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SOME GEOMETRIC CHARACTERIZATIONS OF INNER PRODUCT SPACES

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There are several geometric characterizations of inner product spaces amongst the normed linear spaces. Mahlon M. Day's refinement "rhombi suffice as well as parallelograms", of P. Jordan and J. von Neumann parallelogram law is well known. There are some characterizations which employ various notions of orthogonality. For example, it is known that if in a normed linear space Birkhoff-James orthogonality implies isosceles orthogonality then the space is an inner product space; geometrically it means that if the diagonals of a rectangle, with sides perpendicular in Birkhoff-James sense, are equal then the space is an inner product space. In the main result of this note we improve upon this characterization and show that here unit squares suffice as well as rectangles.

There are several geometric characterizations of inner product spaces amongst the normed linear spaces. The Jordan von Neumann parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2\left[\|x\|^2+\|y\|^2\right] \quad \text{for all } x \text{ and } y ,$$

and its refinement rhombi suffice

$$\|x+y\|^2 + \|x-y\|^2 = 4 \quad \text{for all } x \text{ and } y \text{ with } \|x\| = \|y\| = 1 ,$$

due to Day [2], are among the well known ones. There are some

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characterizations which employ various notions of orthogonality. For example, it is known that if in a normed linear space Birkhoff-James orthogonality implies isosceles orthogonality then the space must be an inner product space. In this note we improve upon this characterization in the same sense as Day did for the parallelogram law. More explicitly, we prove that if in a normed linear space X, $\|x\| = \|y\| = 1$ and x Birkhoff-James orthogonal to y implies x isosceles orthogonal to y then X must be an inner product space. Geometrically it will mean that if the diagonals of any unit square, with sides perpendicular in the Birkhoff-James sense, are equal then X is an inner product space. In the same vein some previously known characterizations due to Day [2], Kapoor and Prasad [7] and Holub [3] have been improved upon as corollaries to our main result. Finally a sufficient condition for strict convexity of a normed linear space has been proved.

We give briefly the definitions and notations. X is a real normed linear space and $S = \{x \in X : \|x\| = 1\}$ is the unit ball of X throughout this note. If $x, y \in X$, x is called isosceles orthogonal to $y\,(x \perp_i y)$ if $\|x+y\| = \|x-y\|$; x is called pythagorean orthogonal to $y\,(x \perp_p y)$ if $\|x+y\|^2 = \|x\|^2 + \|y\|^2$; and x is Birkhoff-James orthogonal to $y\,(x \perp y)$ if $\|x+ky\| \geq \|x\|$ for all real k. For details of these orthogonalities one can refer to James [4], [5]. It is well known, James [5], that if $x \neq 0$ and $y \in X$, then there exist numbers c and d such that $x \perp cx + y$ and $dx + y \perp x$, and that the Birkhoff-James orthogonality is symmetric in normed linear space of dimension greater than or equal to 3 if and only if the space is an inner product space, Day [1]. In [1] Day describes those norms in two-dimensional spaces which have symmetric Birkhoff-James orthogonality.

In [6], Joly calls the number $m(X) = \operatorname{Sup}_{x \perp y} \frac{\|x\|+\|y\|}{\|x+y\|}$, the rectangular constant for the space X. There he notes that $\sqrt{2} \leq m(X) \leq 3$, and that $m(X) = \sqrt{2}$ implies the symmetry of Birkhoff-James orthogonality in X; consequently X is an inner product space if dimension $X \geq 3$. In [8] del Río and Benítez proved that $m(X) = \sqrt{2}$ is a characterizing property of inner product spaces in two dimensions also. The result will be basic for our main theorem which follows.

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THEOREM 1. *For a normed linear space* X *the following are equivalent:*

(i) X *is an inner product space;*

(ii) for $x, y \in S$, $x \perp y$ *implies* $x \perp_i y$;

(iii) $m(X) = \sup\limits_{x \perp y} \frac{\|x\|+\|y\|}{\|x+y\|} = \sqrt{2}$.

Proof. *(i)* implies *(ii)* is obvious, and *(iii)* implies *(i)* is contained in [8]. We have to prove *(ii)* implies *(iii)*. This proof consists of many steps which we give below as lemmas.

LEMMA 1. *If* X *satisfies (ii) of Theorem* 1 *then* X *is strictly convex.*

Proof. Assuming X is not strictly convex choose [5, Theorem 4.3], $x, y \in S$ such that $x \perp y$ and $\alpha y + x \perp y$ where $\alpha > 0$ is chosen to be the largest such number. The function $\phi(t) = \|x+ty\|$ is a convex function of t with $\phi(t) = 1$ for $0 \le t \le \alpha$ and $\phi(t)$ is strictly increasing with t for $t \ge \alpha$. By hypothesis we have $\|x+y\| = \|x-y\|$ and $\|(\alpha+1)y+x\| = \|(\alpha-1)y+x\|$ which implies that $\phi(1) = \phi(-1)$ and $\phi(\alpha+1) = \phi(\alpha-1)$. Thus $0 < \alpha \le 1$. Now

$$\begin{aligned}\phi(\alpha-1) = \|(\alpha-1)y+x\| &= \|(\alpha/2)(x+y)+\big(1-(\alpha/2)\big)(x-y)\| \le \|x+y\| \\ &= \phi(1) = \|\big(1-(\alpha/2)\big)\big((\alpha+1)y+x\big)+(\alpha/2)\big((\alpha-1)y+x\big)\| \le \phi(\alpha-1)\end{aligned}$$

which yields a contradiction. Hence X has to be strictly convex.

LEMMA 2. *If* X *satisfies (ii) of Theorem* 1 *then Birkhoff-James orthogonality is symmetric.*

Proof. If not, let $x \perp y$ and $\alpha x + y \perp x$ for $x, y \in S$ and $\alpha > 0$. Let $\beta = \|\alpha x+y\|$; then $1 = \|y\| = \|\alpha x+y-\alpha x\| \ge \|\alpha x+y\| = \beta \ge \alpha$. Putting $\|x+y\| = \|x-y\| = a$ and $\|(\beta+\alpha)x+y\| = \|(\beta-\alpha)x-y\| = b$ we see that

$$b = \|(\beta-\alpha)x-y\| = \|\big((\beta-\alpha-1)/2\big)(x+y)+\big((\beta-\alpha+1)/2\big)(x-y)\| \le a .$$

Similarly we can obtain $a \le b$. Thus we have

$$\|x+y\| = \|x-y\| = \|(\beta+\alpha)x+y\| = \|(\beta-\alpha)x-y\|$$

which is false since X is strictly convex.

LEMMA 3. *If* X *satisfies (ii) of Theorem* 1 *then it satisfies the*

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condition

$$x,\ y \in S\ ,\quad x \perp y \quad \textit{implies} \quad \|x+y\| = \|x-y\| = \sqrt{2}\ .$$

Proof. Let $\|x\| = \|y\| = 1$, $x \perp y$. Firstly we will show that $x + y \perp x - y$. If not, let $x + ay \perp x - y$, where in view of the symmetry of orthogonality we may assume $0 < a < 1$. Let $\|x+ay\| = \beta_1$ and $\|x-y\| = \beta_2 = \|x+y\|$; clearly $\beta_2 \geq \beta_1$.

We may further assume that X is a plane and introduce a coordinate system with $x = (1, 0)$ and $y = (0, 1)$; then $x + ay = (1, a)$ and $x - y = (1, -1)$. Using the result of Day [*1*, p. 330] that $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are orthogonal if and only if $|u_1v_2-u_2v_1| = \|u\|\|v\|$, we obtain $\beta_1\beta_2 = 1 + a$. Moreover from hypothesis we have

$$(1) \qquad \|(\beta_2+\beta_1)x+(a\beta_2-\beta_1)y\| = \|(\beta_2-\beta_1)x+(a\beta_2+\beta_1)y\|$$

which yields $\beta_1 \leq a\beta_2$:

$$\beta_2 = \|x+y\| = \|\big(2/(1+a)\big)(x+ay)+\big((a-1)/(1+a)\big)(x-y)\|$$
$$\leq \big(2/(1+a)\big)\beta_1 + \big((1-a)/(1+a)\big)\beta_2$$

so that $a\beta_2 \leq \beta_1$.

In view of these inequalities (1) becomes

$$\beta_2 + \beta_1 = \|(\beta_2-\beta_1)x+2\beta_1y\|$$

which contradicts strict convexity unless $\beta_1 = \beta_2$. But then $a = 1$. Therefore, $x + y \perp x - y$ and $\|x+y\| = \|x-y\| = \sqrt{2}$.

Proof of Theorem 1. Let $x \perp y$ be any pair of non-zero vectors. Put

$$F(t) = \frac{\|t^2y+x\|}{t^2\|y\|+\|x\|}\ , \quad 0 \leq t < \infty\ .$$

$F(t)$ is differentiable because X is smooth. Symmetry of orthogonality and strict convexity gives smoothness. Let $q'(x, y)$ denote the Gateaux derivative of the norm at x in the direction of y. For an extreme value of F, t must be such that

$$q'(t^2y+x,\ y) = \|y\| \frac{\|t^2y+x\|}{t^2\|y\|+\|x\|} .$$

By using the symmetry of orthogonality and the fact that $x \perp y$, that is, $q'(x, y) = 0$, we obtain $q'(t^2y+x,\ \|y\|x-\|x\|y) = 0$, that is $(t^2y+x) \perp (\|x\|y-\|y\|x)$.

This shows that there is only one extreme value of $F(t)$ for $t > 0$, which by Lemma 3 corresponds to $t^2 = \frac{\|x\|}{\|y\|}$ and the extreme value is

$$F(t)_{\text{extreme}} = \frac{\|\|x\|y+\|y\|x\|}{2\|x\|\|y\|} = 1/\sqrt{2}$$

which must be a minimum. Thus

$$\frac{\|x\|+\|y\|}{\|x+y\|} \le \sqrt{2} \quad \text{whenever} \quad x \perp y .$$

Then

$$m(X) = \sup_{x \perp y} \frac{\|x\|+\|y\|}{\|x+y\|} \le \sqrt{2} .$$

Hence $m(X) = \sqrt{2}$, which was to be proved.

We now give refinements of some of the earlier known results of Day [2], Kapoor and Prasad [7] and Holub [3] as a corollary to Theorem 1.

COROLLARY 1. *A normed linear space* X *is an inner product space if it has any one of the following properties:*

(i) $x, y \in S$, $x \perp_i y$ *implies* $x \perp y$;

(ii) $x, y \in S$, $x \perp y$ *implies* $x \perp_p y$;

(iii) $x, y \in S$, $x \perp_p y$ *implies* $x \perp y$.

Proof. *(i)* Assuming that the space is not strictly convex, choose $x, y \in S$ and the largest $\alpha > 0$ such that $\beta y + x \perp y$, $\beta y + x \in S$, for all $-\alpha \le \beta \le \alpha$. We claim that $x \perp_i y$. The function $\|x+z\| - \|x-z\|$ varies continuously between -2 and 2 , as z moves from $-x$ to x along the curve S_1 which is the intersection of S and the span of x and y . Hence there is a $z = ax + by$ in S_1 , $b \ge 0$, such that

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$\|x+z\| = \|x-z\|$. The hypothesis implies that $x \perp ax + by$,

$$1 = \|x \pm \alpha y\| = \|(b \mp a\alpha/b)x \pm (\alpha/b)(ax+by)\| \geq |(b \mp a\alpha/b)| ,$$

$$b \geq |b \mp a\alpha| .$$

This leads to a contradiction unless $a = 0$ and $b = 1$, and hence $x \perp_i y$. Similarly it can be proved that $\beta y + x \perp_i y$ for $-\alpha \leq \beta \leq \alpha$. It is easily seen that $\alpha \leq 1$.

Putting $\phi(t) = \|ty+x\|$ we note that $\phi(t)$ is a convex function with $\phi(t) \geq 1$, $\phi(-\alpha) = \phi(\alpha) = 1$, $\phi(1) = \phi(-1)$, $\phi(t)$ is strictly decreasing for $-\infty < t \leq -\alpha$ and strictly increasing for $\alpha \leq t < \infty$, and $\phi(\alpha+1) = \phi(\alpha-1)$ which is not possible since $\alpha + 1 > 1$ and $-1 \leq \alpha-1 \leq 0$. Therefore, the space must be strictly convex.

To complete the proof we show that our hypothesis implies the hypothesis *(ii)* of Theorem 1. Let $x, y \in S$, $x \perp y$, choose a and b as above such that $ax + by \perp_i y$ with $a \geq 0$; hence $ax + by \perp y$ and from strict convexity it follows that $a = 1$ and $b = 0$.

The proof of *(ii)* is immediate from Theorem 1, and *(iii)* implies *(ii)* can be proved following the lines of the proof of *(i)* above.

REMARK. The property *(i)* of Corollary 1, without the condition that $x, y \in S$, has been given as property (M) by Day [2, p. 155].

COROLLARY 2. *If* $x, y \in S$, $x \perp y$ *implies* $\|x+y\|^2 + \|x-y\|^2 = 4$, *then* X *is an inner product space, provided Birkhoff-James orthogonality is assumed to be symmetric.*

Proof. Let $x, y \in S$ and $x \perp y$; choose [9, Lemma 1] a where $0 < a \leq 1$ such that $x + ay \perp x - ay$. Put $\|x+ay\| = \beta_1$ and $\|x-ay\| = \beta_2$. The hypothesis gives

$$\|(\beta_2+\beta_1)x+(\beta_2-\beta_1)ay\|^2 + \|(\beta_2-\beta_1)x+(\beta_2+\beta_1)ay\|^2 = 4\beta_1^2\beta_2^2$$

and

$$\|x+y\|^2 + \|x-y\|^2 = 4$$

whence

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(2) $$4\beta_1^2\beta_2^2 \geq \left(\beta_2^2+\beta_1^2\right)\left(1+a^2\right) .$$

Also,

$$\|x+y\| = \|\left((a+1)/2a\right)(x+ay)+\left((a-1)/2a\right)(x-ay)\|$$

gives $\|x+y\| \geq \left((a+1)/2a\right)\beta_1$; similarly, $\|x-y\| \geq \left((a+1)/2a\right)\beta_2$ and, therefore,

(3) $$\left((a+1)/2a\right)^2\left(\beta_1^2+\beta_2^2\right) \leq 4 .$$

(2) and (3) imply $a = 1$ and $\beta_1 = \beta_2 = \sqrt{2}$.

Thus we have $x, y \in S$ and $x \perp y$ implies $\|x+y\| = \|x-y\| = \sqrt{2}$ and hence, by Theorem 1, we get the result.

If the symmetry is not there in Corollary 2 we do not know how to prove it, though we feel it should be true. Without symmetry of orthogonality we can still prove that the space is strictly convex in the following theorem.

THEOREM 2. *If* $x \perp y$ *implies* $\|x+y\|^2 + \|x-y\|^2 = 2\left[\|x\|^2+\|y\|^2\right]$ *then* X *is strictly convex.*

Proof. Let $\|x\| = \|y\| = \|(x+y)/2\| = 1$. It can be easily seen that $(x+y)/2 \not\perp x$. Choose α such that $(x+y)/2 \perp \alpha\left((x+y)/2\right) + x$. But it can be shown that $\alpha = -1$ and then $x + y \perp x - y$. Now we have

$$8 = \|x+y+x-y\|^2 + \|x+y-x+y\|^2 = 2\left[\|x+y\|^2+\|x-y\|^2\right] = 8 + 2\|x-y\|^2$$

which implies that $x = y$, and the proof is complete.

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