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GROTHENDIECK GROUPS OF BASS ORDERS

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Commutative Bass rings, which form a special class of Gorenstein rings, have been thoroughly investigated by Bass [**1**]. The definitions do not carry over to non-commutative rings. However, in case one deals with orders in separable algebras over fields, Bass orders can be defined. Drozd, Kiričenko, and Roĭter [**3**] and Roĭter [**6**] have clarified the structure of Bass orders, and they have classified them. These Bass orders play a key role in the question of the finiteness of the non-isomorphic indecomposable lattices over orders (cf. [**2**; **8**]). We shall use the results of Drozd, Kiričenko, and Roĭter [**3**] to compute the Grothendieck groups of Bass orders locally. Locally, the Grothendieck group of a Bass order (with the exception of one class of Bass orders) is the epimorphic image of the direct sum of the Grothendieck groups of the maximal orders containing it. Using this, one can compute the local Grothendieck groups. However, globally, the above property does not hold any more, as can be seen by considering hereditary orders. Nevertheless, there exists a positive integer n, depending only on the algebra, such that the n-fold of the Grothendieck group of a Bass order is contained in the image of the Grothendieck groups of the maximal orders containing it.

1. Preliminaries on Bass orders. Let R be a Dedekind domain with quotient field K, and Λ an R-order in the separable finite-dimensional K-algebra A. Λ is called a left *Gorenstein order* if the injective dimension of Λ as left Λ-module is finite or equivalently (this property is more useful in praxis) $\Lambda^* = \mathrm{Hom}_R(\Lambda, R)$ is a projective right Λ-module. As for hereditary orders, Λ is left Gorenstein if and only if it is right Gorenstein. Thus we may simply talk about Gorenstein orders.

Λ is called a *Bass order* if Λ and every R-order in A containing Λ is a Gorenstein order. A Bass order is characterized by the property that the full two-sided Λ-ideals form a groupoid with respect to the proper multiplication.

Let $\hat{R}$ be a complete discrete rank-one valuation ring with quotient field $\hat{K}$, and $\hat{\Lambda}$ an $\hat{R}$-order in the finite-dimensional separable $\hat{K}$-algebra $\hat{A}$. Moreover, we assume that $\hat{R}$ *has a finite residue class field.*

We list without proof some essential properties of Bass orders. The proofs may be found in [**2**] or [**9**, IX, §§ 5, 6].

LEMMA 1. *Let $\hat{\Lambda}$ be a Bass order in $\hat{A}$. If $\hat{M}$ is an indecomposable $\hat{\Lambda}$-lattice, which is not a lattice over an $\hat{R}$-order $\hat{\Lambda}_1$ in $\hat{A}$, with $\hat{\Lambda}_1 \supset \hat{\Lambda}$, $\hat{\Lambda}_1 \neq \hat{\Lambda}$, then $\hat{M}$ is a projective*

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$\hat{\Lambda}$-lattice. In addition, if $\hat{M}$ is faithful, then it is a progenerator for the category of left $\hat{\Lambda}$-lattices.

For the convenience of the reader, let us recall some of the definitions. A $\hat{\Lambda}$-*lattice* $\hat{M}$ is a left $\hat{\Lambda}$-module, which is finitely generated and projective over $\hat{R}$. $\hat{M}$ is *faithful* if $\hat{K} \otimes_{\hat{R}} \hat{M}$ is a faithful $\hat{A}$-module. $\hat{M}$ is called a *progenerator* if it is a projective lattice, and if $\operatorname{Hom}_{\hat{\Lambda}}(\hat{M}, -)$ is a faithful functor. We say that $\hat{\Lambda}$ is *completely primary* if $\hat{\Lambda}$ is indecomposable as left $\hat{\Lambda}$-lattice. Because of the method of "lifting idempotents", $\hat{\Lambda}$ is completely primary, if modulo its radical, it is a division algebra.

THEOREM 1 (*Classification theorem*). *Let $\hat{\Lambda}$ be a Bass order in $\hat{A}$, which is indecomposable as a ring. Then one of the following cases must occur:*

(I) *$\hat{A}$ is simple and $\hat{\Lambda}$ is hereditary;*

(II) *$\hat{A}$ is simple, say $\hat{A} \cong (\hat{D})_n$, where $\hat{D}$ is a separable division algebra and $\hat{\Lambda}$ is Morita-equivalent to an $\hat{R}$-order of the form*

$$\begin{pmatrix} \hat{\Omega} & \hat{N}^d \\ \hat{\Omega} & \hat{\Omega} \end{pmatrix}, \qquad d \geqq 2,$$

where $\hat{\Omega}$ with $\operatorname{rad} \hat{\Omega} = \hat{N}$ *is the maximal $\hat{R}$-order in $\hat{D}$* ($\operatorname{rad} \hat{\Omega}$ *is the Jacobson radical of $\hat{\Omega}$);*

(III) *$\hat{A} = (\hat{D})_n$ is simple, and $\hat{\Lambda}$ is Morita-equivalent to a Bass order in $\hat{D}$;*

(IV) *$\hat{A} = (\hat{D})_n$ is simple, and $\hat{\Lambda}$ is Morita-equivalent to a completely primary Bass order in $(\hat{D})_2$;*

(V) *$\hat{A} = (\hat{D}_1)_{n_1} \oplus (\hat{D}_2)_{n_2}$, $\hat{D}_i$ a separable division algebra over $\hat{K}$, $i = 1, 2$, and $\hat{\Lambda}$ is Morita-equivalent to a completely primary Bass order in $\hat{D}_1 \oplus \hat{D}_2$, which is a subdirect sum of $\hat{\Omega}_1 \oplus \hat{\Omega}_2$, where $\hat{\Omega}_i$ is the maximal $\hat{R}$-order in $\hat{D}_i$, $i = 1, 2$.* (*We recall that two $\hat{R}$-orders $\hat{\Lambda}_i$ in $\hat{A}_i$, $i = 1, 2$, are Morita-equivalent, if there exists a $\hat{\Lambda}_1$-lattice $\hat{M}_1$ which is a progenerator for the category of $\hat{\Lambda}_1$-lattices such that* $\operatorname{End}_{\hat{\Lambda}_1}(\hat{M}_1) = \hat{\Lambda}_2$. *Being Morita-equivalent is an equivalence relation, which preserves Bass orders.*)

LEMMA 2. *Every non-maximal completely primary Bass order $\hat{\Lambda}$ in $\hat{A}$ is contained in a unique minimal over-order $\hat{\Lambda}_1 \supset \hat{\Lambda}$ such that* $\hat{\Lambda}_1 \operatorname{rad} \hat{\Lambda} \subset \operatorname{rad} \hat{\Lambda}$.

In case $\hat{A}$ is a skew field with maximal $\hat{R}$-order $\hat{\Omega}$, there is a unique strictly ascending chain of orders

$$\hat{\Lambda} = \hat{\Lambda}_0 \subset \hat{\Lambda}_1 \subset \ldots \subset \hat{\Lambda}_s = \hat{\Omega},$$

such that $\hat{\Lambda}_i$ is the unique minimal over-order of $\hat{\Lambda}_{i-1}$, $1 \leqq i \leqq s$; moreover, $\hat{\Lambda}/\operatorname{rad} \hat{\Lambda} \cong \hat{\Lambda}_i/\operatorname{rad} \hat{\Lambda}_i$, $1 \leqq i \leqq s - 1$. For $i = s$, one of the following cases must occur, which are subcases of (III):

(IIIa) $\hat{\Omega}/\operatorname{rad} \hat{\Omega} \cong \hat{\Lambda}/\operatorname{rad} \hat{\Lambda}$, or

(IIIb) $\hat{\Omega}/\operatorname{rad} \hat{\Omega} = \mathfrak{k}_1$ is a two-dimensional extension field of $\hat{\Lambda}/\operatorname{rad} \hat{\Lambda} = \mathfrak{k}$.

2. Local theory of Grothendieck groups of Bass orders. Again $\hat{R}$ with quotient field $\hat{K}$ is a complete discrete rank-one valuation ring with finite residue class field, and $\hat{\Lambda}$ is an $\hat{R}$-order in the finite-dimensional separable $\hat{K}$-algebra $\hat{A}$. By $\mathfrak{G}_0(\hat{\Lambda})$ we denote the Grothendieck group of the $\hat{\Lambda}$-lattices relative to short exact sequences, and by $\mathfrak{G}_0{}^f(\hat{\Lambda})$, the Grothendieck group of all finitely generated $\hat{\Lambda}$-modules, relative to short exact sequences; $\mathfrak{G}_0{}^T(\hat{\Lambda})$ is the Grothendieck group of all $\hat{R}$-torsion $\hat{\Lambda}$-modules of finite type, relative to short exact sequences, and $\mathfrak{K}_1(\hat{A})$ denotes the Whitehead group of $\hat{A}$. For the computation of the Grothendieck group of $\hat{\Lambda}$, $\mathfrak{G}_0(\hat{\Lambda})$, we may assume that $\hat{\Lambda}$ is indecomposable as a ring.

THEOREM 2. *Let $\hat{\Lambda}$ be a Bass order, which is indecomposable as a ring. If $\{\hat{\Gamma}_i\}$ $(1 \leqq i \leqq s)$ are the different maximal $\hat{R}$-orders in $\hat{A}$ containing $\hat{\Lambda}$, then the map*

$$\hat{\varphi}: \bigoplus_{i=1}^{s} \mathfrak{G}_0(\hat{\Gamma}_i) \to \mathfrak{G}_0(\hat{\Lambda}),$$

induced by restriction of the operator domain is an epimorphism, unless $\hat{\Lambda}$ is Morita-equivalent to an $\hat{R}$-order of type (IIIb).

Proof. We first recall that there are only finitely many different maximal $\hat{R}$-orders in $\hat{A}$ containing $\hat{\Lambda}$ (cf. [**10**]). To prove the theorem we shall first establish that no $\hat{R}$-order in $\hat{A}$ containing $\hat{\Lambda}$ can be Morita-equivalent to an order of type (IIIb), unless this is true for $\hat{\Lambda}$. Once this is shown, we can use induction. Let us therefore assume that $\hat{\Lambda}_1$ is an $\hat{R}$-order in $\hat{A}$, which is Morita-equivalent to an $\hat{R}$-order $\hat{\Omega}_1$ of type (IIIb). Then $\hat{A} = (\hat{D})_n$ and $\hat{\Omega}_1$ is an $\hat{R}$-order of type (IIIb) in the separable division algebra $\hat{D}$. If $\hat{\Lambda} \subset \hat{\Lambda}_1$, it follows from the classification theorem that $\hat{\Lambda}$ must also be Morita-equivalent to an $\hat{R}$-order $\hat{\Omega}_2$ in $\hat{D}$. (For otherwise $\hat{\Lambda}$ would contain the maximal order in $\hat{D}$.) Then we may assume that $\hat{\Lambda}_1 = (\hat{\Omega}_1)_n$ and $\hat{\Lambda} = a(\hat{\Omega}_2)_n a^{-1}$ for some regular element a in $\hat{A}$. We have to show that $\hat{\Omega}_2$ is of type (IIIb) if this is true for $\hat{\Omega}_1$. Let $\hat{\Omega}$ be the maximal $\hat{R}$-order in $\hat{D}$ and put $\hat{\Omega}/\text{rad}\,\hat{\Omega} = \mathfrak{k}_1$, where $\mathfrak{k}_1$ is a two-dimensional field extension of $\mathfrak{k} = \hat{\Omega}_1/\text{rad}\,\hat{\Omega}_1$, $\hat{\Omega}_1$ being of type (IIIb). We assume now that $\hat{\Omega}_2$ is of type (IIIa). Then $\hat{\Omega}_2/\text{rad}\,\hat{\Omega}_2 \cong \mathfrak{k}_1$. If $\mathfrak{k}$ is an s-dimensional extension field of $\bar{R} = \hat{R}/\text{rad}\,\hat{R}$, then for a simple left $\hat{\Lambda}_1$-module U, we have

$$\dim_{\bar{R}}(U) = ns.$$

But U is also an $\hat{R}$-torsion $\hat{\Lambda}$-module since $\hat{\Lambda}_1 \supset \hat{\Lambda}$, and if it has a composition series of m terms as $\hat{\Lambda}$-module, then

$$\dim_{\bar{R}}(U) = 2msn,$$

and we have obtained a contradiction to the assumption that $\hat{\Lambda}_1$ is Morita-equivalent to an $\hat{R}$-order of type (IIIb).

Therefore it suffices by induction to show that

$$\varphi' : \bigoplus_{\hat{\Lambda}'} \mathfrak{G}_0(\hat{\Lambda}') \to \mathfrak{G}_0(\hat{\Lambda})$$

is an epimorphism, where the sum is taken over all $\hat{R}$-orders in $\hat{A}$, properly containing $\hat{\Lambda}$. It should be observed that this sum is finite since $\hat{\Lambda}$ is noetherian, and since there are only finitely many maximal orders containing $\hat{\Lambda}$. (Moreover, Γ/Λ is a finite ring for every maximal order Γ containing Λ.) It is even enough to show that for every irreducible $\hat{\Lambda}$-lattice $\hat{M}$, $[\hat{M}] \in \operatorname{Im} \varphi$, where $[\hat{M}]$ denotes the class of $\hat{M}$ in $\mathfrak{G}_0(\hat{\Lambda})$. In view of Lemma 1, we may assume that $\hat{M}$ is an irreducible projective $\hat{\Lambda}$-lattice, which is not a lattice over an $\hat{R}$-order $\hat{\Lambda}_1$ properly containing $\hat{\Lambda}$. In particular, if $\operatorname{ann}_{\hat{A}}(\hat{K} \otimes_{\hat{R}} \hat{M}) = \hat{A}/\hat{A}e$ for some central idempotent e of $\hat{A}$, then $e \in \hat{\Lambda}$ and $\hat{M}$ is faithful, $\hat{\Lambda}$ being indecomposable as a ring; i.e. $\hat{M}$ is a progenerator by Lemma 1. Thus $\hat{\Lambda} \cong (\hat{\Omega}_1)_n$, where $\hat{\Omega}_1$ is a Bass order of type (IIIa) in a division algebra $\hat{D}$, $\hat{M}$ being a progenerator, $\hat{\Omega}_1 = \operatorname{End}_{\hat{\Lambda}}(\hat{M})$. Since $\hat{\Lambda} \cong \hat{M}^{(n)}$ as left $\hat{\Lambda}$-lattice, where $M^{(n)}$ denotes the direct sum of n copies of M, we have $\hat{\Lambda}a = \hat{M}^{(n)}$ for some regular element $a \in \hat{A}$. But then $a^{-1}\hat{\Lambda}a = (\hat{\Omega}_1)_n$, and conjugation with a transforms the $\hat{R}$-orders containing $\hat{\Lambda}$ into the $\hat{R}$-orders containing $(\hat{\Omega}_1)_n$. Therefore we may assume that $\hat{\Lambda} = (\hat{\Omega}_1)_n$ and $\hat{M} = (\hat{\Omega}_1)_n\mathfrak{E}_{11}$, where $\mathfrak{E}_{ij}$ is the matrix with 1 at the (i, j)-position and zeros elsewhere. Let $\hat{\Omega}$ be the maximal $\hat{R}$-order containing $\hat{\Omega}_1$. Since $\hat{\Omega}_1$ is of type (IIIa), we have

$$(*) \qquad \hat{\Omega}/\operatorname{rad} \hat{\Omega} \cong \hat{\Omega}_1/\operatorname{rad} \hat{\Omega}_1 \cong \mathfrak{k},$$

where $\mathfrak{k}$ is a finite field. In particular, $\mathfrak{k}$ is an $\hat{\Omega}$-module. Moreover, by Lemma 2, there exists a unique Bass order $\hat{\Omega}_2$ properly containing $\hat{\Omega}_1$ such that rad $\hat{\Omega}_1$ is an $\hat{\Omega}_2$-module. (We point out that our theorem is only of interest if $\hat{\Omega}_1$ is not maximal.) Then

$$\hat{M}/(\operatorname{rad} \hat{\Lambda})\hat{M} = (\mathfrak{k})_n\mathfrak{E}_{11}$$

is an $(\hat{\Omega})_n$-module and $(\operatorname{rad} \hat{\Lambda})\hat{M}$ is an $(\hat{\Omega}_2)_n$-lattice. The exact sequence

$$0 \to (\operatorname{rad} \hat{\Lambda})\hat{M} \to \hat{M} \to \hat{M}/(\operatorname{rad} \hat{\Lambda})\hat{M} \to 0$$

shows that

$$[\hat{M}] = [(\operatorname{rad} \hat{\Lambda})\hat{M}] + [\hat{M}/(\operatorname{rad} \hat{\Lambda})\hat{M}] \quad \text{in } \mathfrak{G}_0{}^f(\hat{\Lambda}),$$

and the commutative diagram (cf. [**11**])

$$\begin{array}{ccc}
\bigoplus_{\substack{\hat{\Lambda}' \supset \hat{\Lambda};\\ \hat{\Lambda}' \neq \hat{\Lambda}}} \mathfrak{G}_0{}^f(\hat{\Lambda}') & \xrightarrow{\varphi_f{}'} & \mathfrak{G}_0{}^f(\hat{\Lambda}) \\
\wr\Vert & & \wr\Vert \\
\bigoplus_{\substack{\hat{\Lambda}' \supset \hat{\Lambda};\\ \hat{\Lambda}' \neq \hat{\Lambda}}} \mathfrak{G}_0(\hat{\Lambda}') & \xrightarrow{\varphi'} & \mathfrak{G}_0(\hat{\Lambda})
\end{array}$$

(where the vertical maps are isomorphisms) shows that $[\hat{M}] \in \operatorname{Im} \varphi'$. Now we can use induction to conclude that φ is an epimorphism. (One has to distinguish between the case that $\hat{\Lambda}'$ decomposes as a ring and the one that $\hat{\Lambda}'$ is indecomposable as a ring.)

We remark briefly what happens in the remaining case.

COROLLARY 1. *If $\hat{\Lambda}$ is a Bass order in $\hat{A}$, indecomposable as a ring, which is Morita-equivalent to an $\hat{R}$-order of type* (IIIb), *then* $\operatorname{Im} \varphi \supset 2\mathfrak{G}_0(\hat{\Lambda})$.

Proof. In the notation of Theorem 2 and its proof, we would have instead of (*):

$$\hat{\Omega}/\operatorname{rad} \hat{\Omega} = \mathfrak{k}_1, \qquad \hat{\Omega}_1/\operatorname{rad} \hat{\Omega}_1 = \mathfrak{k},$$

where $\mathfrak{k}_1$ is a two-dimensional extension field of $\mathfrak{k}$ and $2[\mathfrak{k}] = [\mathfrak{k}_1]$ in $\mathfrak{G}_0{}^f(\hat{\Lambda})$. This then shows that $2[\hat{M}] \in \operatorname{Im} \varphi$.

Example 1. φ need not be an epimorphism if $\hat{\Lambda}$ is a Bass order which is of type (IIIb). Let $\hat{\Lambda}$ be of type (IIIb) in the separable division algebra $\hat{D}$, with maximal $\hat{R}$-order $\hat{\Omega}$. The commutative diagram (cf. [**5**])

$$\begin{array}{ccccccc}
\mathfrak{K}_1(\hat{D}) & \rightarrow & \mathfrak{G}_0{}^T(\hat{\Omega}) & \rightarrow & \mathfrak{G}_0(\hat{\Omega}) & \rightarrow & \mathfrak{G}_0(\hat{D}) \\
\| & & \downarrow\sigma & & \downarrow\varphi & & \| \\
\mathfrak{K}_1(\hat{D}) & \rightarrow & \mathfrak{G}_0{}^T(\hat{\Lambda}) & \rightarrow & \mathfrak{G}_0(\hat{\Lambda}) & \rightarrow & \mathfrak{G}_0(\hat{D})
\end{array}$$

shows that σ is an epimorphism if φ is epic. Here σ is also induced from the restriction of the operators. It remains to show that σ cannot be epic. The Jordan-Hölder theorem shows that

$$\mathfrak{G}_0{}^T(\hat{\Omega}) \simeq \mathfrak{G}_0(\hat{\Omega}/\operatorname{rad} \hat{\Omega}) \simeq \mathbf{Z}, \qquad \mathfrak{G}_0{}^T(\hat{\Lambda}) \simeq \mathfrak{G}_0(\hat{\Lambda}/\operatorname{rad} \hat{\Lambda}) \simeq \mathbf{Z},$$

and $\mathfrak{G}_0{}^T(\hat{\Omega})$ is freely generated by $[\mathfrak{k}_1] = [\hat{\Omega}/\operatorname{rad} \hat{\Omega}]$ and $\mathfrak{G}_0{}^T(\hat{\Lambda})$ is freely generated by $[\mathfrak{k}_1] = [\hat{\Lambda}/\operatorname{rad} \hat{\Lambda}]$, where $\mathfrak{k}_1$ is a two-dimensional extension field of $\mathfrak{k}$. Thus

$$\sigma\colon [\mathfrak{k}_1] \mapsto 2[\mathfrak{k}]$$

and σ is not epic; hence φ cannot be epic.

3. Computation of the Grothendieck groups of local Bass orders. We shall use Theorems 1 and 2 to compute $\mathfrak{G}_0(\hat{\Lambda})$ explicitly, in case $\hat{\Lambda}$ is a Bass order which is indecomposable as a ring. Again we assume that $\hat{R}$ with quotient field $\hat{K}$ is a complete discrete rank-one valuation ring with finite residue class field, and $\hat{A}$ is a separable finite-dimensional $\hat{K}$-algebra. Since Morita-equivalent $\hat{R}$-orders have isomorphic Grothendieck groups, we need only consider the following cases (cf. Theorem 1, Lemma 2).

(I) $\hat{\Lambda}$ is hereditary of type n in $\hat{A} = (\hat{D})_n$, where $\hat{D}$ is a separable division algebra. (A hereditary $\hat{R}$-order is said to be of *type s* if it has s non-isomorphic

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irreducible lattices, or, equivalently, if it is contained in s different maximal $\hat{R}$-orders.)

$$\text{(II)} \qquad \hat{\Lambda} = \begin{pmatrix} \hat{\Omega} & \hat{N}^d \\ \hat{\Omega} & \hat{\Omega} \end{pmatrix}, \qquad d \geqq 2,$$

where $\hat{\Omega}$ with rad $\hat{\Omega} = \hat{N}$ is the maximal $\hat{R}$-order in $\hat{D}$.

(IIIa) $\hat{\Lambda}$ is a Bass order in $\hat{D}$ of type (IIIa).

(IIIb) $\hat{\Lambda}$ is a Bass order in $\hat{D}$ of type (IIIb).

(IV) $\hat{\Lambda}$ is a completely primary Bass order in $(\hat{D})_2$, and the unique minimal over-order of $\hat{\Lambda}$, that decomposes as a module, is either

(IVa) maximal, or

(IVb) non-maximal hereditary.

(V) $\hat{\Lambda}$ is a completely primary Bass order in $\hat{D}_1 \oplus \hat{D}_2$, $\hat{D}_i$ a separable division algebra, $i = 1, 2$.

In the next theorem, (I)–(V) refer to the above list.

THEOREM 3. *If $\hat{\Lambda}$ is a Bass order which is indecomposable as a ring, then one of the following cases must occur:*

(I) $\mathfrak{G}_0(\hat{\Lambda}) \cong \mathbf{Z}^{(n)}$;

(II) $\mathfrak{G}_0(\hat{\Lambda}) \cong \mathbf{Z}^{(2)}$;

(IIIa) $\mathfrak{G}_0(\hat{\Lambda}) \cong \mathbf{Z}$;

(IIIb) $\mathfrak{G}_0(\hat{\Lambda}) \cong \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$;

(IVa) $\mathfrak{G}_0(\hat{\Lambda}) \cong \mathbf{Z}$;

(IVb) $\mathfrak{G}_0(\hat{\Lambda}) \cong \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$;

(V) $\mathfrak{G}_0(\hat{\Lambda}) \cong \mathbf{Z}^{(2)}$.

Proof. (I) Because of the Krull-Schmidt theorem and since $\hat{\Lambda}$ is hereditary of type n in $(\hat{D})_n$, we have $\mathfrak{G}_0(\hat{\Lambda}) \cong \mathbf{Z}^{(n)}$.

$$\text{(II)} \qquad \hat{\Lambda} = \begin{pmatrix} \hat{\Omega} & \hat{N}^d \\ \hat{\Omega} & \hat{\Omega} \end{pmatrix}, \qquad d \geqq 2.$$

$\hat{\Omega}/\text{rad}\,\hat{\Omega} = \mathfrak{k}$ is a finite field and $\hat{\Lambda}/\text{rad}\,\hat{\Lambda} \cong \mathfrak{k} \oplus \mathfrak{k}$ as a $\hat{\Lambda}$-module. If

$$\hat{\Lambda}_0 = \begin{pmatrix} \hat{\Omega} & \hat{N} \\ \hat{\Omega} & \hat{\Omega} \end{pmatrix}$$

is a hereditary $\hat{R}$-order containing $\hat{\Lambda}$, then $\hat{\Lambda}_0/\text{rad}\,\hat{\Lambda}_0 \cong \mathfrak{k} \oplus \mathfrak{k}$ as a $\hat{\Lambda}_0$-module, and in the commutative diagram

$$\begin{array}{ccccccccc}
\mathfrak{K}_1((\hat{D})_2) & \to & \mathfrak{G}_0^T(\hat{\Lambda}_0) & \to & \mathfrak{G}_0(\hat{\Lambda}_0) & \to & \mathfrak{G}_0(\hat{A}) & \to & 0 \\
\| & & \downarrow\sigma & & \downarrow\varphi & & \| & & \\
\mathfrak{K}_1((\hat{D})_2) & \to & \mathfrak{G}_0^T(\hat{\Lambda}) & \to & \mathfrak{G}_0(\hat{\Lambda}) & \to & \mathfrak{G}_0(\hat{A}) & \to & 0
\end{array}$$

σ is an isomorphism. Diagram chasing shows that φ has to be an isomorphism, and by (I) we conclude that $\mathfrak{G}_0(\hat{\Lambda}) \cong \mathbf{Z}^{(2)}$.

(IIIa) $\hat{\Lambda}$ is a Bass order of type (IIIa) in $\hat{D}$, and by Theorem 2,

$$\varphi\colon \mathfrak{G}_0(\hat{\Omega}) \to \mathfrak{G}_0(\hat{\Lambda})$$

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is an epimorphism, where $\hat{\Omega}$ is the maximal $\hat{R}$-order in $\hat{D}$. Since we also have an epimorphism $\mathfrak{G}_0(\hat{\Lambda}) \to \mathfrak{G}_0(\hat{D})$, we conclude that $\mathbf{Z} \cong \mathfrak{G}_0(\hat{\Omega}) \cong \mathfrak{G}_0(\hat{\Lambda})$.

(IIIb) $\hat{\Lambda}$ is a Bass order of type (IIIb) in $\hat{D}$. Let

$$\hat{\Lambda} = \hat{\Lambda} \subset \hat{\Lambda}_1 \subset \ldots \subset \hat{\Lambda}_s = \hat{\Omega}$$

be the unique strictly ascending chain of Bass orders containing $\hat{\Lambda}$ (cf. Lemma 2). Since

$$\hat{\Lambda}/\text{rad } \hat{\Lambda} \cong \hat{\Lambda}_i/\text{rad } \hat{\Lambda}_i, \qquad 1 \leqq i \leqq s-1,$$

we conclude, as in the proof of Theorem 2, that $\mathfrak{G}_0(\hat{\Lambda}) \cong \mathfrak{G}_0(\hat{\Lambda}_{s-1})$. (Observe that $s \geqq 1$, $\hat{\Lambda}$ being of type (IIIb).) Hence we may assume that $\hat{\Lambda} = \hat{\Lambda}_{s-1}$, and thus, every non-projective indecomposable $\hat{\Lambda}$-lattice $\hat{M}$ is an $\hat{\Omega}$-lattice, i.e., in the homomorphism $\varphi: \mathfrak{G}_0(\hat{\Omega}) \to \mathfrak{G}_0(\hat{\Lambda})$, we have $[\hat{M}] \in \operatorname{Im} \varphi$. Because of the Krull-Schmidt theorem, and since $\hat{\Lambda}$ is completely primary, $\hat{\Lambda}$ is the only projective $\hat{\Lambda}$-lattice, which is also irreducible. Corollary 1 and Example 1 show that $[\hat{\Lambda}] \notin \operatorname{Im} \varphi$ and $2[\hat{\Lambda}] \in \operatorname{Im} \varphi$. Hence $\mathfrak{G}_0(\hat{\Lambda})$ is generated by $([\hat{\Omega}], [\hat{\Omega}] - [\hat{\Lambda}])$ and so $\mathfrak{G}_0(\hat{\Lambda}) \cong \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$.

(IV) If $\hat{\Lambda}$ is a completely primary Bass order in $(\hat{D})_2$, there exists a unique properly ascending chain of orders

$$\hat{\Lambda} = \hat{\Lambda}_0 \subset \hat{\Lambda}_1 \subset \ldots \subset \hat{\Lambda}_s,$$

where $\hat{\Lambda}_i$ is the unique minimal over-order of $\hat{\Lambda}_{i-1}$, $1 \leqq i \leqq s$, and where $\hat{\Lambda}_i$ is completely primary, $0 \leqq i \leqq s-1$ (cf. Lemma 2). $\hat{\Lambda}_s$ decomposes as a module, and thus it is hereditary (cf. [**3**]).

(IVa) If $\hat{\Lambda}_s$ is maximal, then it is the unique maximal $\hat{R}$-order containing $\hat{\Lambda}$, and since $\varphi: \mathfrak{G}_0(\hat{\Lambda}_s) \to \mathfrak{G}_0(\hat{\Lambda})$ is an epimorphism, we must have $\mathfrak{G}_0(\hat{\Lambda}) \cong \mathbf{Z}$.

(IVb) $\hat{\Lambda}_s$ is a non-maximal hereditary $\hat{R}$-order. Since the above chain of orders is unique, the proof of Theorem 2 shows that we have an epimorphism

$$\varphi: \mathfrak{G}_0(\hat{\Lambda}_s) \to \mathfrak{G}_0(\hat{\Lambda}).$$

The commutative diagram

$$\begin{tikzcd}
\mathfrak{K}_1((\hat{D})_2) \arrow[r, "\vartheta_0"] \arrow[d, equal] & \mathfrak{G}_0^T(\hat{\Lambda}_s) \arrow[r, "\rho_0"] \arrow[d, "\sigma"] & \mathfrak{G}_0(\hat{\Lambda}_s) \arrow[r] \arrow[d, "\varphi"] & \mathfrak{G}_0((\hat{D})_2) \arrow[r] \arrow[d, equal] & 0 \\
\mathfrak{K}_1((\hat{D})_2) \arrow[r, "\vartheta"] & \mathfrak{G}_0^T(\hat{\Lambda}) \arrow[r, "\rho"] & \mathfrak{G}_0(\hat{\Lambda}) \arrow[r] & \mathfrak{G}_0((\hat{D})_2) \arrow[r] & 0
\end{tikzcd}$$

shows that σ is an epimorphism, φ being epic. Since every irreducible $\hat{\Lambda}$-lattice is also an irreducible $\hat{\Lambda}_s$-lattice by Lemma 1, and since isomorphism is preserved, there are exactly two non-isomorphic irreducible $\hat{\Lambda}_s$-lattices, $\hat{M}_1$ and $\hat{M}_2$. Then $(\text{rad } \hat{\Lambda}_s)\hat{M}_1 \cong \hat{M}_2$ and $(\text{rad } \hat{\Lambda}_s)\hat{M}_2 \cong \hat{M}_1$ (cf. [**7**]). If $\hat{\Omega}$ is the maximal $\hat{R}$-order in $\hat{D}$, and $\hat{\Omega}/\text{rad } \hat{\Omega} \cong \mathfrak{k}$, then

$$\hat{\Lambda}_s/\text{rad } \hat{\Lambda}_s \cong \mathfrak{k}e_1 \oplus \mathfrak{k}e_2,$$

where $\bar{M}_i = \hat{M}_i/(\text{rad } \hat{\Lambda}_s)\hat{M}_i \cong \mathfrak{k}e_i$, $i = 1, 2$. Since $\hat{\Lambda}$ is completely primary, there exists exactly one simple $\hat{\Lambda}$-module $U = \hat{\Lambda}/\text{rad } \hat{\Lambda}$, and in $\mathfrak{G}_0{}^T(\hat{\Lambda})$, we have

$$[\bar{M}_i] = [U^{(s_i)}],$$

and consideration of the dimensions shows that $s_1 = s_2$. However, since σ is an epimorphism, $s_1 = s_2 = 1$ and σ: $[\bar{M}_i] \mapsto [U]$, $i = 1, 2$. Thus, diagram chasing shows that

$$\varphi: 2([\hat{M}_1] - [\hat{M}_2]) \mapsto 0.$$

Therefore, to prove that $\mathfrak{G}_0(\hat{\Lambda}) \cong \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$, we need only show that $\varphi([\hat{M}_1] - [\hat{M}_2]) \neq 0$. Since $\rho_0([\bar{M}_1]) = [\hat{M}_1] - [\hat{M}_2]$, and since $\sigma([\bar{M}_1])$ generates $\mathfrak{G}_0{}^T(\hat{\Lambda})$, we conclude, by letting $\varphi[\hat{M}_1] = \varphi[\hat{M}_2]$, that ϑ must be an epimorphism, say

$$\vartheta(x) = [\hat{\Lambda}/\text{rad } \hat{\Lambda}] = \sigma([\bar{M}_1]).$$

The commutativity of our diagram then shows that $\sigma\vartheta_0(x) = \sigma([\bar{M}_1])$. However, Ker $\sigma = \{n([\bar{M}_1] - [\bar{M}_2]) : n \in \mathbf{Z}\}$; i.e., $\vartheta_0(x) = [\bar{M}_1] + n([\bar{M}_1] - [\bar{M}_2])$, and consequently

$$0 = \rho_0\vartheta_0(x) = (2n + 1)([\hat{M}_1] - [\hat{M}_2]) \quad \text{in } \mathfrak{G}_0(\hat{\Lambda}_s).$$

Since $\mathfrak{G}_0(\hat{\Lambda}_s)$ is a free abelian group, and since $[\hat{M}_1] \neq [\hat{M}_2]$ in $\mathfrak{G}_0(\hat{\Lambda}_s)$, we have obtained a contradiction, i.e., $\mathfrak{G}_0(\hat{\Lambda}) \cong \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$.

(V) If $\hat{\Lambda}$ is a completely primary Bass order in $\hat{D}_1 \oplus \hat{D}_2$, then $\mathfrak{G}_0(\hat{\Lambda}) \cong \mathbf{Z}^{(2)}$, since φ: $\mathfrak{G}_0(\hat{\Gamma}) \to \mathfrak{G}_0(\hat{\Lambda})$ is an epimorphism, where $\hat{\Gamma}$ is the unique maximal $\hat{R}$-order containing $\hat{\Lambda}$, and since $\mathfrak{G}_0(\hat{D}_1 \oplus \hat{D}_2) \cong \mathbf{Z}^{(2)}$.

4. Grothendieck groups of global Bass orders. Let R be a Dedekind domain with quotient field K and Λ an R-order in the separable finite-dimensional K-algebra A. We assume that the Jordan-Zassenhaus theorem holds for Λ-lattices. By max(R) we denote the spectrum of maximal ideals in R, and if $\mathfrak{p} \in \max(R)$, we write $X_\mathfrak{p}$ $(\hat{X}_\mathfrak{p})$ for the localization (completion) of X at $\mathfrak{p}$.

Since hereditary R-orders are Bass orders, the analogue of Theorem 2 cannot hold globally, as shown by the following example.

Example 2. Let A be a central division algebra which is unramified at $\mathfrak{q} \in \max(R)$, say $\hat{A}_\mathfrak{q} \cong (\hat{K}_\mathfrak{q})_2$, and let $\hat{\Lambda}_\mathfrak{q}$ be a hereditary $\hat{R}$-order of type 2 in $\hat{A}_\mathfrak{q}$ and put $\Lambda_\mathfrak{q} = A \cap \hat{\Lambda}_\mathfrak{q}$. Then $\Lambda_\mathfrak{q}$ is a hereditary $R_\mathfrak{q}$-order in A, and $\Lambda_\mathfrak{q}$ is contained in exactly two maximal $R_\mathfrak{q}$-orders $\Gamma_\mathfrak{q}$ and $\Gamma_\mathfrak{q}'$, and

$$\varphi: \mathfrak{G}_0(\Gamma_\mathfrak{q}) \oplus \mathfrak{G}_0(\Gamma_\mathfrak{q}') \to \mathfrak{G}_0(\Lambda_\mathfrak{q})$$

is not an epimorphism. In fact, let $\hat{M}_\mathfrak{q}$ $(\hat{M}_\mathfrak{q}')$ be the irreducible $\hat{\Gamma}_\mathfrak{q}$-lattice $(\hat{\Gamma}_\mathfrak{q}'$-lattice$)$; then

$$[N_\mathfrak{q}] = [A \cap (\hat{M}_\mathfrak{q} \oplus \hat{M}_\mathfrak{q}')] \notin \text{Im } \varphi.$$

(We may assume without loss of generality that $\hat{K}_\mathfrak{q}(\hat{M}_\mathfrak{q} \oplus \hat{M}_\mathfrak{q}') = \hat{A}_\mathfrak{q}$.)

We shall generalize this phenomenon to arbitrary Bass orders. Assume that $A = \bigoplus_{i=1}^{n} (D_i)_{n_i}, \{D_i\}$ $(1 \leqq i \leqq n)$ separable skew fields; then for $\mathfrak{p} \in \max(R)$, we have

$$\hat{D}_{i\mathfrak{p}} = \bigoplus_{j=1}^{n_i(\mathfrak{p})} (\hat{D}_{ij}(\mathfrak{p}))_{n_{ij}(\mathfrak{p})},$$

where $\{\hat{D}_{ij}(\mathfrak{p})\}$ are separable skew fields. We let $v_\mathfrak{p}$ be the smallest common multiple of $\{n_{ij}(\mathfrak{p})\}$ $(1 \leqq i \leqq n, 1 \leqq j \leqq n_i(\mathfrak{p}))$. For a given R-order Λ in A, we let $\mathfrak{S}_0 = \{\mathfrak{p} \in \max(R)\colon \Lambda_\mathfrak{p}$ is not maximal$\}$ and let v be the smallest common multiple of $\{v_\mathfrak{p}\}$ $(\mathfrak{p} \in \mathfrak{S}_0)$. If v_0 is the smallest common multiple of the $\{v_\mathfrak{p}\}$ $(\mathfrak{p} \in \max(R))$, then v_0 is independent of Λ and $v \leqq v_0 < \infty$.

THEOREM 4. *Let $\{\Gamma_i\}$ $(1 \leqq i \leqq s)$ be the maximal R-orders in A containing Λ, and assume that for every $\mathfrak{p} \in \mathfrak{S}_0$,*

$$\varphi_\mathfrak{p}\colon \bigoplus_{i=1}^{s} \mathfrak{G}_0(\hat{\Gamma}_{i\mathfrak{p}}) \to \mathfrak{G}_0(\hat{\Lambda}_\mathfrak{p})$$

is an epimorphism. Then in the homomorphism $\varphi\colon \bigoplus_{i=1}^{s} \mathfrak{G}_0(\Gamma_i) \to \mathfrak{G}_0(\Lambda)$, we have Im $\varphi \supset v\mathfrak{G}_0(\Lambda)$.

As the example of hereditary orders shows, this result is best possible as a general statement.

Proof. We may assume that Λ is not maximal, and hence $\mathfrak{S}_0 \neq \emptyset$. We first show that the result is true for $\Lambda_\mathfrak{p}$, $\mathfrak{p}$ a fixed maximal ideal in R. We have the commutative diagram of Grothendieck groups

$$\begin{array}{ccc}
\bigoplus_{i=1}^{s} \mathfrak{G}_0(\Gamma_{i\mathfrak{p}}) & \xrightarrow{\bigoplus_{i=1}^{s} \hat{\alpha}_i} & \bigoplus_{i=1}^{s} \mathfrak{G}_0(\hat{\Gamma}_{i\mathfrak{p}}) \\
\varphi_\mathfrak{p}\big\downarrow & & \big\downarrow\hat{\varphi}_\mathfrak{p} \\
\mathfrak{G}_0(\Lambda_\mathfrak{p}) & \xrightarrow{\hat{\alpha}} & \mathfrak{G}_0(\hat{\Lambda}_\mathfrak{p}),
\end{array}$$

where $\{\hat{\alpha}_i\}$ $(1 \leqq i \leqq s)$ and $\hat{\alpha}$ are induced from the functor $\hat{R}_\mathfrak{p} \otimes_{R_\mathfrak{p}} —$. We claim that $\hat{\alpha}$ is a monomorphism. Let $\hat{\alpha}([M_1] - [M_2]) = 0$. Then there exist two exact sequences of $\hat{\Lambda}_\mathfrak{p}$-lattices (cf. [**4**])

$$\hat{E}_1\colon 0 \to \hat{X} \to \hat{M}_{1_\mathfrak{p}} \oplus \hat{Z} \to \hat{Y} \to 0$$
$$\hat{E}_2\colon 0 \to \hat{X} \to \hat{M}_{2_\mathfrak{p}} \oplus \hat{Z} \to \hat{Y} \to 0$$

However, it is easily seen that we may assume that there are $\Lambda_\mathfrak{p}$-lattices X, Y, and Z with $\hat{X}_\mathfrak{p} \cong \hat{X}$, $\hat{Y}_\mathfrak{p} \cong \hat{Y}$, and $\hat{Z}_\mathfrak{p} \cong \hat{Z}$. But if N and N' are $\Lambda_\mathfrak{p}$-lattices, then

$$\mathrm{Ext}_{\hat{\Lambda}_\mathfrak{p}}{}^1(\hat{N}_\mathfrak{p}, \hat{N}_\mathfrak{p}') \cong^{\text{nat}} \hat{R}_\mathfrak{p} \oplus_{R_\mathfrak{p}} \mathrm{Ext}_{\Lambda_\mathfrak{p}}{}^1(N, N'),$$

and since $\mathrm{Ext}_{\Lambda_\mathfrak{p}}{}^1(N, N')$ is an $R_\mathfrak{p}$-torsion module of finite type, we conclude that

$$\mathrm{Ext}_{\Lambda_\mathfrak{p}}{}^1(N, N') \cong \mathrm{Ext}_{\hat{\Lambda}_\mathfrak{p}}{}^1(\hat{N}_\mathfrak{p}, \hat{N}_\mathfrak{p}'),$$

and the isomorphism is induced by $\hat{R}_\mathfrak{p} \otimes_{R_\mathfrak{p}} -$. Applying this to our situation, we can find two exact sequences of $\Lambda_\mathfrak{p}$-lattices:

$$E_1: 0 \to X \to M \to Y \to 0$$
$$E_2: 0 \to X \to N \to Y \to 0$$

such that $\hat{R}_\mathfrak{p} \oplus_{R_\mathfrak{p}} E_i \equiv \hat{E}_i, i = 1, 2$. Then $\hat{M}_\mathfrak{p} \cong \hat{M}_{1_\mathfrak{p}} \oplus \hat{Z}_\mathfrak{p}$ and $\hat{N}_\mathfrak{p} \cong \hat{M}_{2_\mathfrak{p}} \oplus \hat{Z}_\mathfrak{p}$; i.e., $M \cong M_1 \oplus Z$ and $N \cong M_2 \oplus Z$. Thus $[M] = [N]$ in $\mathfrak{G}_0(\Lambda_\mathfrak{p})$ (cf. [**4**]), and $\hat{\alpha}$ is monic. To show that

$$\operatorname{Im} \varphi_\mathfrak{p} \supset v_\mathfrak{p}\mathfrak{G}_0(\Lambda_\mathfrak{p}),$$

it therefore suffices to establish that

$$\operatorname{Im}\left(\hat{\varphi}_\mathfrak{p}\left(\bigoplus_{i=1}^{s} \hat{\alpha}_i\right)\right) \supset v_\mathfrak{p} \operatorname{Im} \hat{\alpha}.$$

Let $x \in \mathfrak{G}_0(\Lambda_\mathfrak{p})$. Then $\hat{\alpha}x \in \operatorname{Im} \hat{\varphi}_\mathfrak{p}$; but

$$\mathfrak{G}_0(\Gamma_{i_\mathfrak{p}}) \cong^{\text{nat}} \mathfrak{G}_0(A) \quad \text{and} \quad \mathfrak{G}_0(\hat{\Gamma}_{i_\mathfrak{p}}) \cong^{\text{nat}} \mathfrak{G}_0(\hat{A}_\mathfrak{p}).$$

In addition, φ $(\hat{\varphi}_\mathfrak{p})$ composed with the epimorphism $\mathfrak{G}_0(\Lambda) \to \mathfrak{G}_0(A)$ $(\mathfrak{G}_0(\hat{\Lambda}_\mathfrak{p}) \to \mathfrak{G}_0(\hat{A}_\mathfrak{p}))$ induces the codiagonal map

$$\bigoplus_{1}^{s} \mathfrak{G}_0(A) \to \mathfrak{G}_0(A) \qquad \left(\bigoplus_{1}^{s} \mathfrak{G}_0(\hat{A}_\mathfrak{p}) \to \mathfrak{G}_0(\hat{A}_\mathfrak{p})\right).$$

Consequently,

$$v_\mathfrak{p}\hat{\alpha}x \in \operatorname{Im}\left(\hat{\varphi}_\mathfrak{p}\left(\bigoplus_{i=1}^{s} \hat{\alpha}_i\right)\right);$$

i.e., $vx \in \operatorname{Im} \hat{\varphi}_\mathfrak{p}$. To show globally that $\operatorname{Im} \varphi \supset v\mathfrak{G}_0(\Lambda)$, we consider the commutative diagram

$$\begin{array}{ccc}
\bigoplus_{i=1}^{s} \mathfrak{G}_0(\Gamma_i) & \xrightarrow{\bigoplus_{i=1}^{s} \alpha_i} & \bigoplus_{\mathfrak{p}\in\mathfrak{S}_0} \left(\bigoplus_{i=1}^{s} \mathfrak{G}_0(\Gamma_{i_\mathfrak{p}})\right) \\
\varphi \downarrow & & \downarrow \bigoplus_{\mathfrak{p}\in\mathfrak{S}_0} \varphi_\mathfrak{p} \\
\mathfrak{G}_0(\Lambda) & \xrightarrow{\alpha} & \bigoplus_{\mathfrak{p}\in\mathfrak{S}_0} \mathfrak{G}_0(\Lambda_\mathfrak{p})
\end{array}$$

where $\{\alpha_i\}$ $(1 \leqq i \leqq s)$ and α are induced from $R_\mathfrak{p} \otimes_R -$. We shall show next that for $[M] \in \mathfrak{G}_0(\Lambda)$, there exists $x \in \bigoplus_{i=1}^{s} \mathfrak{G}_0(\Gamma_i)$ such that

$$v\alpha[M] = \left(\bigoplus_{\mathfrak{p}\in\mathfrak{S}_0} \varphi_\mathfrak{p}\right)\left(\bigoplus_{i=1}^{s} \alpha_i\right)x.$$

Rather than presenting the proof in the general case, where it is very much obscured by a multitude of indices, we only present a proof in case $\mathfrak{S}_0$ consists of two ideals. (Apart from the indices, this proof is the same as the general proof.) We assume that $\Lambda_\mathfrak{q} = \Gamma_\mathfrak{q}$ is maximal for all

$$\mathfrak{q} \in \max(R), \qquad q \neq \mathfrak{p}_1, \mathfrak{p}_2, \qquad \mathfrak{S}_0 = \{\mathfrak{p}_1, \mathfrak{p}_2\}.$$

Let $\{\Gamma_i(\mathfrak{p}_j)\}$ $(1 \leqq i \leqq n_j)$ be the different maximal $R_{\mathfrak{p}_j}$-orders containing $\Lambda_{\mathfrak{p}_j}$, $j = 1, 2$. Then

$$\Gamma_{ij} = \left(\bigcap_{\mathfrak{q} \in \max(R)\setminus\mathfrak{S}_0} \Gamma_{\mathfrak{q}}\right) \cap \Gamma_i(\mathfrak{p}_1) \cap \Gamma_j(\mathfrak{p}_2), \qquad 1 \leqq i \leqq n_1, 1 \leqq j \leqq n_2,$$

are the different maximal R-orders containing Λ, and we have the commutative diagram:

$$\begin{array}{ccc}
\bigoplus_{\substack{1\leqq i \leqq n_1;\\ 1 \leqq j \leqq n_2}} \mathfrak{G}_0(\Gamma_{ij}) & \xrightarrow{\bigoplus \alpha_{ij}} & \bigoplus_{i,j} \mathfrak{G}_0((\Gamma_{ij})_{\mathfrak{p}_1}) \oplus \left(\bigoplus_{i,j} \mathfrak{G}_0(\Gamma_{ij})_{\mathfrak{p}_2}\right) \\
\Big\downarrow{\scriptstyle\varphi} & & \Big\downarrow{\scriptstyle\varphi_{\mathfrak{p}_1} \oplus \varphi_{\mathfrak{p}_2}} \\
\mathfrak{G}_0(\Lambda) & \xrightarrow{\alpha} & \mathfrak{G}_0(\Lambda_{\mathfrak{p}_1}) \oplus \mathfrak{G}_0(\Lambda_{\mathfrak{p}_2})
\end{array}$$

We observe that $(\Gamma_{ij})_{\mathfrak{p}_1} = (\Gamma_{ij'})_{\mathfrak{p}_1}$ and $(\Gamma_{ij})_{\mathfrak{p}_2} = (\Gamma_{i'j})_{\mathfrak{p}_2}$ for all i, i', j, j'. Since $\mathfrak{G}_0(\Lambda)$ is generated by the irreducible Λ-lattices, it is enough to show that $v[M] \in \operatorname{Im}\varphi$, whenever M is irreducible. According to the first part of the proof, $v\alpha[M] \in \operatorname{Im}(\varphi_{\mathfrak{p}_1} \oplus \varphi_{\mathfrak{p}_2})$; i.e., there exist $\Gamma_i(\mathfrak{p}_k)$-lattices M_{ik}^1 and M_{ik}^2, $1 \leqq i \leqq n_k$, $k = 1, 2$, such that

$$v[M_{\mathfrak{p}_k}] = \varphi_{\mathfrak{p}_k}\left(\sum_{1\leqq i \leqq n_k} [M_{ik}^1] - [M_{ik}^2]\right), \qquad k = 1, 2.$$

In $\mathfrak{G}_0(A)$ we must have

$$\sum_{1 \leqq i \leqq n_k} ([KM_{ik}^1] - [M_{ik}^2]) = v[KM], \qquad k = 1, 2.$$

However, KM is simple, say it is a faithful Ae_1-module, e_1 a central primitive idempotent in A. Then we may assume that KM_{ik}^l are Ae_1-modules, for all i, k, l. However, the orders $\{\Gamma_{ij}\}$ are maximal, and for the sake of simplicity, we shall assume that A is simple. But then there exists (up to isomorphism) only one irreducible $\Gamma_i(\mathfrak{p}_k)$-lattice. Moreover, by adding and subtracting suitable elements in $\mathfrak{G}_0(\Gamma_i(\mathfrak{p}_k))$, we may assume that

$$\sum_{1 \leqq i \leqq n_1} [KM_{i1}^s] = \sum_{1 \leqq i \leqq n_2} [KM_{i2}^s], \qquad s = 1, 2.$$

Therefore it remains to show the following: Let $M_i(\mathfrak{p}_k)$ be the irreducible $\Gamma_i(\mathfrak{p}_k)$-lattice; then for every pair $\{M_i(\mathfrak{p}_1), M_j(\mathfrak{p}_2)\}$, there exists a Γ_{ij}-lattice N such that $N_{\mathfrak{p}_1} \cong M_i(\mathfrak{p}_1)$ and $N_{\mathfrak{p}_2} \cong M_j(\mathfrak{p}_2)$. But

$$N = \left(\bigcap_{\mathfrak{q} \in \max(R)\setminus\mathfrak{S}_0} M(\mathfrak{q})\right) \cap M_i(\mathfrak{p}_1) \cap M_j(\mathfrak{p}_2),$$

where $M(\mathfrak{q})$ is the irreducible $\Gamma_{\mathfrak{q}}$-lattice, has the desired properties. Hence we have shown that

$$v\alpha\mathfrak{G}_0(\Lambda) \subset \operatorname{Im}\left(\bigoplus_{\mathfrak{p} \in \mathfrak{S}_0} \varphi_{\mathfrak{p}}\right)\left(\bigoplus_{i=1}^{s} \alpha_i\right).$$

To complete the proof, we consider the commutative diagram (cf. [**11**])

$$\begin{tikzcd}
\bigoplus_{i=1}^{s} \bigoplus_{\mathfrak{q}\notin\mathfrak{S}_0} \mathfrak{G}_0(\Gamma_i/\mathfrak{q}\Gamma_i) \arrow[r, "\alpha"] \arrow[d, "\bigoplus_{\mathfrak{q}\notin\mathfrak{S}_0} \tau_{\mathfrak{q}}"'] & \bigoplus_{i=1}^{s} \mathfrak{G}_0(\Gamma_i) \arrow[r] \arrow[d, "\varphi"'] & \bigoplus_{\mathfrak{p}\in\mathfrak{S}_0} \left(\bigoplus_{i=1}^{s} \mathfrak{G}_0(\Gamma_{i\mathfrak{p}}) \right) \arrow[d, "\bigoplus_{\mathfrak{p}\in\mathfrak{S}_0} \varphi_{\mathfrak{p}}"] \\
\bigoplus_{\mathfrak{q}\notin\mathfrak{S}_0} \mathfrak{G}_0(\Lambda/\mathfrak{q}\Lambda) \arrow[r, "\beta"] & \mathfrak{G}_0(\Lambda) \arrow[r, "\gamma"] & \bigoplus_{\mathfrak{p}\in\mathfrak{S}_0} \mathfrak{G}_0(\Lambda_{\mathfrak{p}}),
\end{tikzcd}$$

where the rows are exact sequences and the columns are induced from the restriction of the operators. Since $\Lambda/\mathfrak{q}\Lambda \cong \Lambda_{\mathfrak{q}}/\mathfrak{q}\Lambda_{\mathfrak{q}} \cong \Gamma_{i\mathfrak{q}}/\mathfrak{q}\Gamma_{i\mathfrak{q}} \cong \Gamma_i/\mathfrak{q}\Gamma_i$, for $\mathfrak{q} \notin \mathfrak{S}_0$, we see that $\tau_{\mathfrak{q}}$ is an isomorphism for every $\mathfrak{q} \notin \mathfrak{S}_0$. Given now $v[M] \in \mathfrak{G}_0(\Lambda)$. Then we have shown above that there exists $x \in \bigoplus_{i=1}^{s} \mathfrak{G}(\Gamma_i)$ such that $\varphi x - v[M] \in \operatorname{Ker}\gamma = \operatorname{Im}\beta$. Since $\bigoplus_{\mathfrak{q}\notin\mathfrak{S}_0} \tau_{\mathfrak{q}}$ is an epimorphism, we have $y \in \bigoplus_{\mathfrak{q}\notin\mathfrak{S}_0} \mathfrak{G}_0(\Gamma_i/\mathfrak{q}\Gamma_i)$ such that $\varphi x - v[M] = \varphi\alpha y$; i.e.,

$$v[M] = \varphi(x - \alpha y) \in \operatorname{Im}\varphi.$$

THEOREM 5. *Let Λ be a Bass order in A and $\{\Gamma_i\}$ $(1 \leqq i \leqq s)$ the different maximal R-orders in A containing Λ. In the homomorphism*

$$\varphi: \bigoplus_{i=1}^{s} \mathfrak{G}_0(\Gamma_i) \to \mathfrak{G}_0(\Lambda),$$

we have $\operatorname{Im}\varphi \supset 2v\,\mathfrak{G}_0(\Lambda)$. *Moreover, if for every $\mathfrak{p} \in \mathfrak{S}_0$, no ring direct summand of $\Lambda_{\mathfrak{p}}$ is Morita-equivalent to a Bass order of type* (IIIb), *then $2v$ can be replaced by v. If A is commutative, then $v = 1$.*

Proof. This follows immediately from the previous theorems. We *remark* that in case φ is an epimorphism, $\mathfrak{G}_0(\Lambda)$ can be computed explicitly, using a technique developed by Heller and Reiner [**5**].

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