Fourth Meeting, 10*th February* 1905.

Mr W. L. Thomson, President, in the Chair.

The Ambiguous Cases in the Solution of Spherical Triangles.

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Although the following note makes no pretence at novelty so far as the results are concerned, yet the method employed does not seem to occur in the ordinary text-books on Spherical Trigonometry. I have found this process very useful in explaining to beginners how to distinguish between the various possibilities, and I hope it may be of some interest to other teachers.

As a starting point, suppose the known parts (two sides and an opposite angle) to be a, b, A; we have then the equation for c

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

When c is known, the triangle is determined without further ambiguity, since the angles are uniquely determined when the sides are known; thus the only ambiguity arises through the determination of c from the given parts.

Write now $t = \tan\frac{1}{2}c$, and after a little reduction we find the quadratic for t:

$$(1) \qquad (\cos a + \cos b)t^2 - 2t \sin b \cos A + (\cos a - \cos b) = 0.$$

If we suppose, as is usual in the elementary theory, that all sides and angles are less than two right angles, it follows that the admissible roots of (1) are limited by the further condition of being *real* and *positive.*

Take first the case when $(\cos a + \cos b)$ and $(\cos a - \cos b)$ have *opposite* signs; then the two roots of (1) are real, but only one of them is positive.

But $\qquad (\cos a + \cos b)(\cos a - \cos b) = (\sin b - \sin a)(\sin b + \sin a),$

so that, since $\sin a$ and $\sin b$ are positive, this case occurs if (and only if) $\sin a$ is greater than $\sin b$. Hence :—

If $\sin a > \sin b$, *one and only one triangle exists with the given parts.*

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If $\sin a$ is less than $\sin b$, the quadratic (1) has real roots only if

$$\sin^2 b \cos^2 A \geqq \cos^2 a - \cos^2 b$$

or, if $$\sin^2 a \geqq \sin^2 b \sin^2 A.$$

Under the restriction already assumed as to sides and angles, the last condition is equivalent to

$$\sin a \geqq \sin b \sin A.$$

The roots t_1, t_2 of the quadratic have then the same sign, which is the sign of

$$\tfrac{1}{2}(t_1 + t_2) = \sin b \cos A/(\cos a + \cos b).$$

Now $(\cos a + \cos b)$ and $(\cos a - \cos b)$ have the same sign, which must therefore be the same as the sign of $\cos a$; and consequently t_1, t_2 have the same sign as $\cos A/\cos a$. Thus we have the result:—

If $\sin b > \sin a \geqq \sin b \sin A$, *two triangles exist with the given parts, provided that* $\cos A/\cos a$ *is positive; the two triangles being coincident in case* $\sin a = \sin b \sin A$. *But if* $\sin b > \sin a$, *and either of the other conditions is broken, there is no triangle with the given parts.*

We have now exhausted all cases except those for which

$$\sin a = \sin b.$$

Then $$(\cos a + \cos b)(\cos a - \cos b) = 0.$$

If $\cos a = \cos b$, the roots of the quadratic (1) are 0 and $\sin b \cos A/\cos a$; thus there is one triangle if $\cos A/\cos a$ is positive, and no triangle otherwise.

We find the same result if $\cos a = -\cos b$. Thus:—

If $\sin a = \sin b$, *there is one triangle with the given parts, if* $\cos A/\cos a > 0$; *no triangle if* $\cos A/\cos a \leqq 0$.

The tests obtained for the cases $\sin a \leqq \sin b$, imply that $\cos a$ is not zero; if it happens that a is a right angle, b must also be a right angle so as to satisfy $\sin a \leqq \sin b$. In this case the quadratic (1) reduces to

$$t \cos A = 0$$

implying that there is no such triangle unless A is also a right angle. Thus:—

The only case not included in the previous tests is $a = b = \tfrac{1}{2}\pi$; *and there can then be no triangle with the given parts, unless* $A = \tfrac{1}{2}\pi$; *if* $A = \tfrac{1}{2}\pi$, *the triangle is indeterminate.*

We have now completed the discussion of the first ambiguous case. To deal with the second ambiguous case (two angles and an opposite side, say, A, B, a, being given), we may start from the relation

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

and transform it to the quadratic

(2) $$(\cos A - \cos B)T^2 - 2T \sin B \cos a + (\cos A + \cos B) = 0$$

by taking $T = \tan \frac{1}{2}C$. But the results can be obtained more quickly by using those already found, and transforming them by the polar triangle properties.

In tabular form, the results are* :—

a, b, A given

Two Triangles - -	$\sin b > \sin a > \sin b \sin A$ *and* $\cos A/\cos a > 0$
Two coincident Triangles	$\sin a = \sin b \sin A$ *and* $\cos A/\cos a > 0$
One Triangle - -	(i) $\sin a > \sin b$ (ii) $\sin a = \sin b$ *and* $\cos A/\cos a > 0$
Infinity of Triangles -	$a = b = A = \frac{1}{2}\pi$
No Triangle - - -	Any case not included in the preceding

If A, B, a are given, the results may be obtained by simply interchanging sides and angles in the table.

* Compare Prof. Lloyd Tanner, *Messenger of Mathematics*, vol. 14, 1885, p. 153.