I

Inputs to the Standard Model

This book is about the Standard Model of elementary particle physics. If we set the beginning of the modern era of particle physics in 1947, the year the pion was discovered, then the ensuing years of research have revealed the existence of a consistent, self-contained layer of reality. The energy range which defines this layer of reality extends up to about 1 TeV or, in terms of length, down to distances of order 10^{-17} cm. The Standard Model is a field-theoretic description of strong and electroweak interactions at these energies. It requires the input of as many as 28 independent parameters.[1] These parameters are not explained by the Standard Model; their presence implies the need for an understanding of Nature at an even deeper level. Nonetheless, processes described by the Standard Model possess a remarkable insulation from signals of such New Physics. Although the strong interactions remain a calculational challenge, the Standard Model (generalized from its original form to include neutrino mass) would appear to have sufficient content to describe all existing data.[2] Thus far, it is a theoretical structure which has worked splendidly.

I–1 Quarks and leptons

The Standard Model is an $SU(3) \times SU(2) \times U(1)$ gauge theory which is spontaneously broken by the Higgs potential. Table I–1 displays mass determinations [RPP 12] of the Z^0 and $W^{\pm}$ gauge bosons, the Higgs boson H^0, and the existing mass limit on the photon γ.

In the Standard Model, the fundamental fermionic constitutents of matter are the quarks and the leptons. Quarks, but not leptons, engage in the strong interactions as a consequence of their color charge. Each quark and lepton has spin one-half.

[1] There are six lepton masses, six quark masses, three gauge coupling constants, three quark-mixing angles and one complex phase, three neutrino-mixing angles and as many as three complex phases, a Higgs mass and quartic coupling constant, and the *QCD* vacuum angle.

[2] Admittedly, at this time the sources of *dark matter* and of *dark energy* are unknown.

Table I–1. *Boson masses.*

Particle	Mass (GeV/c^2)
γ	$< 1 \times 10^{-27}$
$W^{\pm}$	80.385 ± 0.015
Z^0	91.1876 ± 0.0021
H^0	126.0 ± 0.4

Collectively, they display conventional Fermi–Dirac statistics. No attempt is made in the Standard Model either to explain the variety and number of quarks and leptons or to compute any of their properties. That is, these particles are taken at this level as truly elementary. This is not unreasonable. There is no experimental evidence for quark or lepton compositeness, such as excited states or form factors associated with intrinsic structure.

Quarks

There are six quarks, which fall into two classes according to their electrical charge Q. The u, c, t quarks have $Q = 2e/3$ and the d, s, b quarks have $Q = -e/3$, where e is the electric charge of the proton. The u, c, t and d, s, b quarks are eigenstates of the hamiltonian ('mass eigenstates'). However, because they are believed to be permanently confined entities, some thought must go into properly defining quark mass. Indeed, several distinct definitions are commonly used. We defer a discussion of this issue and simply note that the values in Table I–2 provide

Table I–2. *The quarks.*

Flavor	Mass[a] (GeV/c^2)	Charge	I_3	S	C	B	T
u	$(2.55^{+0.75}_{-1.05}) \times 10^{-3}$	2e/3	1/2	0	0	0	0
d	$(5.04^{+0.96}_{-1.54}) \times 10^{-3}$	−e/3	−1/2	0	0	0	0
s	$0.105^{+0.025}_{-0.035}$	−e/3	0	−1	0	0	0
c	$1.27^{+0.07}_{-0.11}$	2e/3	0	0	1	0	0
b	$4.20^{+0.17}_{-0.07}$	−e/3	0	0	0	−1	0
t	173.4 ± 1.6	2e/3	0	0	0	0	1

[a]The t-quark mass is inferred from top quark events. All others are determined in $\overline{\mathrm{MS}}$ renormalization (cf. Sect. II–1) at scales $m_{u,d,s}(2\ \mathrm{GeV}/c^2)$, $m_c(m_c)$ and $m_b(m_b)$ respectively.

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Table I–3. *The leptons.*

Flavor	Mass (GeV/c^2)	Charge	L_e	L_μ	L_τ
ν_e	$< 0.2 \times 10^{-8}$	0	1	0	0
e	$5.10998928(11) \times 10^{-4}$	$-$e	1	0	0
ν_μ	$< 1.9 \times 10^{-4}$	0	0	1	0
μ	0.1056583715(35)	$-$e	0	1	0
ν_τ	< 0.0182	0	0	0	1
τ	1.77682(16)	$-$e	0	0	1

an overview of the quark mass spectrum. A useful benchmark for quark masses is the energy scale Λ_{QCD}($\simeq$ several hundred MeV) associated with the confinement phenomenon. Relative to Λ_{QCD}, the $u,\ d,\ s$ quarks are light, the $b,\ t$ quarks are heavy, and the c quark has intermediate mass. The dynamical behavior of light quarks is described by the chiral symmetry of massless particles (cf. Chap. VI) whereas heavy quarks are constrained by the so-called Heavy Quark Effective Theory (cf. Sect. XIII–3). Each quark is said to constitute a separate *flavor*, i.e. six quark flavors exist in Nature. The $s,\ c,\ b,\ t$ quarks carry respectively the quantum numbers of strangeness (S), charm (C), bottomness (B), and topness (T). The $u,\ d$ quarks obey an $SU(2)$ symmetry (isospin) and are distinguished by the three-component of isospin (I_3). The flavor quantum numbers of each quark are displayed in Table I–2.

Leptons

There are six leptons which fall into two categories according to their electrical charge. The charged leptons $e,\ \mu,\ \tau$ have $Q = -e$ and the neutrinos $\nu_e,\ \nu_\mu,\ \nu_\tau$ have $Q = 0$. Leptons are also classified in terms of three lepton types: electron (ν_e, e), muon (ν_μ, μ), and tau (ν_τ, τ). This follows from the structure of the charged weak interactions (cf. Sect. II–3) in which these charged-lepton/neutrino pairs are coupled to $W^\pm$ gauge bosons. Associated with each lepton type is a lepton number L_e, L_μ, L_τ. Table I–3 summarizes lepton properties.

At this time, there is only incomplete knowledge of neutrino masses. Information on the mass parameters $m_{\nu_e}, m_{\nu_\mu}, m_{\nu_\tau}$ is obtained from their presence in various weak transition amplitudes. For example, the single beta decay experiment ${}^3\text{H} \to {}^3\text{He} + e^- + \overline{\nu}_e$ is sensitive to the mass m_{ν_e}. In like manner, one constrains the masses m_{ν_μ} and m_{ν_τ} in processes such as $\pi^+ \to \mu^+ + \nu_\mu$ and $\tau^- \to 2\pi^- + \pi^+ + \nu_\tau$ respectively. Existing bounds on these masses are displayed in Table I–3.

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It is known experimentally that upon creation the neutrinos $\{\nu_\alpha\} \equiv (\nu_e.\nu_\mu, \nu_\tau)$ will not propagate indefinitely but will instead mix with each other. This means that the basis of states $\{\nu_\alpha\}$ cannot be eigenstates of the hamiltonian. Diagonalization of the leptonic hamiltonian is carried out in Sect. VI–2 and yields the basis $\{\nu_i\} \equiv \{\nu_1, \nu_2, \nu_3\}$ of mass eigenstates. Information on the neutrino mass eigenvalues m_1, m_2, m_3 is obtained from neutrino oscillation experiments and cosmological studies. Oscillation experiments (cf. Sects. VI–3,VI–4) are sensitive to squared-mass differences.[3] Throughout the book, we adhere to the following relations,

$$\text{definition: } \Delta m^2_{ij} \equiv m_i^2 - m_j^2, \qquad \text{convention: } m_2 > m_1. \tag{1.1}$$

From the compilation in [RPP 12], the squared-mass difference $|\Delta m^2_{32}|$ deduced from the study of atmospheric and accelerator neutrinos gives

$$|\Delta m^2_{32}| = 2.32^{+0.12}_{-0.08} \times 10^{-3}\ \text{eV}^2, \tag{1.2a}$$

whereas data from solar and reactor neutrinos imply a squared-mass difference roughly 31 times smaller,

$$\Delta m^2_{21} = (7.50 \pm 0.20) \times 10^{-5}\ \text{eV}^2. \tag{1.2b}$$

Thus the neutrinos ν_1 and ν_2 form a quasi-doublet. One speaks of a *normal* or *inverted* neutrino mass spectrum, respectively, for the cases[4]

$$\text{normal: } m_3 > m_{1,2}, \qquad \text{inverted: } m_{1,2} > m_3. \tag{1.2c}$$

Since the largest neutrino mass m_{lgst}, be it m_2 or m_3, cannot be lighter than the mass splitting of Eq. (1.2), we have the bound $m_{\text{lgst}} > 0.049$ eV. Finally, a combination of cosmological inputs can be employed to bound the neutrino mass sum $\sum_{i=1}^{3} m_i$, the precise bound depending on the chosen input data set. In one example [deP *et al.* 12], photometric redshifts measured from a large galaxy sample, cosmic microwave background (CMB) data and a recent determination of the Hubble parameter are used to obtain the bound

$$m_1 + m_2 + m_3 < 0.26\ \text{eV}, \tag{1.3a}$$

whereas data from the CMB combined with that from baryon acoustic oscillations yields [Ad *et al.* (Planck collab.) 13]

$$m_1 + m_2 + m_3 < 0.23\ \text{eV}. \tag{1.3b}$$

A further discussion of the neutrino mass spectrum appears in Sect. VI–4.

[3] Only two of the mass differences can be independent, so $\Delta m^2_{12} + \Delta m^2_{23} + \Delta m^2_{31} = 0$.

[4] There is also the possibility of a *quasi-degenerate* neutrino mass spectrum ($m_1 \simeq m_2 \simeq m_3$), which can be thought of as a limiting case of both the normal and inverted cases in which the individual masses are sufficiently large to dwarf the $|\Delta m^2_{32}|$ splitting.

Quark and lepton numbers

Individual quark and lepton numbers are known to be not conserved, but for different reasons and with different levels of nonconservation. Individual quark number is not conserved in the Standard Model due to the charged weak interactions (cf. Sect. II–3). Indeed, quark transitions of the type $q_i \to q_j + W^{\pm}$ induce the decays of most meson and baryon states and have led to the phenomenology of *Flavor Physics*. Individual lepton number is not conserved, as evidenced by the observed $\nu_\alpha \leftrightarrow \nu_\beta$ $(\alpha, \beta = e, \mu, \tau)$ oscillations. This source of this phenomenon is associated with nonzero neutrino masses. There is currently no additional evidence for the violation of individual lepton number despite increasingly sensitive limits such as the branching fraction $\mathrm{B}_{\mu^- \to e^- e^- e^+} < 1.0 \times 10^{-12}$.

Existing data are consistent with conservation of total quark and total lepton number, e.g. the proton lifetime bound $\tau_p > 2.1 \times 10^{29}$ yr [RPP 12] and the nuclear half-life limit $t_{1/2}^{0\nu\beta\beta}[^{136}\mathrm{Xe}] > 1.6 \times 10^{25}$ yr [Ac *et al.* (EXO-200 collab.) 11]. These conservation laws are empirical. They are not required as a consequence of any known dynamical principle and in fact are expected to be violated by certain nonperturbative effects within the Standard Model (associated with quantum tunneling between topologically inequivalent vacua – see Sect. III–6).

I–2 Chiral fermions

Consider a world in which quarks and leptons have no mass at all. At first, this would appear to be a surprising supposition. To an experimentalist, mass is the most palpable property a particle has. It is why, say, a muon behaves differently from an electron in the laboratory. Nonetheless, the massless limit is where the Standard Model begins.

The massless limit

Let $\psi(x)$ be a solution to the Dirac equation for a massless particle,

$$i \not\partial \psi = 0. \tag{2.1}$$

We can multiply this equation from the left by γ_5 and use the anticommutativity of γ_5 with γ^μ to obtain another solution,

$$i \not\partial \gamma_5 \psi = 0. \tag{2.2}$$

We superpose these solutions to form the combinations

$$\psi_L = \frac{1}{2}(1 + \gamma_5)\psi, \qquad \psi_R = \frac{1}{2}(1 - \gamma_5)\psi, \tag{2.3}$$

where '1' represents the unit 4×4 matrix. The quantities ψ_L and ψ_R are solutions of definite *chirality* (i.e. handedness). For a massless particle moving with precise momentum, these solutions correspond respectively to the spin being anti-aligned (left-handed) and aligned (right-handed) relative to the momentum. In other words, chirality coincides with helicity for zero-mass particles. The matrices $\Gamma_{\substack{L\\R}} = (1 \pm \gamma_5)/2$ are chirality projection operators. They obey the usual projection operator conditions under addition,

$$\Gamma_L + \Gamma_R = 1, \tag{2.4}$$

and under multiplication,

$$\Gamma_L\Gamma_L = \Gamma_L, \qquad \Gamma_R\Gamma_R = \Gamma_R, \qquad \Gamma_L\Gamma_R = \Gamma_R\Gamma_L = 0. \tag{2.5}$$

In the massless limit, a particle's chirality is a Lorentz-invariant concept. For example, a particle which is left-handed to one observer will appear left-handed to all observers. Thus chirality is a natural label to use for massless fermions, and a collection of such particles may be characterized according to the separate numbers of left-handed and right-handed particles.

It is simple to incorporate chirality into a lagrangian formalism. The lagrangian for a massless noninteracting fermion is

$$\mathcal{L} = i\overline{\psi}\,\partial\!\!\!/\,\psi, \tag{2.6}$$

or in terms of chiral fields,

$$\mathcal{L} = \mathcal{L}_L + \mathcal{L}_R, \tag{2.7}$$

where

$$\mathcal{L}_{L,R} = i\overline{\psi}_{L,R}\partial\!\!\!/\,\psi_{L,R}. \tag{2.8}$$

The lagrangians $\mathcal{L}_{L,R}$ are invariant under the global chiral phase transformations

$$\psi_{L,R}(x) \to \exp(-i\alpha_{L,R})\psi_{L,R}(x), \tag{2.9}$$

where the phases $\alpha_{L,R}$ are constant and real-valued but otherwise arbitrary. Anticipating the discussion of Noether's theorem in Sect. I–4, we can associate conserved particle-number current densities $J^\mu_{L,R}$,

$$J^\mu_{L,R} = \overline{\psi}_{L,R}\gamma^\mu\psi_{L,R} \qquad (\partial_\mu J^\mu_{L,R} = 0), \tag{2.10}$$

with this invariance. From these chiral current densities, we can construct the vector current $V^\mu(x)$,

$$V^\mu = J^\mu_L + J^\mu_R \tag{2.11}$$

and the axial-vector current $A^\mu(x)$,

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$$A^\mu = J_L^\mu - J_R^\mu. \tag{2.12}$$

Chiral charges $Q_{L,R}$ are defined as spatial integrals of the chiral charge densities,

$$Q_{L,R}(t) = \int d^3x\ J^0_{L,R}(x), \tag{2.13}$$

and represent the number operators for the chiral fields $\psi_{L,R}$. They are time-independent if the chiral currents are conserved. One can similarly define the vector charge Q and the axial-vector charge Q_5,

$$Q(t) = \int d^3x\ V^0(x), \qquad Q_5(t) = \int d^3x\ A^0(x). \tag{2.14}$$

The vector charge Q is the total number operator,

$$Q = Q_R + Q_L, \tag{2.15}$$

whereas the axial-vector charge is the number operator for the difference

$$Q_5 = Q_L - Q_R. \tag{2.16}$$

The vector charge Q and axial-vector charge Q_5 simply count the sum and difference, respectively, of the left-handed and right-handed particles.

Parity, time reversal, and charge conjugation

The field transformations of Eq. (2.9) involve parameters $\alpha_{L,R}$ which can take on a continuum of values. In addition to such continuous field mappings, one often encounters a variety of *discrete* transformations as well. Let us consider the operations of parity

$$x = (x^0, \mathbf{x}) \to x_P = (x^0, -\mathbf{x}), \tag{2.17}$$

and of time reversal

$$x = (x^0, \mathbf{x}) \to x_T = (-x^0, \mathbf{x}), \tag{2.18}$$

as defined by their effects on spacetime coordinates. The effect of discrete transformations on a fermion field $\psi(x)$ will be implemented by a unitary operator P for parity and an antiunitary operator T for time reversal. In the representation of Dirac matrices used in this book, we have

$$P\psi(x)P^{-1} = \gamma^0\psi(x_P), \qquad T\psi(x)T^{-1} = i\gamma^1\gamma^3\psi(x_T). \tag{2.19}$$

An additional operation typically considered in conjunction with parity and time reversal is that of charge conjugation, the mapping of matter into antimatter,

$$C\psi(x)C^{-1} = i\gamma^2\gamma^0\overline{\psi}^T(x), \tag{2.20}$$

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Table I–4. *Response of Dirac bilinears to discrete mappings.*

C	P	T
$S(x)$	$S(x_P)$	$S(x_T)$
$P(x)$	$-P(x_P)$	$-P(x_T)$
$-J^\mu(x)$	$J_\mu(x_P)$	$J_\mu(x_T)$
$J_5^\mu(x)$	$-J_{5\mu}(x_P)$	$J_{5\mu}(x_T)$
$-T^{\mu\nu}(x)$	$T_{\mu\nu}(x_P)$	$-T_{\mu\nu}(x_T)$

where $\overline{\psi}_\beta^T \equiv \psi_\alpha^\dagger \gamma_{\alpha\beta}^0$ $(\alpha, \beta = 1, \ldots, 4)$. The spacetime coordinates of field $\psi(x)$ are unaffected by charge conjugation.

In the study of discrete transformations, the response of the normal-ordered Dirac bilinears

$$\begin{aligned} S(x) &= :\overline{\psi}(x)\psi(x): & P(x) &= :\overline{\psi}(x)\gamma_5\psi(x): \\ J^\mu(x) &= :\overline{\psi}(x)\gamma^\mu\psi(x): & J_5^\mu(x) &= :\overline{\psi}(x)\gamma^\mu\gamma_5\psi(x): \\ T^{\mu\nu}(x) &= :\overline{\psi}(x)\sigma^{\mu\nu}\psi(x): \end{aligned} \tag{2.21}$$

is of special importance to physical applications. Their transformation properties appear in Table I–4. Close attention should be paid there to the location of the indices in these relations. Another example of a field's response to these discrete transformations is that of the photon $A^\mu(x)$,

$$\begin{aligned} C\ A^\mu(x)\ C^{-1}\ c &= -A^\mu(x), & P\ A^\mu(x)\ P^{-1} &= \ A_\mu(x_P), \\ T\ A^\mu(x)\ T^{-1}\ c &= \ A_\mu(x_T). \end{aligned} \tag{2.22}$$

Beginning with the discussion of Noether's theorem in Sect. 1–4, we shall explore the topic of invariance throughout much of this book. It suffices to note here that the Standard Model, being a theory whose dynamical content is expressed in terms of hermitian, Lorentz-invariant lagrangians of local quantum fields, is guaranteed to be invariant under the combined operation CPT. Interestingly, however, these discrete transformations are individually symmetry operations only of the strong and electromagnetic interactions, but not of the full electroweak sector. We see already the possibility for such behavior in the occurrence of chiral fermions $\psi_{L,R}$, since parity maps the fields $\psi_{L,R}$ into each other,

$$\psi_{L,R} \to P\ \psi_{L,R}(x)\ P^{-1} = \gamma^0 \psi_{R,L}(x_P). \tag{2.23}$$

Thus any effect, like the weak interaction, which treats left-handed and right-handed fermions differently, will lead inevitably to parity-violating phenomena.

I–3 Fermion mass

Although the discussion of chiral fermions is cast in the limit of zero mass, fermions in Nature do in fact have nonzero mass and we must account for this. In a lagrangian, a mass term will appear as a hermitian, Lorentz-invariant bilinear in the fields. For fermion fields, these conditions allow realizations referred to as *Dirac* mass and *Majorana* mass.[5]

Dirac mass

The Dirac mass term for fermion fields $\psi_{L,R}$ involves the bilinear coupling of fields with opposite chirality

$$-\mathcal{L}_D = m_D[\,\overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L\,] = m_D\,\overline{\psi}\psi \tag{3.1}$$

where $\psi \equiv \psi_L + \psi_R$ and m_D is the Dirac mass. The Dirac mass term is invariant under the phase transformation $\psi(x) \to \exp(-i\alpha)\psi(x)$ and thus does not upset conservation of the vector current $V^\mu = \overline{\psi}\gamma^\mu\psi$ and the corresponding number fermion operator Q of Eq. (2.15). All fields in the Standard Model, save possibly for the neutrinos, have Dirac masses obtained from their interaction with the Higgs field (cf. Sects. II–3, II–4). Although right-handed neutrinos have no couplings to the Standard Model gauge bosons, there is no principle prohibiting their interaction with the Higgs field and thus generating neutrino Dirac masses in the same manner as the other particles.

Majorana mass

A Majorana mass term is one which violates fermion number by coupling two fermions (or two antifermions). In the Majorana construction, use is made of the *charge-conjugate* fields,

$$\psi^c \equiv C\gamma^0\psi^*, \qquad (\psi_{L,R})^c = (\Gamma_{L,R}\psi)^c, \tag{3.2}$$

where C is the charge-conjugation operator, obeying

$$C = -C^{-1} = -C^\dagger = -C^T. \tag{3.3}$$

In the Dirac representation of gamma matrices (cf. App. C), one has $C = i\gamma^2\gamma^0$. Some useful identities involving ψ^c include

[5] We suppress spacetime dependence of the fields in this section.

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$$\begin{aligned}
\overline{(\psi_i^c)}\,\psi_j &= \psi_i^T\,C\,\psi_j, & \overline{\psi_i}\,\psi_j^c &= -\psi_i^{*T}\,C\,\psi_j^*,\\
\left(\overline{(\psi_i^c)}\,\psi_j\right)^\dagger &= \overline{\psi_j}\,\psi_i^c, & \left(\psi_i^T\,C\,\psi_j\right)^\dagger &= -\psi_j^{*T}\,C\,\psi_i^*,\\
\overline{(\psi_i^c)}\,\psi_j^c &= \overline{\psi_j}\,\psi_i, & \overline{(\psi_i^c)}\,\gamma^\mu\,\psi_j^c &= -\overline{\psi_j}\,\gamma^\mu\,\psi_i,\\
\overline{(\psi_R^c)}\,\psi_L &= 0, & \overline{(\psi_R^c)}\,\gamma^\mu\,\psi_R &= 0.
\end{aligned} \tag{3.4}$$

The two identities in the bottom line follow from $\Gamma_R C \Gamma_L = 0$.

The possibility of a Majorana mass term follows from the fact that a combination of two fermion fields $\psi^T C \psi$ is an invariant under Lorentz transformations. Two equivalent expressions for a Majorana mass term involving chiral fields $\psi_{L,R}$ are[6]

$$\begin{aligned}
-\mathcal{L}_M &= \frac{m_{L,R}}{2}\left[\overline{(\psi_{L,R})^c}\,\psi_{L,R} + \overline{\psi_{L,R}}\,(\psi_{L,R})^c\right]\\
&= \frac{m_{L,R}}{2}\left[(\psi_{L,R})^T C \psi_{L,R} - (\psi_{L,R}^*)^T C \psi_{L,R}^*\right].
\end{aligned} \tag{3.5}$$

Because the cross combination $(\psi_R)^T C \psi_L = 0$, the Majorana mass terms involves either two left-chiral fields or two right-chiral fields, and the left-chiral and right-chiral masses are independent. Treating ψ and ψ^* as independent variables, the resulting equations of motion are

$$i\not\partial\,\psi_R - m_R\,\psi_R^c = 0, \qquad i\not\partial\,\psi_R^c - m_R\,\psi_R = 0, \tag{3.6}$$

with a similar set of equations for ψ_L. Iteration of these coupled equations shows that m_R indeed behaves as a mass.

A Majorana mass term clearly does not conserve fermion number and mixes the particle with its antiparticle. Indeed, a Majorana fermion can be identified with its own antiparticle. This can be seen, using ψ_R as an example, by rewriting the lagrangian in terms of the self-conjugate field

$$\psi_M = \frac{1}{\sqrt{2}}\left[\psi_R + \psi_R^c\right], \tag{3.7}$$

which, given the equations of motion above, will clearly satisfy the Dirac equation. The total Majorana lagrangian can be simply rewritten in terms of this self-conjugate field as

$$\begin{aligned}
\mathcal{L}_{KE}^{(R)} + \mathcal{L}_M^{(R)} &= \overline{\psi}_R i\not\partial\,\psi_R - \frac{m_R}{2}\left[\overline{(\psi_R)^c}\,\psi_R + \overline{\psi_R}\,(\psi_R)^c\right]\\
&= \overline{\psi}_M i\not\partial\,\psi_M - m_R\overline{\psi}_M\,\psi_M\\
&= \psi_M^T C i\not\partial\,\psi_M - m_R\psi_M^T C\,\psi_M,
\end{aligned} \tag{3.8}$$

[6] The factor of $1/2$ with the Majorana mass parameters $m_{L,R}^{(M)}$ compensates for a factor of 2 encountered in taking the matrix element of the Majorana mass term.

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where again the identity $\Gamma_R\ C\Gamma_L = 0$ plays a role in this construction. To avoid the possibility of nonconservation of charge, any Majorana mass term must be restricted to a field with is neutral under the gauge charges. Thus, among the particles of the Standard Model, we will see that only right-handed neutrinos satisfy this condition.

Finally, we note that a Dirac field can be written as two Majorana fields with opposite masses via the construction

$$\psi_a = \frac{1}{\sqrt{2}}\left[\psi_R + \psi_L^c\right], \qquad \psi_b = \frac{1}{\sqrt{2}}\left[\psi_L - \psi_R^c\right], \tag{3.9}$$

in which case we find

$$\begin{aligned} -\mathcal{L}_D &= m_D\left[\overline{\psi_L}\ \psi_R\ + \overline{\psi_R}\ \psi_L\right] \\ &= \frac{m_D}{2}\ \overline{(\psi_a^c)}\ \psi_a - \frac{m_D}{2}\ \overline{(\psi_b)^c}\ \psi_b\ +\ \text{h.c.} \end{aligned} \tag{3.10}$$

The apparent violation of lepton number that looks like it would arise from this framework does not actually occur, because the effects proportional to the mass of these fields will cancel due to the minus sign between the two mass terms. To make matters look even more puzzling, we can flip the sign on the mass term for the second field, by the field redefinition $\psi_b\ \to\ i\ \psi_b$ in which case both masses appear positive. However in this case, the weak current would pick up an unusual factor of i, since the left-handed field would then become

$$\psi_L \to \frac{1}{\sqrt{2}}\left[i\ \psi_b + \psi_a^c\right]. \tag{3.11}$$

In this case, potential lepton-number violating processes would cancel between the two fields because of the occurrence of a factor of $i^2 = -1$ from the application of the weak currents. These algebraic gymnastics become more physically relevant when we combine both Dirac and Majorana mass terms in Chap. VI.

I–4 Symmetries and near symmetries

A symmetry is said to arise in Nature whenever some change in the variables of a system leaves the essential physics unchanged. In field theory, the dynamical variables are the fields, and symmetries describe invariances under transformations of the fields. For example, one associates with the spacetime translation $x_\mu \to x_\mu + a_\mu$ a transformation of the field $\psi(x)$ to $\psi(x+a)$. In turn, the 'essential physics' is best described by an action, at least in classical physics. If the action is invariant, the equations of motion, and hence the classical physics, will be unchanged. The invariances of quantum physics are identified by consideration of matrix elements or, equivalently, of the path integral. We begin the study of symmetries here by exploring several lagrangians which have invariances and by considering some of the consequences of these symmetries.

Noether currents

The classical analysis of symmetry focusses on the lagrangian, which in general is a Lorentz-scalar function of several fields, denoted by φ_i, and their first derivatives $\partial_\mu \varphi_i$, i.e. $\mathcal{L} = \mathcal{L}(\varphi_i, \partial_\mu \varphi_i)$. *Noether's theorem* states that for any invariance of the action under a continuous transformation of the fields, there exists a classical charge Q which is time-independent ($\dot{Q} = 0$) and is associated with a conserved current, $\partial_\mu J^\mu = 0$. This theorem covers both internal and spacetime symmetries. For most[7] internal symmetries, the lagrangian is itself invariant. Given a continuous field transformation, one can always consider an infinitesimal transformation

$$\varphi_i'(x) = \varphi_i(x) + \epsilon f_i(\varphi), \tag{4.1}$$

where ϵ is an infinitesimal parameter and $f_i(\varphi)$ is a function of the fields in the theory. The procedure for constructing the Noether current of an internal symmetry is to temporarily let ϵ become a function of x and to define the quantity

$$\hat{\varphi}_i(x) = \varphi_i(x) + \epsilon(x) f_i(\varphi), \tag{4.2}$$

such that in the restriction back to constant ϵ, $\mathcal{L}$ becomes invariant and $\hat{\varphi}_i(x) \to \varphi_i'(x)$. For an internal symmetry, the Noether current is then defined by

$$J^\mu(x) \equiv \frac{\partial}{\partial(\partial_\mu \epsilon(x))} \mathcal{L}(\hat{\varphi}, \partial\hat{\varphi}). \tag{4.3}$$

Use of the equation of motion together with the invariance of the lagrangian under the transformation in Eq. (4.1) yields $\partial_\mu J^\mu = \partial \mathcal{L}/\partial \epsilon(x) = 0$ as desired. The Noether charge $Q = \int d^3x\, J_0$ is time-independent if the current vanishes sufficiently rapidly at spatial infinity, i.e.

$$\frac{dQ}{dt} = \int d^3x\, \partial_0 J_0 = -\int d^3x\, \nabla \cdot \mathbf{J} = 0. \tag{4.4}$$

We refer the reader to field theory textbooks for further discussion, including the analogous procedure for constructing Noether currents of spacetime symmetries.

Identifying the current does not exhaust all the consequences of a symmetry but is merely the first step towards the implementation of symmetry relations. Notice that we have been careful to use the word 'classical' several times. This is because the invariance of the action is not generally sufficient to identify symmetries of a quantum theory. We shall return to this point.

[7] An exception occurs for the so-called topological gauge symmetries.

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Examples of Noether currents

Let us now consider some explicit field theory models in order to get practice in constructing Noether currents.

(i) *Isospin symmetry*: $SU(2)$ isospin invariance of the nucleon–pion system provides a standard and uncomplicated means for studying symmetry currents. Consider a doublet of nucleon fields

$$\psi = \begin{pmatrix} p \\ n \end{pmatrix}, \tag{4.5}$$

and a triplet of pion fields $\boldsymbol{\pi} = \{\pi^i\}$ $(i = 1, 2, 3)$ with lagrangian

$$\mathcal{L} = \bar{\psi}\,(i\partial\!\!\!/ - \mathbf{m})\,\psi + \frac{1}{2}\left[\partial_\mu \boldsymbol{\pi}\cdot\partial^\mu \boldsymbol{\pi} - m_\pi^2 \boldsymbol{\pi}\cdot\boldsymbol{\pi}\right] + ig\bar{\psi}\boldsymbol{\tau}\cdot\boldsymbol{\pi}\gamma_5\psi - \frac{\lambda}{4}\,(\boldsymbol{\pi}\cdot\boldsymbol{\pi})^2 \tag{4.6}$$

where $\mathbf{m}$ is the nucleon mass matrix

$$\mathbf{m} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

and $\tau = \{\tau^i\}(i = 1, 2, 3)$ are the three Pauli matrices. This lagrangian is invariant under the *global* $SU(2)$ rotation of the fields

$$\psi \to \psi' = U\,\psi, \qquad U = \exp\left(-i\boldsymbol{\tau}\cdot\boldsymbol{\alpha}/2\right) \tag{4.7}$$

for any α^i, $(i = 1, 2, 3)$ provided the pion fields are transformed as

$$\boldsymbol{\tau}\cdot\boldsymbol{\pi} \to \boldsymbol{\tau}\cdot\boldsymbol{\pi}' = U\boldsymbol{\tau}\cdot\boldsymbol{\pi}U^\dagger. \tag{4.8}$$

In proving this, it is useful to employ the identity

$$\boldsymbol{\pi}\cdot\boldsymbol{\pi} = \frac{1}{2}\,\mathrm{Tr}\,\left(\boldsymbol{\tau}\cdot\boldsymbol{\pi}\,\boldsymbol{\tau}\cdot\boldsymbol{\pi}\right), \tag{4.9}$$

from which we easily see that $\pi^i\pi^i$ is invariant under the transformation of Eq. (4.8). The response of the individual pion components to an isospin transformation can be found from multiplying Eq. (4.8) by τ^i and taking the trace,

$$\pi'^{\,i} = R^{ij}(\boldsymbol{\alpha})\pi^j, \qquad R^{ij}(\boldsymbol{\alpha}) = \frac{1}{2}\,\mathrm{Tr}\,\left(\tau^i U\tau^j U^\dagger\right). \tag{4.10}$$

To determine the isospin current, one considers the spacetime-dependent transformation with $\boldsymbol{\alpha}$ now *infinitesimal*,

$$\hat{\psi} = (1 - i\boldsymbol{\tau}\cdot\boldsymbol{\alpha}(x)/2)\,\psi, \qquad \hat{\pi}^i = \pi^i - \epsilon^{ijk}\pi^j\alpha^k(x). \tag{4.11}$$

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Performing this transformation on the lagrangian gives

$$\mathcal{L}(\hat{\psi}, \hat{\pi}) = \mathcal{L}(\psi, \pi) + \frac{1}{2}\bar{\psi}\gamma^{\mu}\boldsymbol{\tau}\cdot\partial_{\mu}\boldsymbol{\alpha}\psi - \epsilon^{ijk}(\partial_{\mu}\pi^{i})\pi^{j}\partial^{\mu}\alpha_{k}, \tag{4.12}$$

and applying our expression Eq. (4.3) for the current yields the triplet of currents (one for each α_i)

$$V_{\mu}^{i} = \bar{\psi}\gamma_{\mu}\frac{\tau^{i}}{2}\psi + \epsilon^{ijk}\pi^{j}\partial_{\mu}\pi^{k}. \tag{4.13}$$

By use of the equations of motion for ψ and π, it is straightforward to verify that this current is conserved.

(ii) *The linear sigma model*: With a few modifications the above example becomes one of the most instructive of all field theory models, the sigma model [GeL 60]. One adds to the lagrangian of Eq. (4.6) a scalar field σ with judiciously chosen couplings, and removes the bare nucleon mass,

$$\begin{aligned}\mathcal{L} = {} & \bar{\psi}i\partial\!\!\!/\psi + \frac{1}{2}\partial_{\mu}\boldsymbol{\pi}\cdot\partial^{\mu}\boldsymbol{\pi} + \frac{1}{2}\partial_{\mu}\sigma\partial^{\mu}\sigma \\ & - g\bar{\psi}\left(\sigma - i\boldsymbol{\tau}\cdot\boldsymbol{\pi}\gamma_{5}\right)\psi + \frac{\mu^{2}}{2}\left(\sigma^{2}+\boldsymbol{\pi}^{2}\right) - \frac{\lambda}{4}\left(\sigma^{2}+\boldsymbol{\pi}^{2}\right)^{2}.\end{aligned} \tag{4.14}$$

For $\mu^2 > 0$, the model exhibits the phenomenon of spontaneous symmetry breaking (cf. Sect. I–6). In describing the symmetries of this lagrangian, it is useful to rewrite the mesons in terms of a matrix field

$$\Sigma \equiv \sigma + i\boldsymbol{\tau}\cdot\boldsymbol{\pi}, \tag{4.15}$$

such that

$$\sigma^{2} + \boldsymbol{\pi}^{2} = \frac{1}{2}\operatorname{Tr}\left(\Sigma^{\dagger}\Sigma\right). \tag{4.16}$$

Then we obtain

$$\begin{aligned}\mathcal{L} = {} & \bar{\psi}_{L}i\partial\!\!\!/\psi_{L} + \bar{\psi}_{R}i\partial\!\!\!/\psi_{R} + \frac{1}{4}\operatorname{Tr}\left(\partial_{\mu}\Sigma\partial^{\mu}\Sigma^{\dagger}\right) \\ & + \frac{1}{4}\mu^{2}\operatorname{Tr}\left(\Sigma^{\dagger}\Sigma\right) - \frac{\lambda}{16}\operatorname{Tr}^{2}\left(\Sigma^{\dagger}\Sigma\right) - g\left(\bar{\psi}_{L}\Sigma\psi_{R} + \bar{\psi}_{R}\Sigma^{\dagger}\psi_{L}\right),\end{aligned} \tag{4.17}$$

where $\psi_{L,R}$ are chiral fields (cf. Eq. (2.3)). The left-handed and right-handed fermion fields are coupled together only in the interaction with the Σ field. The purely mesonic portion of the lagrangian is obviously invariant under rotations among the $\sigma, \boldsymbol{\pi}$ fields. The full lagrangian has separate 'left' and 'right' invariances, i.e. $SU(2)_L \times SU(2)_R$,

$$\psi_{L,R} \to \psi'_{L,R} = U_{L,R}\psi_{L,R}, \qquad \Sigma \to \Sigma' = U_{L}\Sigma U_{R}^{\dagger}, \tag{4.18}$$

with U_L and U_R being arbitrary $SU(2)$ matrices,

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$$U_{L,R} = \exp\left(-i\boldsymbol{\alpha}_{L,R} \cdot \boldsymbol{\tau}/2\right). \tag{4.19}$$

The fermion portions of the transformation clearly involve just the $SU(2)$ isospin rotations on the left-handed and right-handed fermions. However, the mesons involve a combination of a pure isospin rotation among the $\boldsymbol{\pi}$ fields together with a transformation between the σ and $\boldsymbol{\pi}$ fields

$$\begin{aligned}
\sigma \to \sigma' &= \frac{1}{2}\,\mathrm{Tr}\,\left(U_L U_R^\dagger\right)\sigma + \frac{i}{2}\,\mathrm{Tr}\,\left(U_L \tau^k U_R^\dagger\right)\pi^k \\
&\simeq \sigma + \frac{1}{2}\left(\boldsymbol{\alpha}_L - \boldsymbol{\alpha}_R\right)\cdot\boldsymbol{\pi}, \\
\pi^k \to \pi'^k &= -\frac{i}{2}\,\mathrm{Tr}\,\left(\tau^k U_L U_R^\dagger\right)\sigma + \frac{1}{2}\,\mathrm{Tr}\,\left(\tau^k U_L \tau^\ell U_R^\dagger\right)\pi^\ell \\
&\simeq \pi^k - \frac{1}{2}\left(\alpha_L^k - \alpha_R^k\right)\sigma - \frac{1}{2}\epsilon^{k\ell m}\pi^\ell\left(\alpha_L^m + \alpha_R^m\right),
\end{aligned} \tag{4.20}$$

where the second form in each case is for infinitesimal $\boldsymbol{\alpha}_L$, $\boldsymbol{\alpha}_R$. For each invariance there is a separate conserved current

$$\begin{aligned}
J_{L\mu}^k &= \bar{\psi}_L\gamma_\mu\frac{\tau^k}{2}\psi_L - \frac{i}{8}\,\mathrm{Tr}\,\left(\tau^k\left(\Sigma\partial_\mu\Sigma^\dagger - \partial_\mu\Sigma\,\Sigma^\dagger\right)\right) \\
&= \bar{\psi}_L\gamma_\mu\frac{\tau^k}{2}\psi_L - \frac{1}{2}\left(\sigma\,\partial_\mu\pi^k - \pi^k\partial_\mu\sigma\right) + \frac{1}{2}\epsilon^{k\ell m}\pi^\ell\partial_\mu\pi^m, \\
J_{R\mu}^k &= \bar{\psi}_R\gamma_\mu\frac{\tau^k}{2}\psi_R + \frac{i}{8}\,\mathrm{Tr}\,\left(\tau^k\left(\partial_\mu\Sigma^\dagger\Sigma - \Sigma^\dagger\partial_\mu\Sigma\right)\right) \\
&= \bar{\psi}_R\gamma_\mu\frac{\tau^k}{2}\psi_R + \frac{1}{2}\left(\sigma\,\partial_\mu\pi^k - \pi^k\partial_\mu\sigma\right) + \frac{1}{2}\epsilon^{k\ell m}\pi^\ell\partial_\mu\pi^m.
\end{aligned} \tag{4.21}$$

These can be formed into a conserved vector current

$$V_\mu^k = J_{L\mu}^k + J_{R\mu}^k = \bar{\psi}\gamma_\mu\frac{\tau^k}{2}\psi + \epsilon^{k\ell m}\pi^\ell\partial_\mu\pi^m, \tag{4.22}$$

which is just the isospin current derived previously, and a conserved axial-vector current

$$A_\mu^k = J_{L\mu}^k - J_{R\mu}^k = \bar{\psi}\gamma_\mu\gamma_5\frac{\tau^k}{2}\psi + \pi^k\partial_\mu\sigma - \sigma\,\partial_\mu\pi^k. \tag{4.23}$$

(iii) *Scale invariance*: Our third example illustrates the case of a spacetime transformation in which the lagrangian changes by a total derivative. Consider classical electrodynamics (cf. Sect. II–1) but with a massless electron,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\,i\not{D}\psi, \tag{4.24}$$

where ψ is the electron field, $D_\mu\psi = (\partial_\mu + ieA_\mu)\psi$ is the covariant derivative of ψ, A_μ is the photon field, and $F_{\mu\nu}$ is the electromagnetic field strength. We shall

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describe the construction of both $D_\mu \psi$ and $F_{\mu\nu}$ in the next section. Since there are no dimensional parameters in this lagrangian, we are motivated to consider the effect of a change in coordinate scale $x \to x' = \lambda x$ together with the field transformations

$$\psi(x) \to \psi'(x) = \lambda^{3/2}\psi(\lambda x), \qquad A_\mu(x) \to A'_\mu(x) = \lambda A_\mu(\lambda x). \tag{4.25}$$

Although the lagrangian itself is not invariant,

$$\mathcal{L}(x) \to \mathcal{L}'(x) = \lambda^4 \mathcal{L}(\lambda x), \tag{4.26}$$

with a change of variable the action is easily seen to be unchanged,

$$S = \int d^4x\, \mathcal{L}(x) \to \int d^4x\, \lambda^4 \mathcal{L}(\lambda x) = \int d^4x'\, \mathcal{L}(x') = S. \tag{4.27}$$

There is nothing in this classical theory which depends on how length is scaled. The Noether current associated with the change of scale is

$$J^\mu_{\text{scale}} \equiv x_\nu \theta^{\mu\nu}, \tag{4.28}$$

where $\theta^{\mu\nu}$ is the energy-momentum tensor of the theory,

$$\theta^{\mu\nu} = -g^{\mu\nu}\left[-\frac{1}{4}F^{\lambda\sigma}F_{\lambda\sigma} + \bar{\psi} i \not{D} \psi\right] - F^{\mu\lambda}F^\nu{}_\lambda + A^\nu \partial_\lambda F^{\mu\lambda} + \frac{i}{2}\bar{\psi}\gamma^\mu \overleftrightarrow{\partial}^\nu \psi. \tag{4.29}$$

Since the energy-momentum tensor is itself conserved, $\partial_\mu \theta^{\mu\nu} = 0$, the conservation of scale current is equivalent to the vanishing of the trace of the energy-momentum tensor,

$$\partial_\mu J^\mu_{\text{scale}} = \theta_\mu{}^\mu = 0. \tag{4.30}$$

This trace property may be easily verified using the equations of motion.

Approximate symmetry

Thus far, we have been describing exact symmetries. Symmetry considerations are equally useful in situations where there is ‘almost’ a symmetry. The very phrase ‘approximate symmetry’ seems self-contradictory and needs explanation. Quite often a lagrangian would have an invariance if certain of the parameters in it were set equal to zero. In that limit the invariance would have a set of physical consequences which, with the said parameters being nonzero, would no longer obtain. Yet, if the parameters are in some sense ‘small’, the predicted consequences are still approximately valid. In fact, when the interaction which breaks the symmetry has a well-defined behavior under the symmetry transformation, its effect can generally be analyzed in terms of the basis of unperturbed particle states by using the

Wigner–Eckart theorem. The precise sense in which the symmetry-breaking terms can be deemed small depends on the problem under consideration. In practice, the utility of an approximate symmetry is rarely known *a priori*, but is only evident after its predictions have been checked experimentally.

If a symmetry is not exact, the associated currents and charges will no longer be conserved. For example, in the linear sigma model, the symmetry is partially broken if we add to the lagrangian a term of the form

$$\mathcal{L}' = a\,\sigma = \frac{a}{2}\,\mathrm{Tr}\,\Sigma, \tag{4.31}$$

where Σ is the matrix defined in Eq. (4.15). With this addition, the vector isospin $SU(2)$ symmetry remains exact but the axial $SU(2)$ transformation is no longer an invariance. The axial-current divergence becomes

$$\partial^\mu A^i_\mu = a\pi^i, \tag{4.32}$$

and the charge is time-dependent,

$$\frac{dQ^i_5}{dt} = a\int d^3x\,\pi^i. \tag{4.33}$$

In the linear sigma model, if the parameters g, λ are of order unity it is clear that the perturbation is small provided $1 \gg a/\mu^3$, as μ is the only other mass scale in the theory. However, if either g or λ happens to be anomalously large or small, the condition appropriate for a 'small' perturbation is not *a priori* evident.

In our example (iii) of scale invariance in massless fermion electrodynamics, the addition of an electron mass

$$\mathcal{L}_{\mathrm{mass}} = -m\bar{\psi}\psi \tag{4.34}$$

would explicitly break the symmetry and the trace would no longer vanish,

$$\theta_\mu^{\ \mu} = m\bar{\psi}\psi \neq 0. \tag{4.35}$$

This is in fact what occurs in practice. Fermion mass is typically *not* a small parameter in *QED* and cannot be treated as a perturbation in most applications.

I–5 Gauge symmetry

In our discussion of chiral symmetry, we considered the effect of global phase transformations, $\psi_{L,R}(x) \to \exp\,(-i\alpha_{L,R})\psi_{L,R}(x)$. Global phase transformations are those which are constant throughout all spacetime. Let us reconsider the system of chiral fermions, but now insist that the phase transformations be *local*. Each transformation is then labeled by a spacetime-dependent phase $\alpha_{L,R}(x)$,

$$\psi_{L,R}(x) \to \exp(-i\alpha_{L,R}(x))\;\psi_{L,R}(x). \tag{5.1}$$

Such local mappings are referred to as *gauge transformations*. The free massless lagrangian of Eq. (2.1) is not invariant under the gauge transformation

$$i\overline{\psi}_{L,R}(x)\partial\!\!\!/\psi_{L,R}(x) \rightarrow i\overline{\psi}_{L,R}(x)\partial\!\!\!/\,\psi_{L,R}(x) + \overline{\psi}_{L,R}(x)\gamma^{\mu}\psi_{L,R}(x)\cdot\partial_{\mu}\alpha_{L,R}(x), \tag{5.2}$$

because of the spacetime dependence of $\alpha_{L,R}$. In order for such a local transformation to be an invariance of the lagrangian, we need an extended kind of derivative D_μ, such that

$$D_\mu\psi_{L,R}(x) \rightarrow \exp(-i\alpha_{L,R}(x))D_\mu\psi_{L,R}(x) \tag{5.3}$$

under the local transformation of Eq. (5.1). The quantity D_μ is a *covariant derivative,* so called because it responds covariantly, as in Eq. (5.3), to a gauge transformation.

Abelian case

Before proceeding with the construction of a covariant derivative, we broaden the context of our discussion. Let $\Theta(x)$ now represent a boson or fermion field of any spin and arbitrary mass. We consider transformations

$$\Theta \rightarrow U(\alpha)\Theta \tag{5.4}$$

$$D_\mu\Theta \rightarrow U(\alpha)D_\mu\Theta, \tag{5.5}$$

with a spacetime-dependent parameter, $\alpha = \alpha(x)$. Suppose these gauge transformations form an abelian group, e.g., as do the set of phase transformations of Eq. (5.1).[8] It is sufficient to consider transformations with just one parameter as in Eqs. (5.4)–(5.5) since we can use direct products of these to construct arbitrary abelian groups.

One can obtain a covariant derivative by introducing a vector field $A_\mu(x)$, called a *gauge field*, by means of the relation

$$D_\mu\Theta = (\partial_\mu + ifA_\mu)\Theta, \tag{5.6}$$

where f is a real-valued coupling constant whose numerical magnitude depends in part on the field Θ. For example, in electrodynamics f becomes the electric charge of Θ. The problem is then to determine how A_μ must transform under a gauge transformation in order to give Eq. (5.5). This can be done by inspection, and we find

$$A_\mu \rightarrow A_\mu + \frac{i}{f}\partial_\mu U(\alpha)\cdot U^{-1}(\alpha). \tag{5.7}$$

[8] An *abelian* group is one whose elements commute. A *nonabelian* group is one which is not abelian.

The gauge field A_μ must itself have a kinetic contribution to the lagrangian. This is written in terms of a field strength, $F_{\mu\nu}$, which is antisymmetric in its indices. A general method for constructing such an antisymmetric second rank tensor is to use the commutator of covariant derivatives,

$$[D_\mu, D_\nu]\,\Theta \equiv i f F_{\mu\nu}\Theta. \tag{5.8}$$

By direct substitution we find

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{5.9}$$

It follows from Eq. (5.7) and Eq. (5.9) that the field strength $F_{\mu\nu}$ is invariant under gauge transformations. A gauge-invariant lagrangian containing a complex scalar field φ and a spin one-half field ψ, chiral or otherwise, has the form

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D^\mu\varphi)^\dagger D_\mu\varphi + i\overline{\psi}\not{D}\psi + \cdots, \tag{5.10}$$

where the ellipses stand for possible mass terms and nongauge field interactions. There is no contribution corresponding to a gauge-boson mass. Such a term would be proportional to $A^\mu A_\mu$, which is not invariant under the gauge transformation, Eq. (5.7).

Nonabelian case

The above reasoning can be generalized to nonabelian groups [YaM 54]. First, we need a nonabelian group of gauge transformations and a set of fields which forms a representation of the gauge group. Then, we must construct an appropriate covariant derivative to act on the fields. This step involves introducing a set of gauge bosons and specifying their behavior under the gauge transformations. Finally, the gauge field strength is obtained from the commutator of covariant derivatives, at which point we can write down a gauge-invariant lagrangian.

Consider fields $\Theta = \{\Theta_i\}$ $(i = 1, \ldots, r)$, which form an r-dimensional representation of a nonabelian gauge group $\mathcal{G}$. The Θ_i can be boson or fermion fields of any spin. In the following it will be helpful to think of Θ as an r-component column vector, and operations acting on Θ as $r \times r$ matrices. We take group $\mathcal{G}$ to have a Lie algebra of dimension n, so that the numbers of group generators, group parameters, gauge fields, and components of the gauge field strength are each n. We write the spacetime-dependent group parameters as the n-dimensional vector $\vec{\alpha} = \{\alpha_a(x)\}$ $(a = 1, \ldots, n)$. A gauge transformation on Θ is

$$\Theta' = \mathbf{U}(\vec{\alpha})\Theta, \tag{5.11}$$

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where the $r \times r$ matrix $\mathbf{U}$ is an element of group $\mathcal{G}$. For those elements of $\mathcal{G}$ which are connected continuously to the identity operator, we can write

$$\mathbf{U}(\vec{\alpha}) = \exp(-i\alpha^a \mathbf{G}^a), \tag{5.12}$$

where $\vec{\mathbf{G}} = \{\mathbf{G}_a\}$ $(a = 1, \ldots, n)$ are the *generators* of the group $\mathcal{G}$ expressed as hermitian $r \times r$ matrices. The set of generators obeys the Lie algebra

$$[\mathbf{G}^a, \mathbf{G}^b] = ic^{abc}\mathbf{G}^c \qquad (a, b, c = 1, \ldots, n), \tag{5.13}$$

where $\{c^{abc}\}$ are the structure constants of the algebra. We construct the covariant derivative $\mathbf{D}_\mu \Theta$ in terms of gauge fields $\vec{B}_\mu = \{B^a_\mu\}$ $(a = 1, \ldots, n)$ as

$$\mathbf{D}_\mu \Theta = (\mathbf{I}\partial_\mu + ig\mathbf{B}_\mu)\Theta, \tag{5.14}$$

where g is a coupling constant analogous to f in Eq. (5.6). In Eq. (5.14), $\mathbf{I}$ is the $r \times r$ unit matrix, and

$$\mathbf{B}_\mu \equiv \mathbf{G}^a B^a_\mu. \tag{5.15}$$

Realizing that the covariant derivative must transform as

$$(\mathbf{D}_\mu \Theta)' = \mathbf{U}(\vec{\alpha})(\mathbf{D}_\mu \Theta), \tag{5.16}$$

we infer from Eqs. (5.12)–(5.14) the response, in matrix form, of the gauge fields,

$$\mathbf{B}'_\mu = \mathbf{U}(\vec{\alpha})\mathbf{B}_\mu \mathbf{U}^{-1}(\vec{\alpha}) + \frac{i}{g}\partial_\mu \mathbf{U}(\vec{\alpha}) \cdot \mathbf{U}^{-1}(\vec{\alpha}). \tag{5.17}$$

The field strength matrix $\mathbf{F}_{\mu\nu}$ is found, as before, from the commutator of covariant derivatives,

$$[\mathbf{D}_\mu, \mathbf{D}_\nu]\Theta \equiv ig\mathbf{F}_{\mu\nu}\Theta, \tag{5.18}$$

implying

$$\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{B}_\nu - \partial_\nu \mathbf{B}_\mu + ig[\mathbf{B}_\mu, \mathbf{B}_\nu]. \tag{5.19}$$

Eqs. (5.17) and (5.19) provide the field strength transformation property,

$$\mathbf{F}'_{\mu\nu} = \mathbf{U}(\vec{\alpha})\mathbf{F}_{\mu\nu}\mathbf{U}^{-1}(\vec{\alpha}). \tag{5.20}$$

Unlike its abelian counterpart, the nonabelian field strength is not gauge invariant. Finally, we write down the gauge-invariant lagrangian

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\,(\mathbf{F}^{\mu\nu}\mathbf{F}_{\mu\nu}) + (\mathbf{D}^\mu \Phi)^* \mathbf{D}_\mu \Phi + i\overline{\Psi}\,\not{\mathbf{D}}\Psi + \cdots, \tag{5.21}$$

where Φ and Ψ are distinct multiplets of scalar and spin one-half fields and the ellipses represent possible mass terms and nongauge interactions. Analogously to the abelian case, there is no gauge-boson mass term.

The most convenient approach for demonstrating the theory's formal gauge structure is the matrix notation. However, in specific calculations it is sometimes more convenient to work with individual fields. To cast the matrix equations into component form, we employ a normalization of group generators consistent with Eq. (5.21),

$$\mathrm{Tr}(\mathbf{G}^a \mathbf{G}^b) = \frac{1}{2}\delta^{ab} \qquad (a, b = 1, \ldots, n). \tag{5.22}$$

To obtain the a^{th} component of the field strength $\vec{F}_{\mu\nu} = \{F^a_{\mu\nu}\}$ $(a = 1, \ldots, n)$, we matrix multiply Eq. (5.19) from the left by $\mathbf{G}^a$ and take the trace to find

$$F^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu - g c^{abc} B^b_\mu B^c_\nu \qquad (a, b, c = 1, \ldots, n). \tag{5.23}$$

The lagrangian Eq. (5.21) can likewise be rewritten in component form,

$$\mathcal{L} = -\frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} + (D^\mu_{km}\varphi_m)^\dagger (D_\mu)_{kn}\varphi_n + i\overline{\psi}_i (\not{D})_{ij}\psi_j + \cdots, \tag{5.24}$$

where $a = 1, \ldots, n$ and the remaining indices cover the dimensionalities of their respective multiplets.

Mixed case

In the Standard Model, it is a combination of abelian and nonabelian gauge groups which actually occurs. To deal with this circumstance, let us consider one abelian gauge group $\mathcal{G}$ and one nonabelian gauge group $\mathcal{G}'$ having gauge fields A^μ and $\vec{B}^\mu = \{B^\mu_a\}$ $(a = 1, \ldots, n)$, respectively. Further assume that $\mathcal{G}$ and $\mathcal{G}'$ commute and that components of the generic matter field Θ transform as an r-dimensional multiplet under $\mathcal{G}'$. The key construction involves the generalized covariant derivative, written as an $r \times r$ matrix,

$$\mathbf{D}_\mu \Theta = \left((\partial_\mu + i f A_\mu)\mathbf{I} + i g \vec{B}_\mu \cdot \vec{\mathbf{G}} \right) \Theta, \tag{5.25}$$

where $\mathbf{I}$ is the unit matrix and f, g are distinct real-valued constants. Given this, much of the rest of the previous analysis goes through unchanged. The field strengths associated with the abelian and nonabelian gauge fields have the forms given earlier. So does the gauge-invariant lagrangian, except now the extended covariant derivative of Eq. (5.25) appears, and both the abelian and nonabelian field strengths must be included. For the theory with distinct multiplets of complex scalar fields Φ and spin one-half fields Ψ, the general form of the gauge-invariant lagrangian is

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$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} \mathrm{Tr}\,(\mathbf{F}^{\mu\nu} \mathbf{F}_{\mu\nu}) + i \overline{\Psi} \not{\mathbf{D}} \Psi - \overline{\Psi} \mathbf{m} \Psi$$
$$+ (\mathbf{D}^{\mu} \Phi)^{\dagger} \mathbf{D}_{\mu} \Phi - V(|\Phi|^2) + \mathcal{L}(\Psi, \Phi), \tag{5.26}$$

where $\mathbf{m}$ is the fermion mass matrix, $V(|\Phi|^2)$ contains the Φ mass matrix and any polynomial self-interaction terms, and $\mathcal{L}(\Psi, \ \Phi)$ describes the coupling between the spin one-half and spin zero fields.

I–6 On the fate of symmetries

Depending on the dynamics of the theory, a given symmetry of the lagrangian can be manifested physically in a variety of ways. Apparently all such realizations are utilized by Nature. Here we list the various possibilities.

(1) The symmetry may remain exact. The electromagnetic gauge $U(1)$ symmetry, the $SU(3)$ color symmetry of *QCD*, and the global 'baryon-number minus lepton-number' $(B - L)$ symmetry are examples in this class.
(2) The apparent symmetry may have an anomaly. In this case it is not really a true symmetry. Within the Standard Model the global axial $U(1)$ symmetry is thus affected. Our discussion of anomalies is given in Sect. III–3.
(3) The symmetry may be explicitly broken by terms (perhaps small) in the lagrangian which are not invariant under the symmetry. Isospin symmetry, broken by electromagnetism and light-quark mass difference, is an example.
(4) The symmetry may be 'hidden' in the sense that it is an invariance of the lagrangian but not of the ground state, and thus one does not 'see' the symmetry in the spectrum of physical states. This can be produced by different physical mechanisms.
   (a) The acquiring of vacuum expectation values by one or more scalar fields in the theory gives rise to a *spontaneously broken* symmetry, as in the breaking of $SU(2)_L$ invariance by Higgs fields in the electroweak interactions.
   (b) Even in the absence of scalar fields, quantum effects can lead to the *dynamical breaking* of a symmetry. Such is the fate of chiral $SU(2)_L \times SU(2)_R$ symmetry in the strong interactions.

The various forms of symmetry breaking in the above are quite different. In particular, the reader should be warned that the word 'broken' is used with very different meanings in case (3) and the cases in (4). The meaning in (3) is literal – what would have been a symmetry in the absence of the offending terms in the lagrangian is *not* a symmetry of the lagrangian (nor of the physical world). Although the usage in (4) is quite common, it is really a malapropism because the symmetry is not actually broken. Rather, it is realized in a special way, one which turns out to have

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important consequences for a number of physical processes. The situation is somewhat subtle and requires more explanation, so we shall describe its presence in a magnetic system and in the sigma model.

Hidden symmetry

The phenomenon of hidden symmetry occurs when the ground state of the theory does not have the full symmetry of the lagrangian. Let Q be a symmetry charge as inferred from Noether's theorem, and consider a global symmetry transformation of the vacuum state

$$|0\rangle \to e^{i\alpha Q}|0\rangle, \tag{6.1}$$

where α is a continuous parameter. Invariance of the vacuum,

$$e^{i\alpha Q}|0\rangle = |0\rangle \qquad (\text{all } \alpha), \tag{6.2a}$$

implies that

$$Q|0\rangle = 0. \tag{6.2b}$$

In this circumstance, the vacuum is unique and the symmetry manifests itself in the 'normal' fashion of mass degeneracies and coupling constant identities. Such is the case for the isospin symmetric model of nucleons and pions discussed in Sect. I–4, where the lagrangian of Eq. (4.6) implies the relations

$$\begin{gathered} m_n = m_p, \qquad m_{\pi^+} = m_{\pi^0} = m_{\pi^-}, \\ g(pp\pi^0) = -g(nn\pi^0) = g(pn\pi^+)/\sqrt{2} = g(np\pi^-)/\sqrt{2}, \end{gathered} \tag{6.3}$$

with $\pi^\pm = (\pi_1 \mp i\pi_2)/\sqrt{2}$.

Alternatively, if *new* states $|\alpha\rangle \neq |0\rangle$ are reached via the transformations of Eq. (6.1), we must have

$$Q|0\rangle \neq 0. \tag{6.4}$$

Since, by Noether's theorem, the symmetry charge is time-independent,

$$\dot{Q} = i[H, Q] = 0, \tag{6.5}$$

all of the new states $|\alpha\rangle$ must have the same energy as $|0\rangle$. That is, if E_0 is the energy of the vacuum state, $H|0\rangle \equiv E_0|0\rangle$, then we have

$$H|\alpha\rangle = H\, e^{i\alpha Q}|0\rangle = e^{i\alpha Q} H|0\rangle = E_0|\alpha\rangle. \tag{6.6}$$

Because the symmetry transformation is continuous, there must occur a continuous family of degenerate states.

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Can one visualize these new states in a physical setting? It is helpful to refer to a ferromagnet, which consists of separate domains of aligned spins. Let us focus on one such domain in its ground state. It is invariant only under rotations about the direction of spin alignment, and hence does not share the full rotational invariance of the hamiltonian. In this context, the degenerate states mentioned above are just the different possible orientations available to the lattice spins in a domain. Since space is rotationally invariant, there is no preferred direction along which a domain must be oriented. By performing rotations, one transfers from one orientation to another, each having the same energy.

Let us try to interpret, from the point of view of quantum field theory, the states which are obtained from the vacuum by a continuous symmetry transformation and which share the energy of the vacuum state. In a quantum field theory any excitation about the ground state becomes quantized and is interpreted as a particle. The minimum excitation energy is the particle's mass. Thus the zero-energy excitations generated from symmetry transformations must be described by massless particles whose quantum numbers can be taken as those of the symmetry charge(s). Thus we are led to Goldstone's theorem [Go 61, GoSW 62] – if a theory has a continuous symmetry of the lagrangian, which is not a symmetry of the vacuum, there must exist one or more massless bosons (*Goldstone bosons*). That is, spontaneous or dynamical breaking of a continuous symmetry will entail massless particles in the spectrum.

This phenomenon can be seen in the magnet analogy, where the excitation is a spin-wave quantum. When the wavelength becomes very large, the spin configuration begins to resemble a uniform rotation of *all* the spins. This is one of the other possible domain alignments discussed above, and to reach it does not cost any energy. Thus, in the limit of infinite wavelength ($\lambda \to \infty$), the excitation energy vanishes ($E \to 0$), yielding a Goldstone boson.[9]

Spontaneous symmetry breaking in the sigma model

We proceed to a more quantitative analysis of hidden symmetry by returning to the sigma model of Sect. I–4. Let us begin by inferring from the sigma model lagrangian of Eq. (4.14) the potential energy

$$V(\sigma, \boldsymbol{\pi}) = -\frac{\mu^2}{2}\left(\sigma^2 + \boldsymbol{\pi}^2\right) + \frac{\lambda}{4}\left(\sigma^2 + \boldsymbol{\pi}^2\right)^2 . \tag{6.7}$$

[9] In the ferromagnet case, the spin waves actually have $E \propto \mathbf{p}^2 \sim \lambda^{-2}$ for low momentum. In Lorentz-invariant theories, the form $E \propto |\mathbf{p}|$ is the only possible behavior for massless single particle states. For a more complete discussion, see [An 84].

With μ^2 negative, minimization of $V(\sigma, \boldsymbol{\pi})$ occurs for the unique configuration $\sigma = \boldsymbol{\pi} = 0$. Hidden symmetry occurs for μ^2 positive, where minimization of $V(\sigma, \boldsymbol{\pi})$ reveals the set of degenerate ground states to be those with

$$\sigma^2 + \boldsymbol{\pi}^2 = \frac{\mu^2}{\lambda}. \tag{6.8}$$

Let us study the particular ground state,

$$\langle \sigma \rangle_0 = \sqrt{\frac{\mu^2}{\lambda}} \equiv v, \qquad \langle \boldsymbol{\pi} \rangle_0 = 0. \tag{6.9}$$

Other choices yield the same physics, but require a relabeling of the fields. For this case, field fluctuations in the pionic direction do not require any energy, so that the pions are the Goldstone bosons. Defining

$$\tilde{\sigma} = \sigma - v, \tag{6.10}$$

we then have for the full sigma model lagrangian

$$\begin{aligned} \mathcal{L} = \bar{\psi}\left(i\not{\partial} - gv\right)\psi + \frac{1}{2}\left[\partial_\mu \tilde{\sigma}\,\partial^\mu \tilde{\sigma} - 2\mu^2 \tilde{\sigma}^2\right] + \frac{1}{2}\partial_\mu \boldsymbol{\pi} \cdot \partial^\mu \boldsymbol{\pi} \\ - g\bar{\psi}\left(\tilde{\sigma} - i\boldsymbol{\tau} \cdot \boldsymbol{\pi}\gamma_5\right)\psi - \lambda v \tilde{\sigma}\left(\tilde{\sigma}^2 + \boldsymbol{\pi}^2\right) - \frac{\lambda}{4}\left[\left(\tilde{\sigma}^2 + \boldsymbol{\pi}^2\right)^2 - v^4\right]. \end{aligned} \tag{6.11}$$

Observe that the pion is massless, while the $\tilde{\sigma}$ and nucleon fields are massive. Thus, at least part of the original symmetry in the sigma model lagrangian of Eq. (4.14) appears to have been lost. Certainly, the mass degeneracy $m_\sigma = m_\pi$ is no longer present, although the normal pattern of isospin invariance survives. However, the full set of original symmetry currents remain conserved. In particular, the axial current of Eq. (4.23), which now appears as

$$A^i_\mu = \bar{\psi}\gamma_\mu\gamma_5\frac{\tau^i}{2}\psi - v\partial_\mu \pi^i + \pi^i \partial_\mu \tilde{\sigma} - \tilde{\sigma}\partial_\mu \pi^i, \tag{6.12}$$

still has a vanishing divergence, $\partial^\mu A^i_\mu = 0$. We warn the reader that to demonstrate this involves a complicated set of cancelations.

For a normal symmetry, particles fall into mass-degenerate multiplets and have couplings which are related by the symmetry. The isospin relations in Eq. (6.3) are an example of this. In a certain sense, a hidden symmetry likewise gives rise to degenerate states whose couplings are related by the symmetry. The degeneracy consists of a state taken alone or accompanied by an arbitrary number of Goldstone bosons. For example, in the sigma model it can be a nucleon and the same nucleon accompanied by a zero-energy massless pion, which are degenerate. Moreover, the couplings of such configurations are restricted by the symmetry. Historically,

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predictions of chiral symmetry were originally formulated in terms of *soft-pion theorems* (cf. App. B–3) relating the couplings of the N states to those of the degenerate πN states.

$$\lim_{q_\mu \to 0} \langle \pi^k(q) N' | O | N \rangle = -\frac{i}{F_\pi} \langle N' | [Q_5^k, O] | N \rangle, \tag{6.13}$$

where O is some local operator and N, N' are nucleons or other states. This captures intuitively the nature of symmetry predictions for a hidden symmetry. In this book, we will explore such chiral relations using the more modern techniques of effective lagrangians.

To summarize, if a symmetry of the theory exists but is not apparent in the single-particle spectrum, it still can have a great deal of importance in restricting particle behavior. What happens is actually quite remarkable – in essence, symmetry becomes dynamics. One obtains information about the excitation or annihilation of particles from symmetry considerations. In this regard, hidden symmetries are neither less 'real' nor less useful than normal symmetries – they simply yield a different pattern of predictions.

Problems

(1) **The Poincaré algebra**

(a) Consider the spacetime (Poincaré) transformations, $x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu$, where $\Lambda^\mu{}_\sigma \Lambda^{\sigma\nu} = g^{\mu\nu}$. Associated with each coordinate transformation (a, Λ) is the *unitary* operator $U(a, \Lambda) = \exp(i a_\mu P^\mu - \frac{i}{2}\epsilon_{\mu\nu} M^{\mu\nu})$. For two consecutive Poincaré transformations there is a closure property, $U(a', \Lambda')$ $U(a, \Lambda) = U(\ldots)$. Fill in the dots.

(b) Prove that $U(a^{-1}, 0)U(a', 0)U(a, 0) = U(a', 0)$, and by taking a'_μ, a_μ infinitesimal, determine $[P^\mu, P^\nu]$.

(c) Demonstrate that $(\Lambda^{-1})_{\lambda\nu} = \Lambda_{\nu\lambda}$, and then show that $U(0, \Lambda^{-1})U(a', \Lambda')U(0, \Lambda) = U(\Lambda^{-1}a', \Lambda^{-1}\Lambda'\Lambda)$.

(d) For infinitesimal transformations we write $\Lambda^\mu{}_\lambda \simeq g^\mu{}_\lambda + \epsilon^\mu{}_\lambda$. Prove that $\epsilon_{\sigma\lambda} = -\epsilon_{\lambda\sigma}$ and hence $M_{\sigma\lambda} = -M_{\lambda\sigma}$. Upon taking primed quantities in (c) to be infinitesimal, prove $U(0, \Lambda^{-1}) P^\mu U(0, \Lambda) = \Lambda^\mu{}_\nu P^\nu$ and $U(0, \Lambda^{-1})$ $M^{\mu\nu} U(0, \Lambda) = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta M^{\alpha\beta}$. Finally, letting unprimed quantities be infinitesimal as well, determine $[M^{\alpha\beta}, P^\mu]$ and $[M^{\alpha\beta}, M^{\mu\nu}]$.

(2) **The Meissner effect in gauge theory** [Sh 81]

The lagrangian for the electrodynamics of a charged scalar field is

$$\mathcal{L}_0 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \varphi)^* (D^\mu \varphi) - V(\varphi)$$

with covariant derivative $D_\mu \equiv \partial_\mu + ieA_\mu$ and potential energy,

$$V(\varphi) = \frac{m^2}{2}\varphi^*\varphi + \frac{\lambda}{4}(\varphi^*\varphi)^2 \qquad (\lambda > 0).$$

(a) Identify the electromagnetic current of the φ field.

(b) For $m^2 > 0$, show that the ground state is $\varphi = 0,\ A_\mu = 0$. In this case, the theory is that of normal electrodynamics.

(c) For $m^2 < 0$ ($m^2 \to -\mu^2$ with $\mu^2 > 0$), we enter a different phase of the system. Show that the ground state is now $\varphi = \text{const.} \equiv v,\ A_\mu = 0$. What is the photon mass in this phase? Calculate the potential between two static point charges each of value Q. What sets the scale of the screening length?

(d) Let us now add an external field to the system,

$$\mathcal{L}_0 \to \mathcal{L}_0 + \frac{1}{2}F_{\mu\nu}F^{\mu\nu}_{\text{ext}}.$$

To see that $F^{\mu\nu}_{\text{ext}}$ indeed acts like an applied field, show that if one disregards the field φ the equations of motion require $F^{\mu\nu} = F^{\mu\nu}_{\text{ext}}$.

(e) Demonstrate that there are two simple solutions to the equations of motion in the presence of a *constant* applied field,

$$\varphi = \begin{cases} 0 & (F^{\mu\nu} = F^{\mu\nu}_{\text{ext}}), \\ v & (F^{\mu\nu} = 0). \end{cases}$$

Again, these correspond to unscreened and screened phases of the electromagnetic field.

(f) Calculate the energy of the two phases if $F^{\mu\nu}_{\text{ext}}$ describes a constant magnetic field. Show that the phase in part (e) with $\varphi = 0$ has the lower energy for $B > B_{\text{critical}}$ whereas for $B < B_{\text{critical}}$ it is the phase with $\varphi = v$ which has the lower energy. Discuss the similarity of this result to the Meissner effect.

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