lemma 7.2 *Let R be a hyperbolic Riemann surface and let $E \subset R$ be a closed set. If E has capacity zero, then $R \setminus E$ is connected and $g_{R\setminus E}(p,a) = g_R(p,a)$ for all $a, z \in R \setminus E$, where $g_{R\setminus E}(p,a)$ and $g_R(p,a)$ denote Green's functions of $R \setminus E$ and R, respectively. Conversely, if $R \setminus E$ is connected and the equality holds for some $a \in R \setminus E$ and all $p \in R \setminus E$, then E is of capacity zero.*

In order to prove Theorem 7.1 we will list a couple of lemmas and partly prove them.

Lemma 7.3 *Under the assumption of Lemma 6.1 with $t_0 = 0$, if the second equality in (3.1) holds for some $p > q > 0$, then*

$$\psi(t) = \frac{\pi\psi(0)}{\psi(0) + (\pi - \psi(0))e^{2t}} \quad \text{for } t \geq 0. \tag{7.1}$$

Conversely, if (7.1) holds, then all equalities in (3.1) hold for any $p > q > 0$.

Proof From the proof of Lemma 6.1, it is easy to see that the second equality in (3.1) for some $p > q > 0$ implies (7.1). Conversely, if (7.1) holds, then

$$\begin{aligned} h_p(0) &= -\int_0^\infty t^p\, d\psi(t) = 2\pi\psi(0)(\pi - \psi(0)) \int_0^\infty \frac{t^p e^{2t}\, dt}{(\psi(0) + (\pi - \psi(0))e^{2t})^2} \\ &= \frac{\lambda_p \psi(0)}{\pi - \psi(0)} \end{aligned}$$

holds for any $p \geq 0$. Thus, all equalities in (3.1) hold for any $p > q > 0$. ∎

The following lemma is direct consequence of Theorem 2.4 and Lemma 7.3.

Lemma 7.4 *Let R be a hyperbolic Riemann surface, $a \in R$ and let $f \in M(R)$ be not a constant with $D^\#(f) < 1$. If the second equality in (4.1) holds for some $p > q > 0$, then the function $\psi(t)$ defined in Theorem 2.4 satisfies (7.1). Conversely, (7.1) implies that all equalities in (4.1) hold for any $p > q > 0$.*

An equivalent formulation of the condition $(*)$ is the following: f is conformal mapping and $f(R)$ is obtained from a spherical disk of center $f(a)$ by deleting at most a set of capacity zero.

Proof of Theorem 7.1 First, assume that R, a, f satisfies the spherical Kobayashi condition. Without loss of generality, assume that $f(a) = 0$. Then, by the equivalent formulation of condition $(*)$, f is univalent and $R' = f(R) = \Delta_\rho \setminus E$, where Δ_ρ is the disk of center 0 and radius ρ, and E is a set of capacity 0. By Lemma 7.2,

$$g_R(p,a) = g_{R'}(f(p), 0) = \log \frac{\rho}{|f(p)|} \quad \text{for } p \in R.$$

Thus,

$$\psi(t) = \iint_{|w| < \rho e^{-t}} \frac{du\,dv}{(1 + |w|^2)^2} = \frac{\pi\rho^2}{\rho^2 + e^{2t}},$$

and so (7.1) holds. By Lemma 7.4, all equalities in (4.1) hold.

Now, assume that $0 < D^\#(f) < 1$ and the second equality in (4.1) holds for some $p > q > 0$. Then, by Lemma 7.4, the function $\psi(t)$ defined in Theorem 2.4 satisfies (7.1) and all equalities in (4.1) hold.

Let $G(p) = g(p,a) + ig^*(p,a)$ and $F(p) = \exp\{-G(p)\}$ for $p \in R$. Here $F(p)$ is a multiple-valued analytic function. However, $|F(p)| = \exp\{-g(p,a)\}$ is single-valued on R, $F(p)$ is single-valued near a, and $F'(a) \neq 0$. So, we may take $\zeta = F(p)$ as a local parameter around a and write $f(\zeta) = b_1\zeta + b_2\zeta^2 + \cdots$ if $|\zeta|$ is sufficiently small (without loss of generality we assume that $f(a) = 0$). On the other hand, $c(a) = 1$, with respect to the parameter ζ, since $g_R(p,a) = \log\frac{1}{F(p)} = \log\frac{1}{\zeta(p)}$ for $|\zeta(p)| < \delta$ according to the definition of $F(p)$, and $f^\#(a) = |b_1|$ with respect to the same parameter. Thus,

$$|b_1|^2 = \frac{f^\#(a)^2}{c(a)^2} = \frac{D^\#}{1 - D^\#} = \frac{\psi(0)}{\pi - \psi(0)}, \tag{7.2}$$

since all equalities in (4.1) hold. Thus, by (7.1) and (7.2),

$$\psi(t) = \pi \sum_{j=0}^{k} (-1)^j |b_1|^{2(j+1)} e^{-2(j+1)t} + O(e^{-2(k+1)t}), \quad t \to \infty. \tag{7.3}$$

We claim that $b_j = 0$ for $j \geq 2$. Assume to the contrary that $b_2 = \cdots = b_{k-1} = 0$ and $b_k \neq 0$ with $k \geq 2$. Then

$$|f(\zeta)|^2 = |b_1|^2|\zeta|^2 + S_1, \quad |f'(\zeta)|^2 = |b_1|^2 + k^2|b_k|^2|\zeta|^{2(k-1)} + S_2,$$

where

$$S_1 = \sum_{j=k}^{2(k-1)} \mathrm{Re}(b_1\overline{b}_j\zeta\overline{\zeta}^j) + O(|\zeta|^{2k}),$$

$$S_2 = \sum_{j=k}^{2k} j\,\mathrm{Re}(b_1\overline{b}_j\zeta\overline{\zeta}^{j-1}) + k(k+1)\mathrm{Re}(b_k\overline{b}_{k+1}\zeta^{k-1}\overline{\zeta}^k) + O(|\zeta|^{2k}).$$

Consequently,

$$\frac{1}{(1+|f(\zeta)|^2)^2} = \sum_{j=0}^{k-1} (j+1)(-1)^j|b_1|^{2j}|\zeta|^{2j} + \sum_{j=1}^{k-1} j(j+1)(-1)^j|b_1|^{2(j-1)}|\zeta|^{2(j-1)} \sum_{j=k}^{2(k-1)} \mathrm{Re}(b_1\overline{b}_j\zeta\overline{\zeta}^j) + O(|\zeta|^{2k})$$

and

$$f^\#(\zeta)^2 = \sum_{j=0}^{k-1} (-1)^j(j+1)|b_1|^{2(j+1)}|\zeta|^{2j} + k^2|b_k|^2|\zeta|^{2(k-1)} + S_3,$$

where

$$S_3 = \sum_{j\neq l} \alpha_{j,l}\zeta^j\overline{\zeta}^l + O(|\zeta|^{2k}).$$

Thus,

$$\psi(t) = \sum_{j=0}^{k} (-1)^j|b_1|^{2(j+1)}e^{-2(j+1)t} + k|b_k|^2e^{-2kt} + O(e^{-2(k+1)t}), \quad t \to \infty. \tag{7.4}$$

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Comparing (7.3) with (7.4) gives $b_k = 0$, a contradiction. This shows that $f(p) = b_1 F(p)$ for p close to a and, consequently, $f(p)$ is equal to $b_1 F(p)$ on R identically and $F(p)$ is actually single-valued on R.

Now we want to prove that f is univalent on R. $R' = f(R)$ is contained in the disk $\Delta_\rho = \{w : |w| < \rho\}$ of center 0 and radius $|b_1|$, since $f(p) = b_1 F(p)$ and $|F(p)| < 1$ for $p \in R$. Let

$$g_{R'}(w,0) = \log \frac{1}{|w|} + u(w), \quad w \in R', \tag{7.5}$$

where u is a harmonic function on R'. Applying Theorem 4.1 to the surface R' and the identity function $h(w) = w$, the point $w = 0$ and parameter w, we have

$$e^{2u(0)} \le \frac{D^\#(h)}{1 - D^\#(h)} = \frac{A^\#(f)}{1 - A^\#(f)}. \tag{7.6}$$

According the definition of F, we have

$$g_R(p,a) = \log \frac{1}{|F(p)|} = \log \frac{1}{|f(p)|} + \log |b_1|, \quad p \in R.$$

Define $g(p) = g_{R'}(f(p), 0)$ for $p \in R$. Then, by (7.5),

$$g(p) = \log \frac{1}{|f(p)|} + u(f(p)), \quad p \in R.$$

We may take $w = f(p)$ as a local parameter around a, since $f(p)$ is local univalent at a. Then $|f(p)|$ in the above equality for $g(p)$ can be replaced by $|w(p)|$ for p close to a. Then $g(p)$ is a positive harmonic function on $R \setminus \{a\}$ and $g(p) = \log \frac{1}{|w(p)|} + O(1)$ as $p \to a$, where w is a local parameter around a with $w(a) = 0$. It is known [1] that the Green function $g_R(p,a)$ is the smallest one among functions with these two properties. Thus, $g_R(p,a) \le g(p)$ for $p \in R$ and, consequently,

$$\log |b_1| \le u(f(p)), \quad p \in R \setminus \{a\}. \tag{7.7}$$

Letting $p \to a$ gives

$$|b_1|^2 \le e^{2u(0)}. \tag{7.8}$$

Now, from (7.2), (7.6), and (7.8), we conclude that $|b_1| = e^{u(0)}$ and $A^\#(f) = D^\#(f)$, which implies the univalence of f.

It follows from (7.7) and the equality $|b_1| = e^{u(0)}$ that $u(w) = \log |b_1|$ in a neighbourhood of the origin and, consequently, for $w \in R'$. This shows that $g_{R'}(w,0) = g_{\Delta_\rho}(w,0)$ for $w \in R'$. By Lemma 7.2, R, f, a satisfies the condition $(*)$ (the equivalent formulation). The proof is complete. ∎

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