ON THE EXISTENCE OF RIGID ARCS

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By an arc we mean a nondegenerate simply ordered set which is compact and connected in its order topology. A space X is *rigid* if the only homeomorphism of X onto itself is the identity map. Examples of rigid totally disconnected compact ordered spaces may be found in [1], [2], and [3]. It is the purpose of this note to prove the existence of an arc such that no two of its subarcs are homeomorphic. The proof makes use of arcs of large cardinality and the technique of inverse limit spaces.

In what follows, we use the convention that each ordinal is the set of all ordinals preceding it and that the cardinals are certain ones of the ordinals. If M is a set, the cardinal of M is denoted by $|M|$. ω denotes the least infinite cardinal, and Ω denotes the least uncountable cardinal. For each regular cardinal $\lambda > \omega$, let L_λ denote the cross product $\lambda \times [0, 1)$ ordered lexicographically and let $A_\lambda = L_\lambda \cup \{\lambda\}$ with the order on L_λ extended to A_λ in such a way that each point of L_λ is less than λ. L_Ω has sometimes been called the *long line*. We call any arc order isomorphic to A_λ a *long arc of order* λ. If α is an infinite regular cardinal and X is a set, then by an *α-sequence* in X we mean a map of α into X. If X is a topological space and x is a point of X, then the *weight* of X at x, denoted by $w(x)$, is the least cardinal α such that X has a base at x of cardinal α. Note that if X is an arc, x is an interior point of X, and there exist an increasing α-sequence converging to x and a decreasing β-sequence converging to x, then $w(x) = \sup\{\alpha, \beta\}$. A long arc X also has the important property that for each point x of X different from the right end point, there exists a decreasing ω-sequence converging to x. We shall need the following lemma, which is probably well known.

LEMMA. *If $\{M_i, f_i\}$ is an inverse limit system, each M_i is an arc, each f_i is onto and order preserving, M is the inverse limit space, and M is ordered lexicographically, then M is simply ordered and its order topology is the same as its inverse limit topology. Furthermore, for each i, the projection of M onto M_i is order preserving.*

The existence of an arc such that no two of its subarcs are homeomorphic follows immediately from the following theorem together with the fact that a homeomorphism preserves weight.

THEOREM. *There exist an arc M and a cardinal $\Lambda > \Omega$ such that* (1) *for each point x of M, $w(x) < \Lambda$,* (2) *for each regular cardinal α such that $\Omega \leqq \alpha < \Lambda$, there is one and only one point x of M such that $w(x) = \alpha$, and* (3) *the set of all points x of M such that $w(x) \geqq \Omega$ is totally disconnected and dense in M.*

PROOF. We describe inductively an inverse limit system $\{M_i, f_i\}$ of arcs and maps. Let $M_0 = A_\Omega$, and let $\mathscr{S}_0 = \{M_0\}$. Suppose M_i is an arc and $\mathscr{S}_i$ is a collection of disjoint long subarcs of M_i such that $\cup \mathscr{S}_i$ is dense in M_i. Let g be a map with domain M_i such that (1) if x is an interior point of some arc in $\mathscr{S}_i$, then $g(x)$ is a long arc, (2) if x is an end point of some arc in $\mathscr{S}_i$ or a point oi $M_i - \cup \mathscr{S}_i$, then $g(x) = \{x\}$, (3) if x and y are distinct points of M_i, then $g(x) \cap g(y) = \emptyset$, (4) no two arcs in the range $R(g)$ of g have the same order, and (5) if α is the least regular cardinal γ such that each arc in $\mathscr{S}_i$ has order less than γ, then for some regular cardinal $\beta > \alpha$, the set of all orders of the arcs in $R(g)$ is the half-open interval $[\alpha, \beta)$ of regular cardinals. Let $M_{i+1} = \cup R(g)$, and let $\mathscr{S}_{i+1}$ be the set of all arcs in $R(g)$. M_{i+1} is ordered as follows. If each of p and q is a point of M_{i+1}, then $p < q$ if and only if (1) for some x and y in M_i, $p \in g(x)$, $q \in g(y)$, and $x < y$ in M_i or (2) p and q belong to the same arc $g(x)$ and either i is even and $p > q$ in $g(x)$, or i is odd and $p < q$ in $g(x)$. Clearly, M_{i+1} is an arc, and $\cup \mathscr{S}_{i+1}$ is dense in M_{i+1}. Let f_{i+1} be the map of M_{i+1} onto M defined as follows. For each point p in M_{i+1}, let x be the point in M_i such that $p \in g(x)$ and let $f_{i+1}(p) = x$. Thus M_{i+1} and f_{i+1} have been obtained from M_i by exploding certain points of M_i into long arcs and defining f_{i+1} in the natural way. Note that the long arcs in M_{i+1} are all oriented the same way but that the orientation is opposite to that of the long arcs in M_i. Let M denote the inverse limit space of the inverse limit system $\{M_i, f_i\}$, and for each i, let p_i denote the projection of M onto M_i. M is a continuum, and it follows from the lemma that M is an arc.

Now let $a = a_0, a_1, \cdots$ be any point of M. We shall determine the weight of M at a. Suppose that for some even integer i, a_i is the left end point of some long arc A in $\mathscr{S}_i$. There is a decreasing ω-sequence $x_0, x_1, \cdots$ in M_i of interior points of A converging to a_i. For each n, let r_n be a point of the arc $p_i^{-1}(x_n)$. $r_0, r_1, \cdots$ is then a decreasing ω-sequence in M and hence converges to some point $b \geqq a$. But if $b > a$, then $p_i^{-1}(a)$ is nondegenerate, which is a contradiction. Therefore $r_0, r_1, \cdots$ converges to a. If $i = 0$, then $a = 0$ and hence a is the left end point of M, so that $w(a) = \omega$. Assume that $i > 0$. a_{i-1} is an interior point of some long arc A_{i-1} in $\mathscr{S}_{i-1}$ with order α. Now the original order of the points in A_{i-1} has been reversed in M_{i-1}. Hence there exists an increasing ω-sequence $y_0, y_1, \cdots$ in M_{i-1} of interior points of A_{i-1} converging to a_{i-1}. For each n, let $s_n \in p_{i-1}^{-1}(y_n)$.

It follows as before that $s_0, s_1, \cdots$ converges to a. Therefore $w(a) = \omega$. Now suppose that for some odd integer i, a_i is the left end point of some long arc A_i in $\mathscr{S}_i$ with order α. There is a decreasing α-sequence $x_0, x_1, \cdots$ in M_i of interior points of A converging to a_i. For each n, let $r_n \in p_i^{-1}(x_n)$. It follows that $r_0, r_1, \cdots$ is a decreasing α-sequence in M converging to a. a_{i-1} is an interior point of some long arc A_{i-1} in $\mathscr{S}_{i-1}$ with order γ. For some cardinal β, there is an increasing β-sequence $y_0, y_1, \cdots$ in M_{i-1} of interior points of A_{i-1} converging to a_{i-1}. For each n, let $s_n \in p_{i-1}(y_n)$. As before, $s_0, s_1, \cdots$ is an increasing β-sequence converging to a. Now $\beta < \gamma < \alpha$. Therefore $w(a) = \alpha$. Similar arguments will show that if i is odd and a is the right end point of a long arc in $\mathscr{S}_i$, then $w(a) = \omega$ and if i is even and a is the right end point of a long arc in $\mathscr{S}_i$ with order α, then $w(a) = \alpha$. Finally, suppose that for each i, a is not an end point of any arc in $\mathscr{S}_i$. For each i, there exists an arc A_i in $\mathscr{S}_i$ such that a_i is contained in the interior U_i of A_i. Then for each i, $p_i^{-1}(U_i)$ is an open set in M, and $\bigcap_i p_i^{-1}(U_i) = \{a\}$. Therefore $w(a) = \omega$. It now follows that if α is a cardinal such that for some i, there is an arc in $\mathscr{S}_i$ of order α, then there is one and only one point a in M such that $w(a) = \alpha$.

Let c denote the cardinal of the real numbers. Now $|M_0| = c$, and the only long arc in $\mathscr{S}_0$ has order $\aleph_1$. Hence the long arcs in $\mathscr{S}_1$ have orders from $\aleph_2$ up to $\aleph_c$. Then $|M_1| = \aleph_c$, and hence the long arcs in $\mathscr{S}_2$ have orders from $\aleph_c$ up to $\aleph_{\aleph_c}$. Therefore if $\Lambda = \sup\{c, \aleph_c, \aleph_{\aleph_c}, \cdots\}$, then the long arcs in $\bigcup_i \mathscr{S}_i$ have all orders from Ω up to Λ. Conditions (1) and (2) of the conclusion of the theorem now follow. In order to prove condition (3), suppose a and b are two points of M such that $a < b$. Let i be an integer such that $a_i < b_i$. There exists a point c_i of M_i such that $a_i < c_i < b_i$ and c_i is an interior point of some arc in $\mathscr{S}_i$. For each $n > i$, let c_n be an interior point of $f_{n-1}^{-1}(c_{n-1})$, and for each $n < i$, let $c_n = f_{n+1}(c_{n+1})$. Then if $c = c_0, c_1, \cdots, a < c < b$ and $w(c) = \omega$. Now let d_{i+1} be the left end point of $f_{i+1}^{-1}(c_i)$ if i is even and the right end point of $f_{i+1}^{-1}(c_i)$ if i is odd. For each $n > i+1$, let $\{d_n\} = f_{n-1}^{-1}(d_{n-1})$, and for each $n < i+1$, let $d_n = f_{n+1}(d_{n+1})$. Then if $d = d_0, d_1, \cdots, a < d < b$ and $w(d) > \omega$. This completes the proof.

COROLLARY. *There exists a nondegenerate rigid totally disconnected compact ordered space.*

PROOF. Let M be the arc of the above theorem, let H be the set of all interior points x of M such that $w(x) = \omega$, and let $K = M - H$. Let M' be the ordered space obtained from M by exploding each point of H into an ordered pair of points, and let f be the natural map of M' onto M. Now $f^{-1}(K)$ is dense in M', and if x and y are distinct points of $f^{-1}(K)$, then $w(x) \neq w(y)$. It follows that the only homemorphism of M' onto itself is the identity map.

References

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