Glasgow Math. J. **60** (2018) 111–121. © Glasgow Mathematical Journal Trust 2017.
doi:10.1017/S0017089516000598.

FROM COMPLETE TO PARTIAL FLAGS IN GEOMETRIC EXTENSION ALGEBRAS

JULIA SAUTER
Faculty of Mathematics, Bielefeld University, PO Box 100 131, 33501 Bielefeld, Germany
e-mail: jsauter@math.uni-bielefeld.de

(Received 3 October 2015; revised 14 June 2016; accepted 12 July 2016;
first published online 13 March 2017)

Abstract. A geometric extension algebra is an extension algebra of a semi-simple perverse sheaf (allowing shifts), e.g., a push-forward of the constant sheaf under a projective map. Particular nice situations arise for collapsings of homogeneous vector bundles over homogeneous spaces. In this paper, we study the relationship between partial flag and complete flag cases. Our main result is that the locally finite modules over the geometric extension algebras are related by a recollement. As examples, we investigate parabolic affine nil Hecke algebras, geometric extension algebras associated with parabolic Springer maps and an example of Reineke of a parabolic quiver-graded Hecke algebra.

2010 *Mathematics Subject Classification.* Primary 14F43; Secondary 20C08, 14M99, 14F05.

1. Introduction. A geometric extension algebra is an algebra $\mathrm{Ext}^*(\mathcal{L}, \mathcal{L}) := \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}(\mathcal{L}, \mathcal{L}[n])$, where $\mathcal{L}$ is a direct sum of shifts of simple equivariant perverse sheaves on a complex algebraic variety equipped with a suitable group action. For example, $\mathcal{L}$ might be a push-forward of the constant sheaf under an equivariant projective map of varieties. Examples of such algebras include affine nil Hecke algebras, skew group algebras of polynomial rings with a Weyl group action and graded parts of quiver Hecke algebras, see for example [**10**]. All of these algebras arise from collapsing of homogeneous vector bundles over complete flag varieties, and in this case it is often possible to find explicit generators and relations using methods of Varagnolo and Vasserot, see for example [**13**, **12**, **9**]. However, there are also cases of interest associated with partial flag varieties, to which these methods do not apply. This includes quiver Schur algebras in [**11**], where calculations can be made via diagram calculus.

In this paper, we study geometric extension algebras $\mathcal{Z}^P$ arising from partial flag varieties. Associated with such an algebra we construct another geometric extension algebra $\mathcal{Z}^B$ arising from complete flags and we show that there is a close relation between these two algebras. More precisely, we show there is an idempotent element $e_P \in \mathcal{Z}^B$ such that $e_P\mathcal{Z}^B e_P \cong \mathcal{Z}^P$. This implies that the locally finite modules over $\mathcal{Z}^B$ and $\mathcal{Z}^P$ are related by a recollement (obtained from the idempotent e_P). Our result applies to arbitrary reductive groups and more general collapsings than studied in loc. cit. As examples, we study parabolic versions of affine nil Hecke algebras, examples from Springer theory and an example of Reineke.

2. A recollement relating parabolic and Borel cases.

DEFINITION 1. Let $(G, P_i, V, F_i)_{i\in I}$ be a tuple with G a reductive group with parabolic subgroups $P_i, i \in I$ (where I is some finite set) such that $\bigcap P_i$ contain a maximal torus T, $\dim P_i = \dim P_j$ for all $i, j \in I$ and V a G-representation with P_i-subrepresentations $F_i \subset V$. We set $E_i^P := G \times^{P_i} F_i$, $E^P := \bigsqcup E_i^P, \pi^P : E^P \to V, \overline{(g, f)} \mapsto gf$. Choose $T \subset B_i \subset P_i, i \in I$ Borel subgroups of G (where $T \subset \bigcap_{i\in I} P_i$) and consider F_i as B_i-representation, and then $(G, B_i, V, F_i)_{i\in I}$ can be used to define E_i^B, E^B, π^B, Z^B analogously.

For any complex algebraic variety X with an action of an algebraic group A, we will denote by $D_A^b(X)$ the A-equivariant derived category introduced by Bernstein and Lunts [2]. These categories carry a six functor calculus, and we will denote the right- or left-derived functors with the same symbol as the functor. They have a dualizing object $\mathbf{D}_X$ used to define a duality called Verdier duality.

We define $\mathrm{Ext}^*_{D_A^b(X)}(U, V) := \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{D_A^b(X)}(U, V[n])$ for $U, V \in D_G^b(X)$, and $\mathcal{C}^B := \bigoplus_{i\in I} \mathbb{C}_{E_i^B}[\dim E_i^B]$ and $\mathcal{C}^P$ analogously and we set

$$\mathcal{Z}^B = \mathrm{Ext}^*_{D_b^A(V)}(\pi_*^B \mathcal{C}^B, \pi_*^B \mathcal{C}^B),$$
$$\mathcal{Z}^P = \mathrm{Ext}^*_{D_b^A(V)}(\pi_*^P \mathcal{C}^P, \pi_*^P \mathcal{C}^P).$$

We remark that the shifts in the grading of the constant sheaves ensures $\mathbf{D}_{E^B}(\mathcal{C}^B) = \mathcal{C}^B$ and $\mathbf{D}_{E^P}(\mathcal{C}^P) = \mathcal{C}^P$. This implies that both graded algebras $\mathcal{Z}^B, \mathcal{Z}^P$ have an anti-involution given by applying Verdier duality. We denote by $* -$ grmod the category of finitely generated graded modules over a $\mathbb{Z}$-graded algebra $*$. Homomorphisms are given by homogeneous maps of degree zero. Since $\mathcal{Z}^B$ and $\mathcal{Z}^P$ are finitely generated modules over the commutative noetherian ring $H_A^*(pt)$ (see [10]), they are noetherian and the categories of graded modules over them are abelian.

THEOREM 2.1. *There is a recollement of abelian categories given by an idempotent element* e_P,

$$\mathcal{Z}_A^B/\mathcal{Z}_A^B e_P \mathcal{Z}_A^B - \mathrm{grmod} \;\; \begin{smallmatrix}\longleftarrow\\ \longrightarrow \\ \longleftarrow\end{smallmatrix} \;\; \mathcal{Z}_A^B - \mathrm{grmod} \;\; \begin{smallmatrix}\longleftarrow\\ \xrightarrow{e_P} \\ \longleftarrow\end{smallmatrix} \;\; \mathcal{Z}_A^P - \mathrm{grmod}.$$

In fact, given any graded algebra B and an idempotent $e \in B$ in degree 0, such that B is a module-finite algebra over a commutative noetherian ring, we get that the standard recollement for the idempotent e restricts to one of the finitely generated graded modules (using if X and Y are finitely generated Z-modules, then so is $\mathrm{Hom}_Z(X, Y)$).

The key tool to find the idempotent e_P in the theorem is a (topological) operation of the Weyl group.

2.1. The Weyl group operation. Let $T \subset B \subset P \subset G$ with T be a maximal torus and B be a Borel in a reductive group G over the $\mathbb{C}$. We choose a maximal compact subgroup $K \subset G$, then $T' := T \cap K$ is a compact torus in K and the inclusion $K \to G$ induces a homoemorphism $K/T' \to G/B$, this is long well known, see, e.g., [1, Section 2]. The Weyl group W associated with G and T coincides with the Weyl group associated with K and T'. The group W operates on K/T' via $nT' \cdot kT' = kn^{-1}T', n \in N_K(T'), k \in K$ without fixpoints. Similarly, let $W_P \subset W$ be the Weyl group of a Levi subgroup $L \subset P$

and T, it operates on $P/B \cong L/(B \cap L) \cong K'/T'$ for a maximal compact subgroup $K' \subset L$. Now given, a P-representation F, then $E^B := G \times^B F \cong K \times^{T'} F$ carries a W-operation and for the map $\alpha\colon E^B \to E^P := G \times^P F, \overline{(g,f)}^B \mapsto \overline{(g,f)}^P$ one has $\alpha \circ w = \alpha$ for all $w \in W_P$. This induces an operation of W_P on $\alpha_*(\mathcal{F})$ for any sheaf $\mathcal{F}$ and therefore we have an induced functor:

$$\alpha_*()^{W_P}\colon D^b_A(E^B) \to D^b_A(E^P).$$

LEMMA 1. *In the notation from before, for the constant sheaf* $\underline{\mathbb{C}}$ *on* E^P *the adjunction map* $\underline{\mathbb{C}} \to \alpha_*\alpha^*\underline{\mathbb{C}}$ *is a monomorphism in* $D^b_A(E^P)$. *Furthermore, it factorizes over an isomorphism* $\underline{\mathbb{C}} \to (\alpha_*\underline{\mathbb{C}})^{W_P}$, *where* W_P *is the Weyl group of a Levi subgroup in* P.

Proof. For any variety, we write $X_A := X \times^A EA$, where EA is a contractible space with a free A-operation. We denote $\alpha := \alpha_A\colon (E^B)_A \to (E^P)_A$—the associated map. It is smooth and proper submersion with fibres all isomorphic to P/B. The decomposition theorem (in the more specific version for a proper submersion, see [**5**, item (3) after remark 1.6.2]) implies

$$\alpha_*\underline{\mathbb{C}} = \bigoplus_{i \in \mathbb{Z}} R^i\alpha_*\underline{\mathbb{C}}[-i].$$

Since $R^i\alpha_*\underline{\mathbb{C}}$ is the sheaf associated with the presheaf,

$$U \mapsto H^i(\alpha^{-1}(U)),$$

this implies $(R^i\alpha_*\underline{\mathbb{C}})_x = H^i(\alpha^{-1}(x)) \cong H^i(P/B)$ for all $x \in (E^P)_A$. Therefore, $R^i\alpha_*\underline{\mathbb{C}}$ is a local system on $(E^P)_A$ and since $\pi_1((E^P)_A, x_0)$ (for any $x_0 \in (E^P)_A$) is trivial, it is the constant local system. Then, one has $\alpha_*\underline{\mathbb{C}} = \bigoplus_{w \in W_P} \underline{\mathbb{C}}[-2\ell(w)]$ because $H^*(P/B) = H^*(L/(L \cap B) = \mathbb{C}[\mathfrak{t}]/I_{W_P}$, where the last isomorphism is graded algebras and as W_P-representations by the Borel isomorphism. But since $(\mathbb{C}[\mathfrak{t}]/I_{W_P})^{W_P} = \mathbb{C}$ in degree 0, one has $(\alpha_*\underline{\mathbb{C}})^{W_P} \cong \underline{\mathbb{C}}$. Furthermore, it is easy to see that the unit of the adjunction is a monomorphism (since α is locally trivial). By taking the trivial W_P-operation on $\underline{\mathbb{C}}$, we can make the unit of the adjunction a W_P-linear map (because the map is locally trivial and W_P operates only on the fibre), and then taking W_P-invariants prove the lemma. □

Now, let us come back to the setup from definition 1. Let $i \in I$. Consider the following commutative triangle:

$$\begin{array}{ccc} E_i^B = G \times^{B_i} F_i & \xrightarrow{\alpha_i} & E_i^P = G \times^{P_i} F_i\,, \\ & {\scriptstyle \pi_i^B} \searrow \quad \swarrow {\scriptstyle \pi_i^P} & \\ & V & \end{array}$$

where $\pi_i^B = \pi^B|_{E_i^B}$, $\pi_i^P = \pi^P|_{E_i^P}$. We denote by W_i the Weyl group of a Levi subgroup in P_i and the torus T. We define two morphisms in $D^b_A(V)$ using the isomorphisms from Lemma 1.

$$\mathrm{Inc}_i\colon (\pi_i^P)_*\underline{\mathbb{C}}_{E_i^P} \cong (\pi_i^P)_*[(\alpha_i)_*\alpha_i^*\underline{\mathbb{C}}_{E_i^P}^{W_i}] \hookrightarrow (\pi_i^P)_*(\alpha_i)_*\alpha_i^*\underline{\mathbb{C}}_{E_i^P} = (\pi_i^B)_*\underline{\mathbb{C}}_{E_i^B},$$
$$\mathrm{Av}_i\colon (\pi_i^B)_*\underline{\mathbb{C}}_{E_i^B} = (\pi_i^P)_*(\alpha_i)_*\alpha_i^*\underline{\mathbb{C}}_{E_i^P} \twoheadrightarrow (\pi_i^P)_*[(\alpha_i)_*\alpha_i^*\underline{\mathbb{C}}_{E_i^P}^{W_i}] = (\pi_i^P)_*\underline{\mathbb{C}}_{E_i^P}.$$

The second map is locally the averaging map (or Reynolds operator) for the finite group W_i. One has $\mathrm{Av}_i \circ \mathrm{Inc}_i = \mathrm{id}$ and $\mathrm{Inc}_i \circ \mathrm{Av}_i =: e_i$ is an idempotent endomorphism. Now let $s := \dim P_i - \dim B_i$ for one (and all) $i \in I$ and observe $\pi_*^P \mathcal{C}^P[s] = \bigoplus_{i \in I} (\pi_i^P)_* \underline{\mathbb{C}}_{E_i^P}[\dim E_i^B]$, we define

$$\mathrm{Inc} := \bigoplus_{i \in I} \mathrm{Inc}_i[\dim E_i^B] \colon \pi_*^P \mathcal{C}^P[s] \to \pi_*^B \mathcal{C}^B,$$
$$\mathrm{Av} := \bigoplus_{i \in I} \mathrm{Av}_i[\dim E_i^B] \colon \pi_*^B \mathcal{C}^B \to \pi_*^P \mathcal{C}^P[s],$$

and we observe that $\mathrm{Av} \circ \mathrm{Inc} = \mathrm{id}$ and $\mathrm{Inc} \circ \mathrm{Av} = e_P$ with

$$e_P := (e_i)_{i \in I} \in \bigoplus_{i \in I} \mathrm{End}_{D_A^b(V)}((\pi_i^B)_* \underline{\mathbb{C}}) = \bigoplus_{i \in I} \mathrm{End}_{D_A^b(V)}((\pi_i^B)_* \underline{\mathbb{C}}[\dim E_i^B]) \subset (\mathcal{Z}^B)_0.$$

Then, we define a map

$$\theta_B^P \colon \mathcal{Z}^P = \mathrm{Ext}^*(\pi_*^P \mathcal{C}^P[s], \pi_*^P \mathcal{C}^P[s]) \to \mathcal{Z}^B,$$

mapping an element f in degree n to $\mathrm{Inc}[n] \circ f \circ \mathrm{Av}$. It is easy to see that this map preserves degrees, $\theta_B^P(1) = e_P$ and products are mapped to products since $\mathrm{Av} \circ \mathrm{Inc} = \mathrm{id}$.

LEMMA 2. *The map θ_B^P is injective and induces an isomorphism of graded algebras:*

$$\mathcal{Z}^P \cong \mathrm{Im}(\theta_B^P) = e_P \mathcal{Z}^B e_P.$$

Proof. Assume $\mathrm{Inc}[n] \circ f \circ \mathrm{Av} = 0$, then $\mathrm{Inc}[n]$ is a monomorphism $f \circ \mathrm{Av} = 0$. But then $f = f \circ (\mathrm{Av} \circ \mathrm{Inc}) = 0$. The rest is clear. □

Now, the lower recollement is the standard recollement induced by the idempotent element. The map e_P is left multiplication with the idempotent e_P. This completes the proof of Theorem 2.1.

Let $W_P := \prod_{i \in I} W_i$, then W_P operates on $\pi_*^B \mathcal{C}^B$ with $(\pi_*^B \mathcal{C}^B)^{W_P} = \pi_*^P \mathcal{C}^P[s]$, see before. This induces a $W_P \times W_P$-operation on $\mathcal{Z}^B$, for f of degree n, $v, w \in W_P$ we set

$$(v, w) \cdot f := v[n] \circ f \circ w^{-1}.$$

This operation is $H_A^*(pt)$-linear but does not preserve products.

PROPOSITION 1. $\theta_B^P(\mathcal{Z}^P) = (\mathcal{Z}^B)^{W_P \times W_P}$.

Proof. Let $f \in \mathcal{Z}^P$ of degree n, then $\theta_B^P(f) = \mathrm{Inc}[n] \circ f \circ \mathrm{Av}$ is $W_P \times W_P$-invariant; therefore, $\Theta_B^P(\mathcal{Z}^P) \subset (\mathcal{Z}^B)^{W_P \times W_P}$. On the other hand, given $h \in \mathcal{Z}^B$ of degree n with $v[n] \circ h \circ w^{-1} = h$, $v, w \in W_P$, one has $h \circ (\mathrm{Inc} \circ \mathrm{Av}) = (\mathrm{Inc} \circ \mathrm{Av})[n] \circ h$, which implies $h((\pi_*^B \mathcal{C}^B)^{W_P}) \subset (\pi_*^B \mathcal{C}^B)^{W_P}[n]$; therefore, restriction induces an element $\overline{h} \in \mathcal{Z}^P$ and by definition $h = \mathrm{Inc} \circ \overline{h} \circ \mathrm{Av}$. □

3. The hypercohomology functor. Let X be a complex variety with the action of an algebraic group A. We set
$H_A^*(X) := H_A^*(X, \mathbb{C}) := \mathrm{Ext}^*_{D_A^b(X)}(\underline{\mathbb{C}}_X, \underline{\mathbb{C}}_X)$ for *A-equivariant cohomology* and

$H_*^A(X) := H_*^A(X, \mathbb{C}) := \mathrm{Ext}^{-*}_{D^b_A(X)}(\mathbb{C}_X, \mathbf{D}_X)$ for *A-equivariant Borel–Moore homology* with complex coefficients, respectively.

We associate to the data from before the Steinberg variety

$$Z^P := \bigsqcup_{i,j\in I} \underbrace{E_i^P \times_V E_j^P}_{Z_{i,j}} \text{ where } E_i^P := (G \times^{P_i} F_i),\, i \in I,$$

and call $H_*^A(Z^P)$, $A \in \{pt, T, G\}$ the Steinberg algebra associated with the data (where the product is given by a convolution construction defined by [**4**, Section 2.7]). It is a graded algebra with respect to

$$H^A_{[p]}(Z^P) := \bigoplus_{i,j\in I} H^A_{r_i+r_j-p}(Z^P_{i,j}), \text{ where } r_i := \dim_{\mathbb{C}} E_i^P;$$

we write $H^A_{[*]}(Z)$ to indicate this grading. We recall the following result.

THEOREM 3.1 ([**4**, chapter 8]). *Let $A \in \{pt, T, G\}$ we write $\pi_i\colon E_i^P \to V, \overline{(g,f)} \mapsto gf$ and there is an isomorphism of graded $\mathbb{C}$-algebras:*

$$H^A_{[*]}(Z^P) \to \mathcal{Z}^P.$$

For every $\mathcal{F}$ in $D^b_A(E^P)$, the hypercohomology $\mathbb{H}^*_A(E^P, \mathcal{F})$ is naturally a bimodule over the equivariant cohomology ring $H^*_A(E^P)$. This is (a generalization of) Sörgel's bifunctor which leads to the definition of Sörgel bimodules. We think that the (lesser known) operation of the Steinberg algebra should also be of interest.

THEOREM 3.2. *For every $\mathcal{F} \in D^b_A(E^P)$, its hypercohomology $\mathbb{H}^*_A(E^P, \mathcal{F})$ is a graded left and right module over $\mathcal{Z}^P$ with finite-dimensional graded parts. This module structure is natural in $\mathcal{F}$.*

The (ungraded) version of the convolution operation on hypercohomology groups of objects in the equivariant derived category can adapted from the main result of [**7**], the grading follows from the definition of the convolution operation, we explain it shortly. We shorten the notation here $E^P =: E$, $Z^P =: Z$. Recall $Z = E \times_V E$, set $E = E \times pt$. Consider the A-equivariant maps

$$\begin{array}{ccccc} & & E_i \times E_j \times pt & & \\ & \swarrow^{p_{12}} & \downarrow^{p_{23}} & \searrow^{p_{13}} & \\ E_i \times E_j & & E_j \times pt & & E_i \times pt \end{array}$$

since $\iota\colon Z_{ij} \subset E_i \times E_j$ is a closed embedding one has $\mathbb{H}^*_A(Z_{i,j}, D_{Z_{ij}}) = \mathbb{H}^*_A(E_i \times E_j, \iota_*(D))$, where D is the dualizing sheaf and we define

$$\mathbb{H}_A^{p-r_i-r_j}(Z_{i,j}, D) \otimes \mathbb{H}_A^{k+r_t}(E_t, \mathcal{F}|_{E_t}) \xrightarrow{\delta_{j,t}(p_{13})_*(p_{12}^*[-]\otimes p_{23}^*[-])} \mathbb{H}_A^{k+r_i}(E_i, \mathcal{F}|_{E_i})$$

using the operators explained in [**7**, Sections 2 and 4]. The proof of the main result in [**7**] adapts straight forward to this situation and proves that this map provides the left operation (for the right operation permute the factors). If we set $\mathbb{H}_A^{[t]}(E, \mathcal{F}) := \bigoplus_{i\in I} \mathbb{H}_A^{t+r_i}(E_i, \mathcal{F}|_{E_i})$, this gives it the structure of a graded module.

For $P_i = B_i$ for all $i \in I$, we write $Z^B, E^B, \dots$ instead of $Z^P, E^P, \dots$. We observe that $W_P \times W_P$ operates on Z^B by homeomorphisms and this corresponds under the automorphism in Theorem 3.1 to the $W_P \times W_P$-operation from the previous section. Furthermore, if we restrict the operation to $W_P \times \{1\}$ and consider the W_P-operation on E^B, then the multiplication above commutes with this W_P-operation on $\mathbb{H}_A^{k+r_i}(E_i^B, \mathcal{F}|_{E_i}) = \mathbb{H}_A^{k+r_i}(E_i^P, (\alpha_i)_*\mathcal{F}|_{E_i})$, i.e., for $x \in \mathbb{H}_A^{p-r_i-r_j}(Z_{i,j}, D), f \in \mathbb{H}_A^{k+r_t}(E_t, \mathcal{F}|_{E_t})$, $w \in W_i$, we have

$$w(xf) = ((w,1)x)f \in \mathbb{H}_A^{k+r_i}(E_i, \mathcal{F}|_{E_i}),$$

and, in particular, $((\mathcal{Z}^B)^{W_P \times \{1\}})\mathbb{H}_A^*(E^B, \mathcal{F}) = \mathbb{H}_A^*(E^P, \alpha_*(\mathcal{F}))^{W_P}$.

Remark. So, taking hypercohomology provides us with functors:

$$\begin{aligned} \mathbb{H}^* &: D_A^b(E^B) \to \mathcal{Z}^B-\text{grmod}, \\ \mathbb{H}^* &: D_A^b(E^P) \to \mathcal{Z}^P-\text{grmod}. \end{aligned}$$

The complex $\underline{\mathbb{C}}$ maps to the equivariant comohomology $H_A^{[*]}(E^B)$ and $H_A^{[*]}(E^B)$.

If one has $F_i^T = \{0\}$ for every $i \in I$ and $A \in \{T, G\}$, then $H_A^{[*]}(E^P)$ and $H_A^{[*]}(E^B)$ are faithful modules and for $A = pt$ this is not the case, see **[10]**.

So, combining our previous result with the hypercohomology functor, we can prove the following.

Proposition 2. *There is a commutative diagram:*

$$\begin{array}{ccc} D_A^b(E^B) & \xrightarrow{\alpha_*()^{W_P}} & D_A^b(E^P) \\ \Big\downarrow{\scriptstyle \mathbb{H}^*} & & \Big\downarrow{\scriptstyle \mathbb{H}^*} \\ \mathcal{Z}^B-\text{grmod} & \xrightarrow{e_P} & \mathcal{Z}^B-\text{grmod}\,. \end{array}$$

Proof. We have by our previous observation $\mathbb{H}^*(\alpha_*(\mathcal{F})^{W_P}) = \mathbb{H}^*(\alpha_*(\mathcal{F}))^{W_P} = (\mathcal{Z}^B)^{W_P \times \{1\}})\mathbb{H}_A^*(\mathcal{F}) = e_P \cdot \mathbb{H}^*(\mathcal{F})$, where the last equality is proved in the proof of Proposition 1. □

4. Examples and applications.

4.1. The parabolic affine nil Hecke algebra. Let G be a reductive group over $\mathbb{C}$ and $B \subset G$ be a Borel subgroup. The affine nil Hecke algebra is defined by the graded vector space:

$$\text{NH} := \bigoplus_{p \in \mathbb{Z}} H_{[p]}^G(G/B \times G/B),$$

 https://doi.org/10.1017/S0017089516000598

where $H^G_{[p]}(G/B \times G/B) := H^G_{2\dim(G/B)-p}(G/B \times G/B)$, and NH is a graded algebra with respect to the convolution product defined by Chriss and Ginzburg (see [**4**]). We define

$$\mathrm{NH}^P := \bigoplus_{p \in \mathbb{Z}} H^G_{[p]}(G/P \times G/P)$$

with $H^G_{[p]}(G/P \times G/P) := H^G_{2\dim(G/P)-p}(G/P \times G/P)$, again this is a graded algebra with respect to the convolution product defined by Chriss and Ginzburg. We call it the *parabolic affine nil Hecke algebra*, it is a graded $H^*_G(pt)$-algebra. The following lemma has been observed in loc. cit. in the not equivariant case.

LEMMA 3. *One has* $\mathrm{NH}^P \cong \mathrm{End}_{H^*_G(pt)}(H^*_G(G/P))$ *as* $\mathbb{Z}$*-graded* $H^*_G(pt)$*-algebras.*

Recall that $H^*_G(pt) = (H^*_T(pt))^W = \mathbb{C}[\mathfrak{t}]^W$ is a commutative and graded $\mathbb{C}$-algebra, where $T \subset P$ is a maximal torus, $\mathfrak{t}$ its Lie algebra and W the Weyl group for (G, T). Also we know that $H^*_G(G/P) \cong \mathbb{C}[\mathfrak{t}]^{W_P}$, where W_P is the Weyl group of (L, T) for the Levi subgroup $L \subset P$. We write $W^P \subset W$ for the minimal coset representatives of the cosets W/W_P.

We give a proof of the previous lemma on the grounds that we could not find in the literature.

Proof. Let EG be a contractible free G-space (or an appropriate approximation of it in the sense of [**2**]). Let $X := G/P, \pi : X_G := X \times^G EG \to BG$ the map obtained from $X \to pt$ by applying $- \times^G EG$. By [**4**, chapter 8], we know $H^G_{[*]}(G/P \times G/P) \cong \mathrm{Ext}^*_{D^b_G(pt)}(\pi_*\underline{\mathbb{C}}, \pi_*\underline{\mathbb{C}})$ as graded $H^*_G(pt)$-algebras. Since π is a proper submersion, we have (by [**5**, item (3) after remark 1.6.2])

$$\pi_*\underline{\mathbb{C}} = \bigoplus_{i \in \mathbb{Z}} R^i\pi_*\underline{\mathbb{C}}[-i]$$

in $D^b_G(pt)$. Since all fibres of π are isomorphic to X and BG is simply connected, we get that

$$\pi_*\underline{\mathbb{C}} = \bigoplus_{w \in W^P} \underline{\mathbb{C}}[-2\ell(w)],$$

where $\ell(w)$ is the length of w. Let $r = \dim_{\mathbb{C}} H^*(X) = \#W^P$. We know, that $\mathbb{C}[\mathfrak{t}]^{W_P}$ is a free module over $\mathbb{C}[\mathfrak{t}]^W$ of rank r generated by elements b_w, $w \in W^P$, $\deg b_w = 2\ell(w)$. Now, the claim follows from the (well known) algebra isomorphism $\mathrm{Ext}^*_{D^b_G(pt)}(\underline{\mathbb{C}}, \underline{\mathbb{C}}) \cong H^*_G(pt)$. □

Our maps from the earlier section give a natural homomorphism of graded $H^*_G(pt)$-modules

$$\Theta\colon \mathrm{NH}^P = \mathrm{End}_{H^*_G(pt)}(H^*_G(G/P)) \to \mathrm{NH} = \mathrm{End}_{H^*_G(pt)}(H^*_G(G/B))$$
$$f \mapsto \mathrm{Inc} \circ f \circ \mathrm{Av},$$

where $\mathrm{Inc}\colon \mathbb{C}[\mathfrak{t}]^{W_P} \subset \mathbb{C}[\mathfrak{t}]$ is the natural inclusion and $\mathrm{Av}\colon \mathbb{C}[\mathfrak{t}] \to \mathbb{C}[\mathfrak{t}]^{W_P}, f \mapsto \frac{1}{\#W_P}\sum_{w \in W_P} w(f)$ is the averaging map. We set $e_P = \mathrm{Inc} \circ \mathrm{Av}(1)$. Furthermore, $W_P \times W_P$ operates on NH via graded $H^*_G(pt)$-module homomorphisms defined by $(v, w) \cdot h(f) := v(h(w^{-1}(f))), \quad v, w \in W_P, h \in \mathrm{NH}, f \in \mathbb{C}[\mathfrak{t}]$.

Our previous results, imply $\mathrm{NH}^P \cong e_P \mathrm{NH} e_P$ as graded $H_G^*(pt)$-algebras. Furthermore, one has $\mathrm{NH}^P = \mathrm{NH}^{W_P \times W_P}$ as graded $H_G^*(pt)$-modules and $\mathrm{NH}^{W\times W} = \mathbb{C}[\mathfrak{t}]^W = H_G^*(pt)$.

PROPOSITION 3.

(1) *Let $s = \# W_P$. We have an isomorphism of $\mathbb{C}[\mathfrak{t}]^W$-algebras*

$$\mathrm{NH} \cong M_s(\mathrm{NH}^P),$$

in particular it is a free module over NH^P of rank s^2.

(2) *Let $r = \# W^P$. We have an isomorphism of $\mathbb{C}[\mathfrak{t}]^W$-algebras*

$$\mathrm{NH}^P \cong M_r(\mathbb{C}[\mathfrak{t}]^W),$$

in particular it is a free module over $\mathbb{C}[\mathfrak{t}]^W$ of rank r^2.
A basis is given by $c_{v,w}, v, w \in W^P$ with $c_{v,w}$ is a lift of $[\overline{BvP/P} \times \overline{BwP/P}] \in H_*(G/P \times G/P)$ to $H_*^G(G/P \times G/P)$, i.e., elements in the fibres of the forgetful map (which is a surjective ring homomorphism),

$$\begin{aligned} H_*^G(G/P \times G/P) &\cong M_r(\mathbb{C}[\mathfrak{t}]^W) \twoheadrightarrow M_r(\mathbb{C}) \cong \mathrm{End}_{\mathbb{C}}(\mathbb{C}[\mathfrak{t}]^{W_P}/I_W) \\ &\cong H_*(G/P \times G/P), \\ &(f_{i,j})_{i,j} \mapsto (f_{i,j}(0))_{i,j}, \end{aligned}$$

where I_W is the ideal generated by the W-invariant polynomials of degree ≥ 1

Proof.

(1) Let $s = \# W_P$, it is the rank of $\mathbb{C}[\mathfrak{t}]$ as module over $\mathbb{C}[\mathfrak{t}]^{W_P}$ and therefore

$$\mathrm{NH} \cong \mathrm{End}_{\mathbb{C}[\mathfrak{t}]^W}((\mathbb{C}[\mathfrak{t}]^{W_P})^{\oplus s}) \cong M_s(\mathrm{NH}^P).$$

(2) $r = \# W^P$ is the rank of $\mathbb{C}[\mathfrak{t}]^{W_P}$ as module over $\mathbb{C}[\mathfrak{t}]^W$ and the dimension as $\mathbb{C}$-vector space of $\mathbb{C}[\mathfrak{t}]^{W_P}/I_W$. The rest follows as in (1).

□

REMARK. Let $S \subset W$ denote the simple reflections with respect to the Borel B and $n = \mathrm{rk}\, T$. For $s \in S$ let $\alpha_s \in \mathbb{C}[x_1, \ldots, x_n]$ be a linear polynomial with $s(\alpha_s) = -\alpha_s$. We denote by $\delta_s \colon \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n]$ the operator $\delta_s(f) := \frac{s(f)-f}{\alpha_s}$ called divided difference operator. It is well known that the affine nil Hecke algebra $\mathrm{NH} = \mathrm{NH}^B$ (with B a Borel group) is isomorphic to the subalgebra of $\mathrm{End}_{\mathbb{C}[x_1,\ldots,x_n]^W}(\mathbb{C}[x_1, \ldots, x_n])$ generated by $x_1\cdot, \ldots, x_n\cdot, \delta_s, s \in S$. But we do not know any description of generators (and relations) for the parabolic nil Hecke algebra NH^P, the analogues of the divided difference operators are missing.

4.2. On the parabolic analogue of the Springer map.

DEFINITION 2. Let (G, B, T) a complex reductive group with Borel subgroup and maximal torus T. Let (W, S) be the associated Coxeter system and $J \subset S$, we set $W_J := \langle J \rangle \subset W$ and write $P_J = BW_JB$ for the standard parabolic group. Recall, the affine nil Hecke algebra is the graded $\mathbb{C}$-algebra $\mathrm{NH} := \mathrm{End}_{\mathbb{C}[\mathfrak{t}]^W}(\mathbb{C}[\mathfrak{t}])$. Now, we define a subalgebra $A_J \subset \mathrm{NH}$ to be the subalgebra generated by multiplication with elements in $\mathbb{C}[\mathfrak{t}]$ and for each $s \in S$ a generator

$$\sigma(s) := \begin{cases} \delta_s, & \text{if } s \in J \\ s - \mathrm{id}, & \text{if } s \notin J \end{cases}.$$

We have for $J \subset K \subset S$ that $A_J \subset A_K$ and for arbitrary two subsets $J, K \subset S$ one has $A_J \cap A_K = A_{J \cap K}$ and the subalgebra generated by A_J and A_K is $A_{J \cup K}$. Since $A_\emptyset = \mathbb{C}[\mathfrak{t}]\#\mathbb{C}W$, $A_S = \mathrm{NH}$, we see these algebras as a lattice of *interpolations* between the Steinberg algebra $A_\emptyset$ corresponding to the Springer map and the nil Hecke algebra A_S.

We shorten the notation $P := P_J$. Now, the *parabolic version of the classical Springer map*, is given by

$$\pi : G \times^P \mathfrak{u}_P \to \mathcal{N}, \overline{(g, f)} \mapsto gf$$

with $\mathfrak{u}_P = \mathrm{Lie}\, U_P$, where $U_P \subset P$ is the unipotent radical. The fibres of this map have been studied in [3]. Let $\mathcal{Z}^P$ be the associated Steinberg algebra (i.e., $\mathcal{Z}^P = H^G_{[*]}((G \times^P \mathfrak{u}_P) \times_{\mathcal{N}} (G \times^P \mathfrak{u}_P)))$. The degree zero part of these algebras has already been described in the main result of Douglas and Röhrle, see [6]. Our result gives us the following.

PROPOSITION 4. *In the notation as before with $P := P_J$ one has the following:*

1. *The algebra A_J is the Steinberg algebra $\mathcal{Z}^B$ associated with the map $\pi : G \times^B \mathfrak{u}_P \to \mathcal{N}, \overline{(g, f)} \mapsto gf$.*
2. *Let $e := \frac{1}{\# W_J} \sum_{w \in W_J} w \in \mathrm{NH}$ be the idempotent element given by the averaging map from the first section. One has $e \in A_J$ and*

$$eA_Je = \mathcal{Z}^P.$$

Furthermore, one has $eA_Je = A_J \cap \mathrm{NH}^P$ with $P = P_J$.

Proof.

1. We have the monomorphism $\mathcal{Z}^B \to \mathrm{End}_{H^*_G(pt)}(H^{[*]}_G(E^B)) = \mathrm{End}_{H^*_G(pt)} (H^*_G(G/B)) = \mathrm{NH}$ from the Remark before Proposition 2. By the main result from [9], we know that the image is the subalgebra generated by the elements $\sigma(s)$, $s \in S$ as given above and multiplication by polynomials in $\mathbb{C}[\mathfrak{t}]$.
2. Follows from (1) using our main result. For the last statement, we have to see $eA_Je = A_J \cap e\mathrm{NH}e$. Clearly, eA_Je is a subset of $A_J \cap e\mathrm{NH}e$. If you take an element $ebe = a \in A_J$, $b \in \mathrm{NH}$, then $ebe = eae \in eA_Je$.

□

4.3. Reineke's example. In the end of [8], Reineke looked for a description of a parabolic Steinberg algebra, we give the answer which we obtained from our main result. Let Q be the quiver $(1 \to 2)$ and let $(d_1, d_2) \in \mathbb{N}_0^{Q_0}$. A directed partition of the Auslander–Reiten quiver of $\mathbb{C}Q$ is given by $I_1 := \{E_2 := (0 \to \mathbb{C})\}$, $I_2 := \{E_{1,2} := (\mathbb{C} \xrightarrow{\mathrm{id}} \mathbb{C})\}$, $I_3 := \{E_1 := (\mathbb{C} \to 0)\}$, i.e., it is a partition of the vertices of the Auslander–Reiten quiver $\{I_t\}_t$ such that $\mathrm{Ext}^1(I_t, I_t) = 0$ and $\forall t < u\ \mathrm{Hom}(I_u, I_t) = 0 = \mathrm{Ext}^1(I_t, I_u)$. Let $M = E_2^{d_2} \oplus E_{1,2} \oplus E_1^{d_1}$. Then, Reineke proved that quiver-graded Springer map corresponding to the dimension filtration $(0, (d_1 - 1, 0), (d_1 - 1, 1), (d_1, 1), (d_1, d_2))$ gives a resolution of singularities (i.e., birational projective map) for the orbit closure of M. Yet, we will consider the even easier dimension filtration $\underline{\mathbf{d}} := (0, (d_1, 1), (d_1, d_2))$. The associated Steinberg variety is

$$Z := \{(A, L_1, L_2) \in M_{d_2 \times d_1}(\mathbb{C}) \times \mathbb{P}^{d_2}(\mathbb{C}) \times \mathbb{P}^{d_2}(\mathbb{C}) \mid \mathrm{Im}(A) \subset L_i, i = 1, 2\};$$

it carries an operation of $\mathbf{Gl}_{\underline{d}} := \mathbf{Gl}_{d_2} \times \mathbf{Gl}_{d_1}$ via

$$(A, L_1, L_2) \mapsto (g_2^{-1} A g_1, g_2^{-1} L_1, g_2^{-1} L_2), \quad (g_2, g_1) \in \mathbf{Gl}_{\underline{d}}.$$

We want to describe the Steinberg algebra $H_*^{\mathbf{Gl}_{\underline{d}}}(Z)$ with our method. We set

$\mathbb{G} := \mathbf{Gl}_d$, $d := d_1 + d_2$,
$T :=$ invertible diagonal matrices,
$\mathbb{B} :=$ invertible upper triangular matrices,
$\mathbb{P} :=$ invertible upper block matrices with diagonal block sizes $(1, d-1)$,
$\mathcal{U} = \mathrm{Lie}\, \mathbb{U}_{\mathbb{P}}$, where $\mathbb{U}_{\mathbb{P}}$ is the unipotent radical of $\mathbb{P}$,
$G := \mathbf{Gl}_{d_2} \times \mathbf{Gl}_{d_1}$ diagonally embedded into $\mathbb{G}$,
$V = M_{d_2 \times d_1}$ embedded into the right upper corner of $\mathcal{G} = \mathfrak{gl}_d$,

as usual set $B := \mathbb{B} \cap G$, $P := \mathbb{P} \cap G$, $F := \mathcal{U} \cap V$.

The algebra $H_*^G(Z^B)$, $Z^B := (G \times^B F) \times_V (G \times^B F)$ can by Theorem 2.1 of [**9**] be described as the algebra $1_e * \mathcal{Z} * 1_e$ for $\mathcal{Z}$ being the Steinberg algebra associated with $(\mathbb{G}, \mathbb{B}, \mathcal{U}, V)$ and $e \in W \setminus \mathbb{W}$ be the coset of the neutral element. If we set $s_i := (i, i+1) \in S_d$ and

$$\delta_i := \delta_{s_i} \colon \mathbb{C}[t_1, \dots, t_d] \to \mathbb{C}[t_1, \dots, t_d], \quad f \mapsto \frac{s_i(f) - f}{t_i - t_{i+1}}$$

Then, $H_G^*(Z^B)$ is the subalgebra of $\mathrm{End}_{\mathbb{C}[t_1,\dots,t_d]^{S_{d_2} \times S_{d_1}}}(\mathbb{C}[t_1, \dots, t_d])$ generated by

$$(t_j \cdot), \ 1 \le j \le d, \quad \delta_i, \ i \in \{2, \dots d_2 - 2, d_2, \dots, d-1\},$$

$$\theta := \prod_{j=d_1+d_2+1}^{d} (t_1 - t_j)\delta_1.$$

Now, Reineke's variety equals $Z = (G \times^P F) \times_V (G \times^P F)$, by the previous section we conclude that it is the corner algebra of $e_P H_*^G(Z^B) e_P$, where

$$e_P \colon \mathbb{C}[t_1, \dots, t_d] \to \mathbb{C}[t_1, \dots, t_d], f \mapsto \frac{1}{(d_2 - 1)! d_1!} \sum_{w \in <s_2, \dots s_{d_2-2}, s_{d_2}, \dots, s_{d-1}>} w(f).$$

Acknowledgements. This article has been part of my PhD thesis at the University of Leeds. I want to thank my supervisor Andrew Hubery for detailed discussions and William Crawley-Boevey for his help and advice. Furthermore, I would like to thank Syu Kato for his remark about graded Morita equivalence. I received financial support through a 6-months-grant from the University of Leeds and I want to thank the CRC 701 for guest stays at the Universität Bielefeld.

REFERENCES

1. A. Arabia, Cohomologie T-équivariante de la variété de drapeaux d'un groupe de Kac-Moody, *Bull. Soc. Math. France* **117**(2) (1989), 129–165.

2. J. Bernstein and V. Lunts, . Bd. 1578: *Equivariant sheaves and functors*, Lecture notes in mathematics, volume 1578 (Springer-Verlag, Berlin, 1994), iv+139 S.

3. W. Borho and R. MacPherson, Partial resolutions of nilpotent varieties, in *Analysis and topology on singular spaces, II, III (Luminy, 1981)*, vol. 101 (Soc. Math. France, Paris, 1983), 23–74.

4. N. Chriss and V. Ginzburg, *Representation theory and complex geometry* (Birkhäuser Boston, Inc., Boston, MA, 1997).

5. M. A. de Cataldo and L. Migliorini, The decomposition theorem, perverse sheaves and the topology of algebraic maps, *Bull. Amer. Math. Soc.* **46** (2009), 535–633.

6. J. M. Douglass and G. Röhrle, The Steinberg variety and representations of reductive groups, *J. Algebra* **321**(11) (2009), 3158–3196.

7. R. Joshua, Modules over convolution algebras from equivariant derived categories. I, *J. Algebra* **203**(2) (1998), 385–446.

8. M. Reineke, Quivers, desingularizations and canonical bases, in *Studies in memory of Issai Schur (Chevaleret/Rehovot, 2000)*, vol. 210 (Birkhäuser Boston, Boston, MA, 2003), 325–344.

9. Sauter, Julia, Generalized quiver Hecke algebras, arXiv:1306.3892 [math.RT].

10. Sauter, Julia, A survey on springer theory, arXiv:1307.0973v3.

11. C. Stroppel and B. Webster, Quiver Schur algebras and q-Fock space, arXiv:1110.1115v2.

12. M. Varagnolo and E. Vasserot, Canonical bases and affine Hecke algebras of type B, *Invent. Math.* **185**(3) (2011), 593–693.

13. M. Varagnolo and E. Vasserot, Canonical bases and KLR-algebras, *J. Reine Angew. Math.* **659** (2011), 67–100.