THE SUPREMUM OF A FAMILY OF ADDITIVE FUNCTIONS

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Summary. Any system S in which an addition is defined for some, but not necessarily all, pairs of elements can be imbedded in a natural way in a commutative semi-group G, although different elements in S need not always determine different elements in G (see § 2). Theorem 2.1 gives necessary and sufficient conditions in order that a functional $p(x)$ on S can be represented as the supremum of some family of additive functionals on S, and one such set of conditions is in terms of possible extensions of $p(x)$ to G. This generalizes the case with S a Boolean ring treated by Lorentz [**4**]. Lorentz imbeds the Boolean ring in a vector space and this could be done for the general S; but we prefer to imbed S in a commutative semi-group and to give a proof (see § 1) generalizing the classical Hahn-Banach theorem to the case of an arbitrary commutative semi-group.

In § 3, S is specialized to be a relatively complemented modular lattice with zero element in which perspectivity is assumed transitive. Lemmas concerning simultaneous decompositions of several elements in S are proved which enable a certain relation in G to be described in terms of canonical decompositions in S (see Theorem 3.1). Theorem 2.1 can then be given in a more direct form for this special case generalizing the concept of "covered m times" given by Lorentz [**4**] for a Boolean ring.

1. The Hahn-Banach theorem for semi-groups. The theorem of Hahn-Banach concerning the extension of a linear functional [**1**, pp. 27-29] assumes a linear vector space. We establish now a general form of this theorem which includes the case of an arbitrary commutative group or semi-group.

T will denote an arbitrary set of real numbers t which includes the positive integers and the sum and product of any two of its elements.

A set G of elements $x, y, z, \ldots$ will be called a T-semi-group (in place of T-commutative-semi-group) if (i) $z_1 + z_2$ is defined and in G for all z_1, z_2 in G and the commutative and associative laws hold, (ii) tz is defined and in G for all z in G and t in T and the following identities hold:

$$t(z_1 + z_2) = tz_1 + tz_2, \qquad (t_1 + t_2)z = t_1z + t_2z,$$

$$t_1(t_2z) = (t_1t_2)z, \qquad 1z = z.$$

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In this paper function will mean one which is single-valued and has values which are finite real numbers.

A function $f(z)$ on G will be called T-additive if

$$f(z_1 + z_2) = f(z_1) + f(z_2) \qquad \text{for all } z_1, z_2 \text{ in } G, \tag{1.1}$$

$$f(tz) = t f(z) \qquad \text{for all } z \text{ in } G \text{ and } t \text{ in } T. \tag{1.2}$$

A function $p(z)$ on G will be called T-subadditive if

$$p(z_1 + z_2) \leqslant p(z_1) + p(z_2) \qquad \text{for all } z_1, z_2 \text{ in } G, \tag{1.3}$$

$$p(tz) \leqslant tp(z) \qquad \text{for all } z \text{ in } G, t \text{ in } T, t > 0. \tag{1.4}$$

In the above nomenclature the letter T may be omitted when T consists precisely of all positive integers.

Suppose now that G, G_1 are T-semi-groups with G_1 contained in G, that x_0 is in G, and that G^* consists of all y which possess a representation of at least one of the forms

$$y = x + tx_0, \tag{1.5}$$
$$y = tx_0, \tag{1.6}$$
$$y = x, \tag{1.7}$$

with x in G_1 and t in T. Suppose $h(z)$ is an arbitrary function on G and $f(x)$ a T-additive function on G_1. A generalization to this situation of the Hahn-Banach extension lemma is given by the following theorem.

THEOREM 1.1. *Suppose that there is a function $M(u)$ on G such that*

$$f(x_2) + \sum_{i=1}^{m} (\beta_i - \alpha_i)h(z_i) \geqslant f(x_1) + \sum_{j=1}^{n} (t_{1j} - t_{2j})M(u_j) \tag{1.8}$$

whenever, for arbitrary positive integers m, n,

$$x_1 + \sum_{j=1}^{n} t_{1j}u_j + \sum_{i=1}^{m} \alpha_i z_i = x_2 + \sum_{j=1}^{n} t_{2j}u_j + \sum_{i=1}^{m} \beta_i z_i \tag{1.9}$$

with x_1, x_2 in G_1, all u_j and z_i in G, all t_{1j}, t_{2j}, α_i, β_i in T, $t_{1j} \geqslant t_{2j}$ for all j, and $\alpha_i \leqslant \beta_i$ for all i. Then there exists a T-additive function $\phi(y)$ on G^*, which coincides with $f(x)$ on G_1, such that (1.8) *holds, with the same $M(u)$, when f, G_1 are replaced by ϕ, G^* respectively.*

Discussion of condition (1.8). The special case of (1.8) with $t_{1j} = t_{2j}$ for all j, can be stated as follows:

$$f(x_1) \leqslant f(x_2) + \sum_{i=1}^{m} (\beta_i - \alpha_i)h(z_i) \tag{1.10}$$

whenever

$$x_1 + \sum_{i=1}^{m} \alpha_i z_i = x_2 + \sum_{i=1}^{m} \beta_i z_i$$

with $\alpha_i \leqslant \beta_i$ for all i.

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This condition (1.10) actually implies the existence of a function $M(u)$ for which (1.8) holds, *if T contains at least one negative number* $-\tau$, $\tau > 0$. Indeed, from (1.9) we obtain, using arbitrary integers $p > 0$, $q_j \geqslant 0$,

$$p\left[x_1 + \sum_{j=1}^{n} t_{1j}u_j + \sum_{i=1}^{m} \alpha_i z_i\right] + \sum_{j=1}^{n} 1[-\tau u_j] + \sum_{j=1}^{n} q_j[-\tau u_j]$$
$$= p\left[x_2 + \sum_{j=1}^{n} t_{2j}u_j + \sum_{i=1}^{m} \beta_i z_i\right] + \sum_{j=1}^{n} (q_j + 1)[-\tau u_j],$$

the term $q_j\,[-\tau\, u_j]$ to be considered absent if $q_j = 0$. Hence

$$px_1 + \sum_{j=1}^{n} (pt_{1j} - q_j\tau)u_j + \sum_{j=1}^{n} 1[-\tau u_j] + \sum_{i=1}^{m} (p\alpha_i)z_i$$
$$= px_2 + \sum_{j=1}^{n} (pt_{2j})u_j + \sum_{j=1}^{n} (q_j + 1)[-\tau u_j] + \sum_{i=1}^{m} (p\beta_i)z_i.$$

The integers p, q_j can be chosen so that for every j,

$$2p(t_{1j} - t_{2j}) \geqslant q_j\tau \geqslant p(t_{1j} - t_{2j});$$

then (1.10) applies and yields

$$pf(x_1) \leqslant pf(x_2) + \sum_{j=1}^{n} [q_j\tau - p(t_{1j} - t_{2j})]h(u_j) + \sum_{j=1}^{n} q_j h(-\tau u_j)$$
$$+ \sum_{i=1}^{m} p(\beta_i - \alpha_i)h(z_i).$$

Hence

$$f(x_2) + \sum_{i=1}^{m} (\beta_i - \alpha_i)h(z_i) \geqslant f(x_1) - \sum_{j=1}^{n} (t_{1j} - t_{2j})\left(\frac{q_j\tau}{p(t_{1j} - t_{2j})} - 1\right)h(u_j)$$
$$- \sum_{j=1}^{n} (t_{1j} - t_{2j})\frac{q_j}{p(t_{1j} - t_{2j})}h(-\tau u_j)$$

so that (1.8) holds with $M(u) = -\,|\,h\,(u)\,| - (2/\tau)\,|\,h\,(-\tau\, u)\,|$.

Thus, in the classical Hahn-Banach lemma, where T includes all real numbers, the function $M(u)$ does not have to be mentioned explicitly in the hypotheses. In the case of a T-semi-group with T containing non-negative numbers only, condition (1.8), and the extension theorem, too, may fail even though (1.10) is valid. An example of this is given below (Example 1).

We note that (1.10), and hence (1.8) too, include the restriction

(1.11) $$f(x_1) = f(x_2)$$

whenever $x_1 + z = x_2 + z$ with z in G. Also, the choice $x_1 = x + x$, $x_2 = x$, $m = 1$, $z_1 = x$, $\alpha_1 = 1$, $\beta_1 = 2$ shows that (1.10) includes the condition

(1.12) $$f(x) \leqslant h(x) \qquad \text{for all } x \text{ in } G_1.$$

If T contains $t_1 - t_2$ whenever it contains t_1, t_2 with $t_1 > t_2$, the condition (1.8) simplifies to

(1.13) $$f(x_2) + \sum_{i=1}^{m} \gamma_i h(z_i) \geqslant f(x_1) + \sum_{j=1}^{n} t_j M(u_j)$$

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whenever, for arbitrary non-negative integers m, n,

$$x_1 + \sum_{j=1}^{n} t_j u_j + v = x_2 + \sum_{i=1}^{m} \gamma_i z_i + v \tag{1.14}$$

with x_1, x_2 in G_1, all t_j, γ_i in T and > 0, v and all u_j, z_i in G (the terms

$$\sum_{i=1}^{m} \gamma_i h(z_i), \quad \sum_{j=1}^{n} t_j M(u_j)$$

to be replaced by 0 when m, n, respectively, take the value 0). For such T, if $h(z)$ happens to be T-subadditive, it is sufficient that there be a function $M(u)$ with the properties

$$M(tu) \geqslant tM(u) \qquad \text{for all } u \text{ in } G,\ t \text{ in } T,\ t > 0, \tag{1.15}$$

$$M(u_1 + u_2) \geqslant M(u_1) + M(u_2) \qquad \text{for all } u_1\ u_2 \text{ in } G, \tag{1.16}$$

such that

$$f(x_2) + h(z) \geqslant f(x_1) + M(u) \tag{1.17}$$

whenever $x_1 + u + v = x_2 + z + v$ with x_1, x_2 in G_1 and u, z, v in G (the terms $h(z)$, $M(u)$ to be replaced by 0 if z, u respectively are absent in the equality). Finally, for such T, if T contains at least one negative number and $h(z)$ happens to be T-subadditive, it is sufficient, without postulating the function $M(u)$, that

$$f(x_1) \leqslant f(x_2) + h(z) \tag{1.18}$$

whenever $x_1 + v = x_2 + z + v$ with x_1, x_2 in G_1 and z, v in G ($h(z)$ to be replaced by 0 if z is absent in the equality).

Proof of Theorem 1.1. Consider separately two cases.

Case 1. For some λ_1, λ_2 in T with $\lambda_1 \neq \lambda_2$ and for some g_1, g_2 in G_1 and v in G,

$$\lambda_1 x_0 + g_1 + v = \lambda_2 x_0 + g_2 + v = w, \tag{1.19}$$

say. We may suppose $\lambda_1 > \lambda_2$. Set $r_0 = [f(g_2) - f(g_1)]/(\lambda_1 - \lambda_2)$ and define

$$\begin{aligned} \phi(y) &= f(x) + tr_0 && \text{if } y \text{ is given by (1.5),} \\ \phi(y) &= tr_0 && \text{if } y \text{ is given by (1.6),} \\ \phi(y) &= f(x) && \text{if } y \text{ is given by (1.7).} \end{aligned} \tag{1.20}$$

That this ϕ is single-valued and satisfies (1.8) on G^* can be seen as follows: suppose, corresponding to (1.9),

$$y_1 + \sum_{j=1}^{n} t_{1j} u_j + \sum_{i=1}^{m} \alpha_i z_i = y_2 + \sum_{j=1}^{n} t_{2j} u_j + \sum_{i=1}^{m} \beta_i z_i \tag{1.21}$$

with y_1, y_2 in G^*. If $y_1 = x_1 + t_1 x_0$ and $y_2 = x_2 + t_2 x_0$ we multiply (1.21) by λ_1 and by λ_2 and combine to obtain

$$\lambda_1\left(x_1 + t_1 x_0 + \sum_{j=1}^{n} t_{1j} u_j + \sum_{i=1}^{m} \alpha_i z_i\right) + \lambda_2\left(x_2 + t_2 x_0 + \sum_{j=1}^{n} t_{2j} u_j + \sum_{i=1}^{m} \beta_i z_i\right) + (t_1 + t_2)(g_1 + g_2 + v)$$

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$$= \lambda_1\left(x_2 + t_2x_0 + \sum_{j=1}^{n} t_{2j}u_j + \sum_{i=1}^{m} \beta_i z_i\right) + \lambda_2\left(x_1 + t_1x_0 + \sum_{j=1}^{n} t_{1j}u_j + \sum_{i=1}^{m} \alpha_i z_i\right) + (t_1 + t_2)(g_1 + g_2 + v),$$

that is,

$$\lambda_1x_1 + \lambda_2x_2 + t_1g_2 + t_2g_1 + \sum_{j=1}^{n} (\lambda_1 t_{1j} + \lambda_2 t_{2j})u_j + \sum_{i=1}^{m}(\lambda_1\alpha_i + \lambda_2\beta_i)z_i + (t_1 + t_2)w$$
$$= \lambda_1x_2 + \lambda_2x_1 + t_1g_1 + t_2g_2 + \sum_{j=1}^{n} (\lambda_1 t_{2j} + \lambda_2 t_{1j})u_j + \sum_{i=1}^{m} (\lambda_1\beta_i + \lambda_2\alpha_i)z_i + (t_1 + t_2)w.$$

Now (1.8) for f on G_1 applies and gives

$$f(\lambda_1x_2 + \lambda_2x_1 + t_1g_1 + t_2g_2) + \sum_{i=1}^{m} (\lambda_1 - \lambda_2)(\beta_i - \alpha_i)h(z_i) \leqslant f(\lambda_1x_1 + \lambda_2x_2 + t_1g_2 + t_2g_1) + \sum_{j=1}^{n} (\lambda_1 - \lambda_2)(t_{1j} - t_{2j})M(u_j).$$

From this follows at once

$$\phi(y_2) + \sum_{i=1}^{m} (\beta_i - \alpha_i)h(z_i) \geqslant \phi(y_1) + \sum_{j=1}^{n} (t_{1j} - t_{2j})M(u_j). \tag{1.22}$$

Similar reasoning shows that (1.21) implies (1.22) if y_1, y_2 have representations of any of the forms (1.5), (1.6), (1.7). This implies that ϕ is single-valued and satisfies (1.8) on G^*. It is evident that ϕ is T-additive and coincides with f on G_1, so that Theorem 1.1 is proved for Case 1.

Case 2. In every relation of the form (1.19), $\lambda_1 = \lambda_2$. Then, with a number r_0 to be assigned later, we define $\phi(y)$ as in (1.20). Irrespective of the value of r_0, this ϕ is single-valued on G^*. For suppose $y_1 = y_2$. If $y_1 = x_1 + t_1x_0$ and $y_2 = x_2 + t_2x_0$ then $t_1x_0 + x_1 + v = t_2x_0 + x_2 + v$ for any v in G, hence (this is Case 2) $t_1 = t_2$ and, using (1.11), $f(x_1) = f(x_2)$, $\phi(y_1) = \phi(y_2)$. Similar reasoning applies if y_1, y_2 have representations of any of the forms (1.5), (1.6), (1.7) to show that ϕ is single-valued on G^*. It is evident that ϕ is T-additive and coincides with f on G_1.

Thus we need only show that an r_0 exists for which (1.21) implies (1.22) with arbitrary y_1, y_2 in G^*. It is easily seen that it is sufficient to do this for the y_1, y_2 with representations $y_1 = x_1 + t_1x_0$, $y_2 = x_2 + t_2x_0$ with $t_1 \neq t_2$. There are therefore two conditions to satisfy, according as $t_1 > t_2$ or $t_2 > t_1$. Explicitly, we require (use a bar to distinguish the two possibilities),

$$\frac{-1}{(\bar{t}_2 - \bar{t}_1)}\left[f(\bar{x}_2) - f(\bar{x}_1) + \sum_{i=1}^{\bar{m}} (\bar{\beta}_i - \bar{\alpha}_i)h(\bar{z}_i) - \sum_{j=1}^{\bar{n}} (\bar{t}_{1j} - \bar{t}_{2j})M(\bar{u}_j)\right] \leqslant r_0 \tag{1.23}$$

whenever

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$$(1.24)\quad \begin{cases} \bar{t}_2 > \bar{t}_1 \\ \bar{x}_1 + \bar{t}_1 x_0 + \sum_{j=1}^{\bar{n}} \bar{t}_{1j}\bar{u}_j + \sum_{i=1}^{\bar{m}} \bar{\alpha}_i \bar{z}_i = \bar{x}_2 + \bar{t}_2 x_0 + \sum_{j=1}^{\bar{n}} \bar{t}_{2j}\bar{u}_j + \sum_{i=1}^{\bar{m}} \bar{\beta}_i \bar{z}_i, \end{cases}$$

and

$$(1.25)\quad r_0 \leqslant \frac{1}{(t_1 - t_2)}\left[f(x_2) - f(x_1) + \sum_{i=1}^{m} (\beta_i - \alpha_i) h(z_i) - \sum_{j=1}^{n} (t_{1j} - t_{2j}) M(u_j)\right]$$

whenever

$$(1.26)\quad \begin{cases} t_1 > t_2 \\ x_1 + t_1 x_0 + \sum_{j=1}^{n} t_{1j} u_j + \sum_{i=1}^{m} \alpha_i z_i = x_2 + t_2 x_0 + \sum_{j=1}^{n} t_{2j} u_j + \sum_{i=1}^{m} \beta_i z_i. \end{cases}$$

That {L.H.S. of (1.23)} $\leqslant$ {R.H.S. of (1.25)} follows from (1.24) and (1.26), using (1.8) for f on G_1. Hence

$$\sup\ \{\text{L.H.S. of } (1.23)\} \leqslant \inf\{\text{R.H.S. of } (1.25)\}$$

showing that r_0 exists, as required, *if there are realizations of* (1.24) *and* (1.26). Now there are realizations of (1.26), for example: $x_1 = x_2$ (an arbitrary element in G_1),

$t_1 = 2,\quad t_2 = 1,\quad n = m = 1,\quad u_1 = z_1 = x_0,\quad t_{11} = t_{21} = 1,\quad \alpha_1 = 1,\quad \beta_1 = 2.$

There are also realizations of (1.24) (it was to ensure this that the function $M(u)$ was postulated[1]), for example: $x_1 = x_2$ (an arbitrary element in G_1),

$\bar{t}_1 = 1,\quad \bar{t}_2 = 2,\quad \bar{n} = \bar{m} = 1,\quad \bar{u}_1 = \bar{z}_1 = x_0,\quad \bar{t}_{11} = 2,\quad \bar{t}_{21} = 1,\quad \bar{\alpha}_1 = \bar{\beta}_1 = 1.$

This proves Theorem 1.1 for Case 2 and completes the proof of the theorem.

COROLLARY. *Under the conditions of Theorem* 1.1 *the T-additive function* $f(x)$ *can be extended by transfinite induction to a T-additive function* $\phi(z)$ *on G such that* (*use* (1.8) *for* ϕ *on G*) $M(z) \leqslant \phi(z) \leqslant h(z)$ *for all z in G.*

THEOREM 1.2. *Let* $h(z)$ *be a function on a T-semi-group G such that, for some function* $M(u)$,

$$(1.27)\qquad (t_2 - t_1)h(z) + \sum_{i=1}^{m} (\beta_i - \alpha_i) h(z_i) \geqslant \sum_{j=1}^{n} (t_{1j} - t_{2j}) M(u_j)$$

whenever, for arbitrary positive integers m, n,

$$(1.28)\qquad t_1 z + \sum_{j=1}^{n} t_{1j} u_j + \sum_{i=1}^{m} \alpha_i z_i = t_2 z + \sum_{j=1}^{n} t_{2j} u_j + \sum_{i=1}^{m} \beta_i z_i,$$

with z, all u_j, *all* z_i *in G,* t_1, t_2, *all* t_{1j}, t_{2j}, α_i, β_i *in T,* $t_{1j} \geqslant t_{2j}$, $\alpha_i \leqslant \beta_i$. *Then for arbitrary* (*but fixed*) x_0 *in G there is a T-additive function* $\phi(z)$ *on G with* $\phi(x_0) = h(x_0)$ *and* $M(z) \leqslant \phi(z) \leqslant h(z)$ *for all z in G.*

[1]In the classical Hahn-Banach theorem for linear vector spaces, $h(z)$ is a subadditive function $p(z)$ with $p(tz) = tp(z)$ for all $t > 0$ and $-p(-u)$ acts as the function $M(u)$ which we postulated explicitly. G. G. Lorentz has independently had the idea of investigating extensions of an additive $f(x)$ satisfying $q(x) \leqslant f(x) \leqslant p(x)$ for given subadditive $p(z)$ and superadditive $q(z)$.

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Remark. The hypotheses imply:

(1.29) *$h(z)$ is T-subadditive and $h(tz) = th(z)$ for all z in G, t in T, $t > 0$,*

(1.30) *$h(z_1) = h(z_2)$ whenever $z_1 + v = z_2 + v$ with z_1, z_2, v in G.*

Proof. Let G_1 be the T-semi-group of all tx_0 with t in T and define $f(x)$, T-additive on G_1, by $f(tx_0) = th(x_0)$. This f is single-valued, for if $t_1x_0 = t_2x_0$ the hypotheses of the theorem imply that $t_1h(x_0) = t_2h(x_0)$. It is also evident from (1.27) that (1.9) implies (1.8) in the present situation. Thus Theorem 1.1 applies to extend f to a ϕ with the required properties.

THEOREM 1.3. *The hypotheses of Theorem 1.2 are necessary and sufficient in order that $h(z)$ admit a representation*

$$h(z) = \sup\{\phi_\lambda(z)\} \tag{1.31}$$

with a family of T-additive functions ϕ_λ for which inf $\{\phi_\lambda(u)\}$ *is finite for every u in G.*

Proof. The hypotheses of Theorem 1.2 imply a representation (1.31), in fact with $h(z) = \max\{\phi(z)\}$ for a family of T-additive $\phi(z)$ with $M(u) \leqslant \phi(u)$ for all ϕ in the family and all u in G.

Conversely, if there is a representation (1.31), then for each λ,

$$t_1\phi_\lambda(z) + \sum_{j=1}^{n} t_{1j}\phi_\lambda(u_j) + \sum_{i=1}^{m} \alpha_i\phi_\lambda(z_i) = t_2\phi_\lambda(z) + \sum_{j=1}^{n} t_{2j}\phi_\lambda(u_j) + \sum_{i=1}^{m} \beta_i\phi_\lambda(z_i).$$

Hence (1.27) holds with $M(u) = \inf\{\phi_\lambda(u)\}$.

COROLLARY 1. *If $h(z)$ admits a representation* (1.31) *it admits such a representation with* sup *replaced by* max *(possibly with a different family of T-additive functions ϕ_λ).*

COROLLARY 2. *The $M(u)$ in* (1.27) *may be restricted to functions satisfying* (1.15), (1.16).

THEOREM 1.4. *If T contains $t_1 - t_2$ whenever it contains t_1, t_2 with $t_1 > t_2$, then necessary and sufficient conditions that $h(z)$ admit a representation* (1.31) *are:* (1.29) *and*

(1.32) *for some $M(u)$ satisfying* (1.15), (1.16), *$h(z_1) \geqslant h(z_2) + M(u)$ whenever $z_1 + v = z_2 + u + v$ (with $h(z_1)$, $h(z_2)$, $M(u)$ replaced by* 0 *if z_1, z_2, u respectively are absent in the equality).*

Proof. This follows easily from Theorem (1.3).

Remark. For the particular case when T consists precisely of all positive integers, (1.29) can be replaced by

$$h(z_1 + z_2) \leqslant h(z_1) + h(z_2) \qquad \text{for all } z_1, z_2 \text{ in } G, \tag{1.33}$$

$$h(z + z) = h(z) + h(z) \qquad \text{for all } z \text{ in } G. \tag{1.34}$$

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To prove this we need only show that (1.33) implies $h(nz) = nh(z)$ for all positive integers n. But repeated applications of (1.33) give $h(nz) \leqslant nh(z)$ and repeated applications of (1.34) give $h(2^m z) = 2^m h(z)$. By choosing $2^m > n$ we obtain

$$h(2^m z) \leqslant h((2^m - n)z) + h(nz)$$

by (1.33), and hence

$$2^m h(z) \leqslant (2^m - n)h(z) + h(nz),$$

from which follows $nh(z) \leqslant h(nz)$ and therefore $h(nz) = nh(z)$, as required.

THEOREM 1.5. *If T contains at least one negative number $-\tau$, $\tau > 0$, then necessary and sufficient conditions that $h(z)$ admit a representation* (1.31), *whether* inf $\{\phi_\lambda(u)\}$ *is required to be finite or not, are the same, namely*:

$$(t_1 - t_2)h(z) \leqslant \sum_{i=1}^{m} (\beta_i - \alpha_i)h(z_i)$$

whenever, for a positive integer m,

$$t_1 z + \sum_{i=1}^{m} \alpha_i z_i = t_2 z + \sum_{i=1}^{m} \beta_i z_i,$$

with z, all z_i in G, t_1, t_2 all α_i, β_i in T, $\alpha_i \leqslant \beta_i$.

Proof. The methods used on page 465 in the discussion of condition (1.8), Theorem 1.1, show that, with the present hypotheses, (1.28) implies (1.27) if $M(u)$ is taken as $-|h(u)| - (2/\tau)|h(-\tau u)|$.

COROLLARY. *If T contains $t_1 - t_2$ whenever it contains t_1, t_2 with $t_1 > t_2$ and T also contains at least one negative number, then necessary and sufficient conditions that $h(z)$ admit a representation* (1.31) *are*:

$$h(z_1 + z_2) \leqslant h(z_1) + h(z_2) \qquad \text{for all } z_1, z_2 \text{ in } G,$$
$$h(tz) = th(z) \qquad \text{for all } z \text{ in } G,\ t \text{ in } T,\ t > 0,$$
$$h(z_1) = h(z_2) \qquad \text{whenever } z_1 + v = z_2 + v,\ z_1, z_2, v \text{ in } G.$$

The following examples show the necessity of postulating the function $M(u)$ in Theorem 1.1, and the finiteness of inf $\{\phi_\lambda(u)\}$ in the representation (1.31).

Example 1. T consists of all real non-negative numbers; G consists of all two-dimensional vectors $[a_1,a_2]$ with $a_1 \geqslant 0$, $a_2 \geqslant 0$; G_1 consists of all $[a_1,0]$ and $x_0 = [0,1]$; $h[a_1,a_2] = a_2$ if $a_2 > 0$ and $h[a_1,a_2] = a_1$ if $a_2 = 0$; $f[a_1,0] = a_1$.

Then f is T-additive on G_1 and condition (1.10) is satisfied. But for any T-additive extension ϕ of f ($G^* = G$ in this example) and for every positive integer n, $\phi[n,1] = n + \phi(x_0)$, whereas $h[n,1] = 1$ so that there is no such ϕ with $\phi[n,1] \leqslant f[n,1]$ for all n. Thus Theorem 1.1 cannot be proved on the basis of (1.10) alone.

In this example h is T-subadditive and satisfies (1.29), (1.30), yet h does not admit a representation (1.31) (even with inf $\{\phi_\lambda(u)\}$ unrestricted). For if ϕ

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is T-additive and $\phi[a_1,a_2] \leqslant h[a_1,a_2]$ then

$$\phi[n,1] = n\phi[1,0] + \phi[0,1] \leqslant 1 \qquad \text{for all } n;$$

hence $\phi[1,0] \leqslant 0$ for all such ϕ, whereas $h[1,0] = 1$.

Example 2. T consists of all non-negative integers; G consists of all infinite-dimensional vectors $a = (a_0,a_1,\ldots,a_m,\ldots)$ with every a_m a non-negative integer and at most a finite number of a_m different from 0; $h(a) = \max\ \{m(a_m - a_0)\}$.

Then $h(a) = \sup\{\phi_\lambda(a)\}$ with $\phi_\lambda(a)$ the T-additive function $\lambda\ (a_\lambda - a_0)$ ($\lambda = 0,1,2,\ldots$). Nevertheless h does not admit a representation (1.31) with $\inf\{\phi_\lambda(a)\}$ finite. To see this, let a^n denote the vector with $(a^n)_m = 0$ for $m \neq n$ and $(a^n)_m = 1$ for $m = n$. If $\phi(a)$ is a T-additive function with $\phi(a) \leqslant h(a)$ for all a then

$$\phi(a^n) + \phi(a^0) = \phi(a^n + a^0) \leqslant h(a^n + a^0) = 0.$$

Hence if $h(a^n) = \sup\ \{\phi(a^n)\}$ for every n it would follow that $\inf\ \{\phi(a^0)\} \leqslant -n$ for every n.

An elegant generalization (in a different way) of the classical Hahn-Banach theorem has been given by Hidegoro Nakano [**5**, pp. 89-91]. Nakano deals with a linear vector space, that is, with all real numbers as scalar multipliers, but for given $h(z)$ and x_0, the requirements that there shall be a T-additive ϕ with $\phi(x_0) = h(x_0)$, $\phi(z) \leqslant h(z)$ for all z, are replaced by the requirements that there shall be a T-additive ϕ with $\phi(z) \leqslant \phi(x_0) - h(x_0) + h(z)$ for all z.

Theorems 1.1 to 1.5 of the present paper can be extended to include Nakano's generalization.

THEOREM 1.6. *In order that* $h(z)$ *admit a representation*

$$h(z) = \sup\{A_\lambda + \phi_\lambda(z)\} \tag{1.35}$$

with a family of T*-additive* ϕ_λ *and constants* A_λ *for which* $|A_\lambda| \leqslant K < \infty$ *for all* λ *and* $\inf\{\phi_\lambda(u)\}$ *is finite for every* u *in* G, *it is necessary and sufficient that functions* $A(u)$, $M(u)$ *exist with* $|A(u)| \leqslant K$ *for all* u *and*

$$\begin{aligned}(t_2 - t_1)h(z) &+ \sum_{i=1}^{m} (\beta_i - \alpha_i)h(z_i) \\ &\geqslant \sum_{j=1}^{n} (t_{1j} - t_{2j})M(u_j) + \left(t_2 - t_1 + \sum_{i=1}^{m} (\beta_i - \alpha_i) - \sum_{j=1}^{n} (t_{1j} - t_{2j})\right)A(z)\end{aligned} \tag{1.36}$$

whenever (1.28) *holds.*

Proof. If (1.28) implies (1.36), the argument used in the proof of Theorem 1.2 shows that for every x_0 in G there is a T-additive ϕ_0 such that $\phi_0(x_0) = h(x_0) - A(x_0)$ and

$$M(z) - A(x_0) \leqslant \phi_0(z) \leqslant h(z) - A(x_0)$$

for all z in G. Hence (1.35) holds with these functions $A(x_0) + \phi_0(z)$.

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Conversely if (1.35) does hold then (1.28) implies (1.36) if $M(u)$ is taken to be $\inf\{A_\lambda + \phi_\lambda(u)\}$ and $A(z)$ is taken to be the limit of A_{λ_n} for any sequence of λ_n for which $A_{\lambda_n} + \phi_{\lambda_n}(z)$ converges to $h(z)$ and A_{λ_n} converges, as n becomes infinite.

Remark. If $h(z)$ admits a representation (1.35) then h is T-convex, that is,

$$h(\alpha x + (1-\alpha)y) \leqslant \alpha h(x) + (1-\alpha)h(y) \tag{1.37}$$

whenever x,y are in G and $\alpha, 1-\alpha$ are in T $(0 \leqslant \alpha \leqslant 1)$.

2. Systems S with partially-defined addition operator. Now let S be any system of elements, $a, b, c, \ldots$ with an addition $a \dot{+} b$ defined, and in S, for some, but not necessarily all, ordered pairs a,b in S. No further properties of $\dot{+}$ will be postulated in this section. We shall call a function $\phi(a)$ on S additive if $\phi(a \dot{+} b) = \phi(a) + \phi(b)$ whenever $a \dot{+} b$ is defined.

Let G be the set of all formal sums $x \equiv a_1 + \ldots + a_r$ with an arbitrary (but finite) number of a_i from S, the order being immaterial by definition and with two sums x,y identified in G $(x \equiv y)$ if x can be transformed into y by a finite number of changes of the form: a is replaced by $a_1 + a_2$ or conversely $a_1 + a_2$ is replaced by a if $a_1 \dot{+} a_2 = a$. If $x \equiv a_1 + \ldots + a_r$ and $y \equiv b_1 + \ldots + b_s$, let the definition of $x + y$ in G be

$$x + y \equiv a_1 + \ldots + a_r + b_1 + \ldots + b_s.$$

Then G is a semi-group and each element a in S determines an element $x \equiv a$ in G. We shall say S determines G.

THEOREM 2.1. *A function $p(a)$ on S admits a representation*

$$p(a) = \sup\{\phi_\lambda(a)\} \tag{2.1}$$

(ϕ *additive on* S, $\inf\{\phi_\lambda(a)\}$ *finite for each a in S*) *if and only if it admits a representation*

$$p(a) = \max\{\psi_\lambda(a)\} \tag{2.2}$$

(ψ_λ *additive on* S, $\inf\{\psi_\lambda(a)\}$ *finite for each a in S*) *and if and only if $p(a)$ has an extension $p_1(x)$ defined for all x in the G determined by S so that p_1 satisfies* (1.32), (1.33), *and* (1.34), *and if and only if $p(a)$ has the two properties*:

$$mp(a) \leqslant p(a_1) + \ldots + p(a_r) \tag{2.3}$$

whenever $ma + u \equiv a_1 + \ldots + a_r + u$ in G;

$$\inf\{m^{-1}(p(a_1) + \ldots + p(a_r) - p(b_1) - \ldots - p(b_n))\} > -\infty \tag{2.4}$$

whenever, for fixed $c_1, \ldots, c_s$, the integers m, r, n and the $a_1, \ldots, a_r, b_1, \ldots, b_n$ vary so that

$$m(c_1 + \ldots + c_s) + b_1 + \ldots + b_n \equiv a_1 + \ldots + a_r.$$

(In connection with this theorem see Lorentz [**4**]. The definition of $p_1(x)$ in (2.5) below was suggested by [**4**].)

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Proof. For each additive $\phi(a)$ on S define $\phi_1(x) = \phi(a_1) + \ldots + \phi(a_r)$ if $x \equiv a_1 + \ldots + a_r$. Then ϕ_1 is single-valued and additive on G and is an extension of ϕ. Hence if $p(a)$ does admit a representation (2.1) then the function

$$p_1(x) = \sup\{\phi_{\lambda 1}(x)\}$$

is an extension of $p(a)$ which, by Theorem 1.4, satisfies (1.32), (1.33), (1.34). On the other hand, if $p(a)$ has any extension $p_1(x)$ which satisfies these conditions, then by Theorem 1.4, $p_1(x)$ admits a representation (1.31) on G, which, when considered on S only, gives a representation (2.2) for $p(a)$ on S.

Again if $p_1(x)$ on G satisfies (1.32), (1.33), (1.34), then clearly it satisfies (2.3) and (2.4). If such a $p_1(x)$ is an extension of $p(a)$ then $p(a)$ must satisfy (2.3) and (2.4). Conversely, if $p(a)$ on S satisfies (2.3) and (2.4) we define

$$p_1(x) = \inf\{m^{-1}(p(a_1) + \ldots + p(a_r))\} \tag{2.5}$$

for all $a_1, \ldots, a_r$ with $mx + u \equiv a_1 + \ldots + a_r + u$ for some positive integer m and some u in G. Then (2.3) ensures that $p_1(x)$ is an extension of $p(a)$, (2.4) ensures that $p_1(x)$ has finite real numbers as values, and from (2.5) it follows that (1.32), (1.33), (1.34), with $M(c_1 + \ldots + c_s) =$ L.H.S. of (2.4), hold for $p_1(x)$.

Remark. If the cancellation law, $x + u \equiv y + u$ implies that $x \equiv y$, holds in G, the condition (2.3) is equivalent to the (apparently) weaker condition

$$mp(a) \leqslant p(a_1) + \ldots + p(a_r) \tag{2.6}$$

whenever $ma \equiv a_1 + \ldots + a_r$ in G (see the definition of multiple subadditivity given in [**4**]).

3. Modular lattices with zero and relative complements. Suppose now that S is a modular (but not necessarily distributive) relatively complemented lattice with zero element 0, so that the von Neumann theory of "independence" (or "independence over 0" in terms of [**3**]) is valid at least for finite collections of elements of S [**6**; **7**; **3**, p. 539; **2**, p. 114]. Suppose too that $e_1 \dot{+} e_2$ is identical with lattice union $e_1 \cup e_2$ restricted to independent elements.

We recall that

$$e = \dot{\bigcup_{i=1}^{n}} e_i$$

is called a direct decomposition if $e_1, \ldots, e_n$ are independent and e is called perspective to f (with axis a) written $e \backsim f$, if $e \cup a = f \cup a$ and $e \cap a = f \cap a = 0$ for some a in S.

In what follows we shall postulate that S has the additional property that perspectivity is transitive, that is,

$$e \backsim f, \quad f \backsim g \quad \text{imply} \quad e \backsim g. \tag{3.1}$$

(In a Boolean ring (3.1) holds trivially since $e \backsim f$ implies $e = f$. But (3.1) holds also for the continuous geometries of von Neumann or more generally [**6**; **7**; **3**]

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if S has certain continuity properties.) With the hypothesis (3.1) we shall show, for given $e_1, \ldots, e_n, f_1, \ldots, f_m$ in S, that equality in G,

$$e_1 + \ldots + e_n + h_1 + \ldots + h_p \equiv f_1 + \ldots + f_m + h_1 + \ldots + h_p$$

for some $h_1, \ldots, h_p$ in S, can be expressed in a simple way in terms of direct decompositions of $e_1, \ldots, e_n, f_1, \ldots, f_m$.

LEMMA 1. *Suppose that*

$$e = \bigcup_{i=1}^{n} e_i$$

is a direct decomposition and that $e \backsim f$. *Then there exists a direct decomposition* $f = \bigcup f_i$ *with* $e_i \backsim f_i$ *for each* i.

LEMMA 2. *Suppose that* $e = e_1 \cup e_2, f = f_1 \cup f_2$ *are direct decompositions with* $e \backsim f$ *and* $e_1 \backsim f_1$. *Then* $e_2 \backsim f_2$.

LEMMA 3 (Additivity of perspectivity). *Suppose that*

$$e = \bigcup_{i=1}^{n} e_i \quad \textit{and} \quad f = \bigcup_{i=1}^{n} f_i$$

are direct decompositions with $e_i \backsim f_i$ *for each* i. *Then* $e \backsim f$.

Under stronger assumptions these lemmas were proved in [**7**] but the proofs are valid without change in the present case. Lemma 1 corresponds to a corollary of [**3**, Lemma 3.3] and Lemmas 2 and 3 correspond to [**3**, Lemmas 6.2, 6.4].

LEMMA 4. *Suppose* $f_1, \ldots, f_m$ *and* e *are arbitrary. Then there exist direct decompositions* $f_j = f_{1j} \cup f_j'$, $e = e_1 \cup \ldots \cup e_{m+1}$ *such that* $e_j \backsim f_{1j}$ *for* $1 \leqslant j \leqslant m$ *and* $e_1 \cup \ldots \cup e_m = e \cap (f_1 \cup \ldots \cup f_m)$.

Proof. The lemma can be verified as follows: Let $a_j = f_1 \cup \ldots \cup f_{j-1}$ for $1 < j \leqslant m + 1$ and let $a_1 = 0$. Replacing f_j for $1 < j \leqslant m$ by a complement of $a_j \cap f_j$ with respect to f_j we may, and shall, suppose that $f_1, \ldots, f_m$ are independent. Set $e_1 = e \cap f_1$; for $1 < j \leqslant m$ set e_j equal to a complement of $e \cap a_j$ with respect to $e \cap a_{j+1}$; set e_{m+1} equal to a complement of $e \cap a_{m+1}$ with respect to e; set $f_{11} = e_1$; for $1 < j \leqslant m$ set $f_{1j} = f_j \cap (e_j \cup a_j)$; for $1 \leqslant j \leqslant m$ set f_j' equal to a complement of f_{1j} with respect to f_j.

We shall show that $e_j \backsim f_{1j}$ with axis a_j. This is trivial for $j = 1$ and for $j > 1$ we have

$$\begin{aligned} f_{1j} \cup a_j &= (f_j \cap (e_j \cup a_j)) \cup a_j \\ &= (f_j \cup a_j) \cap (e_j \cup a_j) = e_j \cup a_j \end{aligned}$$

by the modular law and since $e_j \leqslant f_j \cup a_j = a_{j+1}$ and $a_j \leqslant f_j \cup a_j$. On the other hand,

$$f_{1j} \cap a_j = f_{1j} \cap f_j \cap a_j = 0$$

since the $f_1, \ldots, f_j$ are independent and $e_j \cap a_j = e_j \cap (e \cap a_j) = 0$. This proves that $e_j \backsim f_{1j}$. The other parts of the lemma are easily verified.

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LEMMA 5. *Suppose $e_1, \ldots, e_n$ are arbitrary. Then there are independent elements g_j $(j = 1, \ldots, N_n)$ and direct decompositions*

$$e_1 = g_1 \cup \ldots \cup g_{N_1},$$
$$e_2 = g_1^{(2)} \cup \ldots \cup g_{N_1}^{(2)} \cup g_{N_1+1} \cup \ldots \cup g_{N_2},$$
$$\cdot \quad \cdot \quad \cdot$$
$$e_n = g_1^{(n)} \cup \ldots \cup g_{N_{n-1}}^{(n)} \cup g_{N_{n-1}+1} \cup \ldots \cup g_{N_n},$$

such that

$$\bigcup_{j=1}^{N_{r-1}} g_j^{(r)} = e_r \cap (e_1 \cup \ldots \cup e_{r-1}),$$
$$g_j^{(r)} = 0 \quad or \quad g_j^{(r)} \backsim g_j,$$

for $1 < r \leqslant n$ and $1 \leqslant j \leqslant N_{r-1}$.

Proof. This lemma can be verified by induction on n, using Lemma 4.

LEMMA 6 (Superposition of decompositions). *Suppose that*

$$e = \bigcup_{i=1}^{n} e_i \quad and \quad f = \bigcup_{j=1}^{m} f_j$$

are direct decompositions and that $e \backsim f$. Then there exist direct decompositions $e_1 = \bigcup e_{ij}$, $f_j = \bigcup f_{ij}$ such that $e_{ij} \backsim f_{ij}$ for all i,j.

Proof. We shall assume, as we clearly may by Lemma 1 and the transitivity of perspectivity, that $e = f$. Apply Lemma 4 to $f_1, \ldots, f_m$ and e_1 (in place of e) and obtain the direct decompositions

$$e_1 = \bigcup_{j=1}^{m} e_{1j}, \quad f_j = f_{1j} \cup f_j' \text{ with } e_{1j} \backsim f_{1j}.$$

By Lemma 3, $e_1 \backsim \bigcup f_{1j}$ and hence by Lemma 2 $(e_2 \cup \ldots \cup e_n) \backsim \bigcup f_j'$. This means that the lemma for n has been reduced to the lemma for $n - 1$. By successive reductions the lemma can be reduced to the case $n = 1$ and for this case the lemma holds by Lemma 1.

THEOREM 3.1. *If $x \equiv e_1 + \ldots + e_n$ and $y \equiv f_1 + \ldots + f_m$, then $x + u \equiv y + u$ for some u in G if and only if there exist independent elements $g_1, \ldots, g_N$ and direct decompositions*

$$e_i = \bigcup_{j=1}^{N} e_{ij}, \quad f_i = \bigcup_{j=1}^{N} f_{ij}, \tag{3.2}$$

such that each e_{ij} is either 0 or $\backsim g_j$, each f_{ij} is either 0 or $\backsim g_j$, and for each j the number E_j of i for which $e_{ij} \backsim g_j$ is equal to the number F_j of i for which $f_{ij} \backsim g_j$.

Proof. Write $x \backsim y(d)$ if decompositions (3.2) do exist and write $x \equiv y(c)$ if $x + u \equiv y + u$ for some u in G. Since $e \backsim f$ implies $e + a \equiv f + a$ for some axis of perspectivity a in S, it follows that $e \backsim f$ implies that $e \equiv f(c)$ and hence $x \backsim y(d)$ implies $x \equiv y(c)$.

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The converse, $x \equiv y(c)$ implies $x \backsim y(d)$, will follow by induction if we prove:

(3.3) $$x \backsim x(d);$$

(3.4) if $x \backsim y(d)$, this relation remains valid if f_1 in y is replaced by $f' + f''$, providing that $f_1 = f' \dot{+} f''$;

(3.5) if $x \backsim y(d)$, this relation remains valid if $f_1 + f_2$ in y is replaced by f, providing that $f_1 \dot{+} f_2 = f$;

(3.6) if $x + u \backsim y + u(d)$ then $x \backsim y(d)$.

For $x + u \equiv y + u$ means that $x + u$ can be transformed into $y + u$ by the changes named in (3.3), (3.4), and (3.5) and it will follow that $x + u \backsim y + u(d)$. From (3.6) we will then have $x \backsim y(d)$ as required.

Proof of (3.3). Given arbitrary elements $e_1, \ldots, e_n$ we need only show that there are independent elements $g_1, \ldots, g_N$ and direct decompositions

$$e_i = \bigcup_{j=1}^{N} e_{ij}$$

such that each e_{ij} is either 0 or $\backsim g_j$. But this follows from Lemma 5.

Proof of (3.4). Suppose $x \backsim y(d)$. This implies an independent set $g_1, \ldots, g_N$ and a particular decomposition (we shall call it the previous decomposition) for each f_i in y. If now f_1 is replaced by $f' + f''$, then Lemma 6 can be applied to the previous decomposition of f_1, say $f_1 = \bigcup f_{1j}$, and the decomposition $f' \cup f''$ of f_1. Direct decompositions $f_{1j} = f_{1j}' \cup f_{1j}''$ result, and these, with the help of Lemma 1, lead to direct decompositions $g_j = g_j' \cup g_j''$ with $f_{1j}' \backsim g_j'$, $f_{1j}'' \backsim g_j''$ if f_{1j} is different from 0 and with $g_j' = g_j$, $g_j'' = 0$ if $f_{1j} = 0$. From these decompositions of g_j we obtain direct decompositions, $f_{ij} = f_{ij}' \cup f_{ij}''$ for $i > 1$ and $e_{ij} = e_{ij}' \cup e_{ij}''$ so that $x \backsim y(d)$ remains valid with $g_1', \ldots g_N'$, $g_1'', \ldots, g_N''$ in place of $g_1, \ldots, g_N$.

Proof of (3.5). Suppose $x \backsim y(d)$, that the e_i, f_i, e_{ij}, g_j satisfy (3.2), and that $f_1 + f_2$ in y is replaced by f. (Note that

$$f = \left(\bigcup_{j=1}^{N} f_{1j} \cup \bigcup_{j=1}^{N} f_{2j} \right)$$

is a direct decomposition for f; but this fails to prove that $x \backsim y(d)$ remains valid with the same $g_1, \ldots, g_N$ since, for some j, both f_{1j} and f_{2j} may differ from zero.) We may suppose that all g_j are different from 0, that $f_{1j} = g_j$ for $j = 1, \ldots, p$ (in place of $f_{1j} \backsim g_j$), and that $f_{1j} = 0$ for $j > p$ (apply Lemmas 2 and 1 to the complements of $g_1 \cup \ldots \cup g_p$ and $f_{11} \cup \ldots f_{1p}$ with respect to $g_1 \cup \ldots \cup g_p \cup f_{11} \cup \ldots \cup f_{1p}$). By rearranging indices we may now suppose that $f_{2j} \backsim f_{1j} = g_j$ for $j = 1, \ldots, r$ with $r \leqslant p$, that $f_{2j} \backsim g_j$ for $j = p + 1, \ldots, q$, and that $f_{2j} = 0$ for all other j. Then we may even suppose $f_{2j} = g_j$ for $j = p + 1, \ldots, q$. Next, by changing the g_j with $j > q$ and increasing N if necessary, we

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may suppose that each such g_j satisfies either $g_j \cap (f_1 \cup f_2) = 0$ or

$$g_j \leqslant \bigcup_{j=1}^{r} f_{2j};$$

letting g_{N+1} be a complement of

$$(g_1 \cup \ldots \cup g_N) \cap \left(\bigcup_{j=1}^{r} f_{2j}\right)$$

with respect to

$$\bigcup_{j=1}^{r} f_{2j}$$

and writing N again for the former $N + 1$ we may now suppose that

$$\bigcup_{j=1}^{r} f_{2j} \leqslant \bigcup_{j=q+1}^{N} g_j.$$

Then

$$\bigcup_{j=q+1}^{N} g_j = \bigcup_{j=1}^{r} f_{2j} \cup f_0$$

are two direct decompositions of the same element (with f_0 a suitable complement) and Lemma 6 applies. We derive direct decompositions for all elements used previously, such that (using the previous notation again) we may even suppose that $f_{2j} \backsim g_{q+j}$ for $j = 1, \ldots, r$. Now a direct decomposition for f is

$$\bigcup_{j=1}^{N} f_j$$

with $f_j = f_{1j}$ for $j = 1, \ldots, p$, $f_j = f_{2j}$ for $j = p + 1, \ldots, q$, $f_{q+j} = f_{2j}$ for $j = 1, \ldots, r$, and $f_j = 0$ for all other j. When the decompositions for f_1, f_2 used in (3.2) are replaced by this decomposition for f the number F_j is altered by -1 if $j = 1, \ldots, r$ and by $+1$ if $j = q + 1, \ldots, q + r$. However, the equality of E_j, F_j can be restored as follows. For each fixed $j = 1, \ldots, r$ we have $g_j \backsim g_{q+j}$. If $F_j < 2 + F_{q+j}$ then there must be an $i > 2$ with $f_{ij} = 0$ and $f_{i,q+j} \backsim g_{q+j}$; in this case we interchange these elements so that $f_{ij} \backsim g_j$ and $f_{i,q+j} = 0$. If however $F_j \geqslant 2 + F_{q+j}$, then $E_j \geqslant 2 + E_{q+j}$ and there must be some i for which $e_{ij} \backsim g_j$ and $e_{i,q+j} = 0$; in this case we interchange these two elements so that $e_{ij} = 0$ and $e_{i,q+j} \backsim g_{q+j}$.

This completes the proof of (3.5).

Proof of (3.6). Suppose

(3.7) $e_1 + \ldots + e_n + h_1 + \ldots + h_p \backsim f_1 + \ldots + f_m + h_1 + \ldots + h_p(d)$.

We wish to deduce $e_1 + \ldots + e_n \backsim f_1 + \ldots + f_m(d)$. Proof by induction will apply here and we need only consider (3.7) with $p = 1$. Then, as detailed in (3.2), there are independent $g_1, \ldots, g_N$ and direct decompositions of the e_i, f_i, and h_1 into elements each of which is perspective to one of the g_j. We may replace h_1 in (3.7) by the lattice union of its corresponding set of g_j. We note

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that h_1 may be assigned two different sets L and R of g_j according as h_1 appears on the left or right of (3.7). Since the two replacements for h_1 are perspective by Lemma 3, we may apply Lemmas 2, 1, and 6 to obtain decompositions of the g_j in L but not in R and of the g_j in R but not in L into new elements (which we will again call g_j) which are perspective in pairs. Thus we may suppose (3.7) given in the form

$$(3.8) \qquad e_1 + \ldots + e_n + g_1 + \ldots + g_r \backsim f_1 + \ldots + f_m + g_{r+1} + \ldots + g_{2r} (d)$$

with $g_1, \ldots, g_{2r}$ a subset of the $g_1, \ldots, g_N$ mentioned in (3.2) and with $g_i \backsim g_{r+i}$ for $i = 1, \ldots, r$.

For fixed j let E_j be the number of i for which $e_{ij} \backsim g_j$ and let F_j be the number of i for which $f_{ij} \backsim g_j$. Then for $j > 2r$ we deduce from (3.8) that $E_j = F_j$. If $j \leqslant r$ we obtain $E_j + 1 = F_j$, $E_{r+j} = F_{r+j} + 1$. Hence at least one of $E_j < E_{r+j}$, $F_j > F_{r+j}$ holds. If $E_j < E_{r+j}$ there must be an e_i for which $e_{ij} = 0$ and $e_{i,r+j} \backsim g_{r+j}$; in that case we interchange these elements e_{ij}, $e_{i,r+j}$ so that now $e_{ij} \backsim g_{r+j} \backsim g_j$ and $e_{i,r+j} = 0$, thus obtaining $E_j = F_j$, $E_{r+j} = F_{r+j}$ for the new decompositions. In the same way, if $F_j > F_{r+j}$ we can rearrange the decomposition of some f_i to obtain $E_j = F_j$ and $E_{r+j} = F_{r+j}$. After this is done for each $j \leqslant r$ we obtain decompositions in terms of $g_1, \ldots, g_N$ for which (3.2) can be easily verified.

This completes the proof of Theorem 3.1.

COROLLARY TO THEOREM 3.1. *Two elements e, f in S satisfy $e + u \equiv f + u$ for some u in G if and only if $e \backsim f$.*

(It is easy to prove directly that $e \equiv f$ if and only if $e = f$.)

Remark. The relation $x \equiv y$ in G can also be characterized in terms of decompositions in S but we omit the somewhat involved statement. In the special case of S a Boolean ring, $e \backsim f$ holds if and only if $e = f$, and Theorem 3.1 shows that $x + u \equiv y + u$ if and only if $x \equiv y$. Thus the cancellation law holds in G if S is a Boolean ring but not if S is a general relatively complemented modular lattice.

THEOREM 3.2. *$me + u \equiv e_1 + \ldots + e_n + u$ as in the condition (2.3) if and only if there are direct decompositions*

$$e_i = \bigcup_{j=1}^{m} e_{ij}' \ (i = 1, \ldots, n), \quad e = \bigcup_{i=1}^{n} e_{ij}'' \ (j = 1, \ldots, m)$$

with $e_{ij}' \backsim e_{ij}''$ for all i,j.

Proof. Apply Theorem 3.1 with $f_1 = \ldots = f_m = e$ to obtain the decompositions of (3.2) with $g_1, \ldots, g_N$ which we may suppose all non-zero. For given p, q let $J(p, q)$ be the set of j for which f_{pj} and e_{qj} are both different from zero, and the number of $r < p$ for which f_{rj} is different from zero is equal to the

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number of $r < q$ for which e_{rj} is different from zero. Set

$$e_{pq}'' = \bigcup f_{pj} \ (j \in J(p, q))$$

$$e_{qp}' = \bigcup e_{qj} \ (j \in J(p, q)).$$

With this construction the theorem can be easily verified.

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