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REMARKS ON THE PAPER "TRANSIENT MARKOV CONVOLUTION SEMI-GROUPS AND THE ASSOCIATED NEGATIVE DEFINITE FUNCTIONS"

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Let X be a locally compact and σ-compact abelian group and let $\hat{X}$ denote the dual group of X. We denote by ξ a fixed Haar measure on X and by $\hat{\xi}$ the Haar measure associated with ξ. In [2], we show the following

THEOREM. *Let $(\alpha_t)_{t\geqq 0}$ be a sub-Markov convolution semi-group on X and let ψ be the negative definite function associated with $(\alpha_t)_{t\geqq 0}$. Then $(\alpha_t)_{t\geqq 0}$ is transient if and only if* $\mathrm{Re}(1/\psi)$ *is locally $\hat{\xi}$-summable.*

In this theorem, the "if" part is essential. To prove it, we showed the following

PROPOSITION (see Proposition 11 in [2]). *Let n, m be non-negative integers and let $X = R^n \times Z^m$, where R and Z denote the additive group of real numbers and the additive group of integers. Let σ be a probability measure on X with* $\mathrm{supp}(\sigma) - \mathrm{supp}(\sigma) = X$, *where* $\mathrm{supp}(\sigma)$ *denotes the support of σ. Put $\psi(\hat{x}) = 1 - \hat{\sigma}(\hat{x})$ on X, where $\hat{\sigma}$ is the Fourier transform of σ. If* $\mathrm{Re}(1/\psi)$ *is locally $\hat{\xi}$-summable, then $\sum_{k=1}^{\infty}(\sigma)^k$ converges vaguely, where $(\sigma)^1 = \sigma$ and $(\sigma)^k = (\sigma)^{k-1} * \sigma$ $(k \geqq 2)$.*

For any positive number p, we put

$$N_p = \frac{1}{p+1}\left(\varepsilon + \sum_{k=1}^{\infty}\left(\frac{1}{p+1}\sigma\right)^k\right),$$

where ε denotes the Dirac measure at the origin. The first main step of the proof in [2] is the following assertion:

$(*)$ There exists $f_0 \in C_K^+(X)$ with $f_0 \neq 0$ such that

$$\left(\left(\frac{1}{2}(N_p + \check{N}_p) - pN_p * \check{N}_p\right) * f_0 * \check{f}_0(0)\right)_{p>0}$$

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is bounded, where $\check{f}_0(x) = f_0(-x)$ and $\int f d\check{N}_p = \int \check{f} dN_p$ for all $f \in C_K(X)$.

Here $C_K(X)$ denotes the space of all real-valued continuous functions on X with compact support and $C_K^+(X) = \{f \in C_K(X);\ f \geqq 0\}$. This proof in [2] is not sufficient, which was pointed out by Prof. C. Berg and Prof. Hirsch. So we must show (*).

For any $p > 0$, we have

$$\frac{1}{2}(\hat{N}_p(\hat{x}) + \widehat{\check{N}}_p(\hat{x})) - p\widehat{N_p * \check{N}_p}(\hat{x}) = \frac{\operatorname{Re}\psi(x)}{|p + \psi(x)|^2}$$
$$\leqq \frac{\operatorname{Re}\psi(x)}{|\psi(x)|^2} = \operatorname{Re}\left(\frac{1}{\psi(x)}\right) \leqq \frac{1}{\operatorname{Re}\psi(x)}.$$

Put

$$g_p = \frac{p}{|p + \psi|^2}, \qquad h_p = \frac{\operatorname{Re}\psi}{|p + \psi|^2} \quad (p > 0) \quad \text{and} \quad h = \operatorname{Re}\left(\frac{1}{\psi}\right).$$

Assume $n = 0$. Then $\hat{X}$ is compact, so $\operatorname{Re}(1/\psi)$ is $\hat{\xi}$-summable. Hence, for any $f \in C_K^+(X)$ and any $p > 0$,

$$\left(\frac{1}{2}(N_p + \check{N}_p) - pN_p * \check{N}_p\right) * f * \check{f}(0) \leqq \int |\hat{f}|^2 \operatorname{Re}\left(\frac{1}{\psi}\right) d\hat{\xi}.$$

Assume $n \geqq 1$. It suffices to show (*) in the case of $X = R^n$. Since $\operatorname{supp}(\sigma + \check{\sigma}) = R^n$, we have $\operatorname{Re}\psi(x) > 0$ outside the origin and

$$\operatorname{Re}\psi(x) \geqq a|x|^2$$

in $B_1(0) = \{x \in R^n; |x| < 1\}$ with some constant $a > 0$, where $|x|$ denotes the distance between x and the origin. It is well-known that if $1/\operatorname{Re}\psi$ is summable in a certain neighborhood of the origin, then $\sum_{k=1}^\infty (\frac{1}{2}(\sigma + \check{\sigma}))^k$ converges vaguely and, for any $f \in C_K(R^n)$,

$$\left(\varepsilon + \sum_{k=1}^{\infty}\left(\frac{1}{2}(\sigma + \check{\sigma})\right)^k\right) * f * \check{f}(0) = \int |\hat{f}|^2 \frac{1}{\operatorname{Re}\psi} dx$$

(see [1]). These imply that if $n \geqq 3$, $((\frac{1}{2}(N_p + \check{N}_p) - pN_p * \check{N}_p) * f * f(0))_{p>0}$ is bounded for all $f \in C_K(R^n)$.

Assume $n = 2$. Since, for any $p > 0$,

$$\int_{B_1(0)} g_p dx \leqq \int_{B_1(0)} \frac{p}{(p + a|x|^2)^2} dx < \frac{2}{a}\pi$$

and $p\widehat{N_p * \check{N}_p} = g_p$, $(g_p * f * \check{f}(0))_{p>0}$ is uniformly bounded on R^n for all $f \in C_K(R^2)$. Since $g_p + h_p$ is of positive type and h is locally summable,

$$\begin{aligned}\sup_{x\in R^n}|h*f*\check{f}(x)| &= \sup_{x\in R^n}\lim_{p\to 0}|h_p*f*\check{f}(x)| \\ &\leqq \sup_{x\in R^n}\lim_{p\to 0}|(g_p+h_p)*f*\check{f}(x)| + \sup_{\substack{p>0\\ x\in R^n}}|g_p*f*\check{f}(x)| \\ &\leqq 2\sup_{p>0} g_p*f*\check{f}(0) + h*f*\check{f}(0) < \infty\end{aligned}$$

for all $f\in C_K(R^2)$. This shows that h is temperate, so, for any $f\in C_K^\infty(R^2)$ and any $p>0$,

$$\Big(\frac{1}{2}(N_p+\check{N}_p) - pN_p*\check{N}_p\Big)*f*\check{f}(0) \leqq \int|\hat{f}|^2\,\mathrm{Re}\Big(\frac{1}{\psi}\Big)dx < \infty\,,$$

where $C_K^\infty(R^2)$ denotes the set of all real-valued and infinitely differentiable functions on R^2 with compact support.

Assume $n=1$. For a symmetric positive measure β on R^1 with compact support and with $\beta\neq 0$, $\beta\leqq\frac{1}{2}(\sigma+\check{\sigma})$, we set

$$\gamma_\beta(x) = \frac{1}{2}\Big(\int|x-y|\,d\beta(y) - \Big(\int d\beta\Big)|x|\Big)\,.$$

Then $\gamma_\beta\in C_K^+(R^1)$, $\mathrm{supp}\,(\gamma_\beta)\supset\mathrm{supp}\,(\beta)$ and

$$\hat{\gamma}_\beta(x) = \int e^{-2\pi ixy}\gamma_\beta(y)dy = \frac{\int d\beta - \hat{\beta}(x)}{4\pi^2|x|^2}\,.$$

For any $f\in C_K^+(R^1)$, we put $f_\beta(x) = f*\gamma_\beta(x)$; then

$$\begin{aligned}\int|\hat{f}_\beta|^2\,\mathrm{Re}\Big(\frac{1}{\psi}\Big)dx &= \int_{|x|<1}|\hat{f}_\beta|^2\,\mathrm{Re}\Big(\frac{1}{\psi}\Big)dx \\ &\quad + \int_{|x|\geqq 1}|\hat{f}(x)|^2\frac{\Big(\int d\beta-\hat{\beta}(x)\Big)^2}{16\pi^4|x|^4}\,\frac{\mathrm{Re}\,\psi(x)}{|\psi(x)|^2}\,dx \\ &\leqq \int_{|x|<1}|\hat{f}_\beta|^2\,\mathrm{Re}\Big(\frac{1}{\psi}\Big)dx + \frac{1}{8\pi^4}\int_{|x|\geqq 1}|\hat{f}(\hat{x})|\frac{1}{|x|^4}\,dx < \infty\,,\end{aligned}$$

because $0\leqq\int d\beta - \hat{\beta}(x)\leqq\mathrm{Re}\,\psi(x)\leqq 2$. Therefore

$$\Big(\Big(\frac{1}{2}N_p+\check{N}_p\Big) - pN_p*\check{N}_p\Big)*f_\beta*\check{f}_\beta(0)\Big)_{p>0}$$

is bounded.

Thus $(*)$ is shown.

Bibliography

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