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SUPERRESOLUTION RATES IN PROKHOROV METRIC

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ABSTRACT. Consider the problem of recovering a probability measure supported by a compact subset U of $\mathbb{R}^m$ when the available measurements concern only some of its Φ-moments (Φ being an $\mathbb{R}^k$ valued continuous function on U). When the *true* Φ-moment c lies on the boundary of the convex hull of $\Phi(U)$, generalizing the results of [10], we construct a *small* set $R_{\alpha,\delta(\epsilon)}$ such that any probability measure μ satisfying $\| \int_U \Phi(x)\,d\mu(x) - c\| \leq \epsilon$ is *almost* concentrated on $R_{\alpha,\delta(\epsilon)}$. When Φ is a pointwise T-system (extension of T-systems), the study of the set $R_{\alpha,\delta(\epsilon)}$ leads to the evaluation of the Prokhorov radius of the set $\{\mu : \| \int_U \Phi(x)\,d\mu(x) - c\| \leq \epsilon\}$.

1. Introduction. Let σ be a probability measure supported by a compact subset U of $\mathbb{R}^m$. The measure σ may model scientifically interesting objects (*e.g.* an electronic density in a crystallographic problem [19]). Suppose we only obtain a finite number of noisy generalized Φ-moments of σ:

$$c^{\zeta} := \int_U \Phi(x)\,d\sigma(x) + \zeta. \tag{1}$$

Here, Φ is a given $\mathbb{R}^k$-valued function on U and ζ is a (*possibly random*) perturbation of magnitude ϵ. Typically, in applied areas Φ is a trigonometrical system [19]. A reconstruction method $\mathcal{R}$ is a mapping that associates to each c^{ζ} a probability measure $\mathcal{R}(c^{\zeta})$ whose Φ-moments are close to c^{ζ}. In different applications ([13], [19]) it has been observed that, for particular σ, any reasonable reconstruction method $\mathcal{R}$ leads to a probability measure $\mathcal{R}(c^{\zeta})$ which is very *close* to σ. This is the so-called *superresolution phenomenon*.

Motivated by this previous work, the aim of this paper is to give, when superresolution occurs, and for small magnitudes ϵ, precise rates on the *distance* between σ and $\mathcal{R}(c^{\zeta})$ (for any method $\mathcal{R}$). Our answers will deepen and broaden the partial information about superresolution scattered about the literature. In [12] the phenomenon is studied qualitatively whereas in [11] numerical examples are given. In [5], [7] and [6], rates are given in the case where U is a discrete set and for trigonometrical systems. In [8] and [17], the rates given concern the Markov problem (*i.e.* there is an extra restriction involving bounds on the reconstructed measure [15]).

First of all, let us describe more precisely the mathematical background of the problem. The convex hull $\mathcal{K}$ of the hypersurface $\Phi(U)$ is also the range over probability measures

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on U of the Φ-moments:

$$\mathcal{K} := \left\{ \int_U \Phi(x)\, d\mu(x); \mu \in \mathbb{P}(U) \right\},$$
$$\mathbb{P}(U) := \{\text{probability measures on } U\}.$$

For $c \in \mathcal{K}$ there exist only two possibilities for the set $\mathcal{S}(c,0)$, of probability measures whose Φ-moments are c:

1) $c \in \mathring{\mathcal{K}}$ (the interior of $\mathcal{K}$), the set $\mathcal{S}(c,0)$ contains absolutely continuous measures with respect to P, the uniform probability on U. Moreover, there exists a continuous density f such that the probability measure fP lies in $\mathcal{S}(c,0)$ and $\forall x \in U, f(x) > 0$. Generally speaking, $\mathcal{S}(c,0)$ contains measures that may be very *different*, (*e.g.* if $U = [0,2\pi]$, $k = 1$, $\Phi(x) = \cos x$, $c = 0$ then $P, \delta_{\frac{\pi}{2}}$ and $\delta_{\frac{3\pi}{2}}$ lie in $\mathcal{S}(c,0)$).

2) $c \in \partial\mathcal{K}$ (the boundary of $\mathcal{K}$), the set $\mathcal{S}(c,0)$ contains only singular measures (with respect to P). Moreover, the support of any member of $\mathcal{S}(c,0)$ lies in the zero set of a particular nonnegative $(1,\Phi)$-polynomial (that is on $\{x \in U : \langle \alpha, \Phi(x)\rangle + 1 = 0\}$, for some particular vector α of $\mathbb{R}^k$ such that $\forall x \in U : \langle \alpha, \Phi(x)\rangle + 1 \geq 0$).

In the second case we will say that c is weakly determined. In this paper we focus on weakly determined points. For such a point c and $\epsilon > 0$, we study the size of the set

$$\mathcal{S}(c,\epsilon) := \left\{ \mu \in \mathbb{P}(U) : \left\| \int_U \Phi(x)\, d\mu(x) - c \right\| \leq \epsilon \right\}. \tag{2}$$

An interesting case is when $\mathcal{S}(c,0)$ reduces to a singleton. It is in this case that superresolution occurs, and we say that c is strongly determined. Indeed, the knowledge of c suffices to reconstruct the unique element σ_c of $\mathcal{S}(c,0)$. This notion is connected with Korovkin Theorems (see [1], Chapter 2). Now, if a strongly determined point c is corrupted by a small perturbation of magnitude ϵ and if μ is a probability measure with this corrupted Φ-moment; how far is μ from σ_c? First, we have to specify a measure of the size of $\mathcal{S}(c,\epsilon)$. We will use in this paper the Prokhorov distance [20]. Recall that for $\mu, \nu \in \mathbb{P}(U)$ the Prokhorov distance between μ and ν, $\pi(\mu,\nu)$, is defined as the smallest $\eta \geq 0$ satisfying $\mu(A) \leq \nu(A^\eta) + \eta$ for any closed subset A of U, where A^η denotes the η-neighborhood of A ($A^\eta = \{x \in U : d(x,a) < \eta$, for some $a \in A\}$).

In [3], Anastassiou calculates, in the case of the real line for $\Phi(x) = (x,x^2)$ and $\xi \in \mathbb{R}$, the Prokhorov radius π_ϵ of $\mathcal{S}\big((\xi,\xi^2),\epsilon\big)$, where the Prokhorov radius is defined by

$$\pi_\epsilon := \sup_{\mu \in \mathcal{S}(c,\epsilon)} \pi(\mu,\sigma_c) \tag{3}$$

(here $\mathcal{S}(c,0) = \{\sigma_c\} = \{\delta_\xi\}$, in the same frame the Levy radius is calculated in [2]). Anastassiou obtains the exact asymptotic Const$\cdot\epsilon^{\frac{1}{3}}$ for π_ϵ. Our main result is a generalization of the behavior found by Anastassiou (see Corollary 4.2): if Φ is smooth pointwise T-system (see Section 3.2) then in most cases $\pi_\epsilon = O(\epsilon^{\frac{1}{3}})$ and $\epsilon^{\frac{1}{3}} = O(\pi_\epsilon)$. The main difficulty in extending Anastassiou's result (only concerned with one point mass measures) is to specify the behavior of masses repartitions at each x_j's neighborhood for any $\mu \in \mathcal{S}(c,\epsilon)$ where $x_1,\ldots,x_l$ denote σ_c's masses. The whole paper is devoted to

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provide a suitable framework (pointwise T-systems and superconcentration notions) in order to achieve this goal.

The paper is organized as follows: in Section 2, we introduce our main notations and assumptions, we then locate, when c is a weakly determined point, the support of any measure lying in $\mathcal{S}(c,\epsilon)$. Roughly speaking, for small ϵ any such measure is almost concentrated on a same small P-measure set $R_{\alpha,\delta(\epsilon)}$ (see Theorem 2.3). In Section 2.3 rates of concentration depending on the regularity on Φ are calculated. This improves the previous rates given in [10]. In Section 3, we first introduce in subsection 3.2 a suitable extension of the notion of T-systems while subsection 3.3 introduces a notion of strong superconcentration yielding an extension of the weak superconcentration theorem of Section 2 in terms of metric size (instead of P-measure size) of the set $R_{\alpha,\delta(\epsilon)}$. Using the set $R_{\alpha,\delta(\epsilon)}$ yields upper bounds for π_ϵ. Finally, in Section 4, we provide lower bounds for π_ϵ directly.

2. Support of weakly determined points.

2.1. *Assumptions and notations* In the paper, U is assumed to be a compact subset of $\mathbb{R}^m$ with non empty interior; when it is not specified, the induced topology of $\mathbb{R}^m$ on U is used (*e.g.* U is open for this topology). We assume throughout the paper that $(1,\Phi)$ is pseudo-Haar, that is:

$$\forall(\alpha,\xi)\in\mathbb{R}^{k+1}\setminus\{0\},\quad P\Big(\{x\in U:\langle\alpha,\Phi(x)\rangle=\xi\}\Big)=0. \tag{4}$$

Here, $\langle\cdot,\cdot\rangle$ is the scalar product on $\mathbb{R}^k$, $\|\cdot\|=\sqrt{\langle\cdot,\cdot\rangle}$ will be the usual Euclidean norm on $\mathbb{R}^k$ or $\mathbb{R}^m$ and $B(\xi,r)$ will denote the open ball centered at ξ and with radius r. For $\alpha\in\mathbb{R}^k$, we set $g_\alpha(x):=\langle\alpha,\Phi(x)\rangle+1$, g_α is the normalized $(1,\Phi)$-polynomial constructed with α. We define for $c,\alpha\in\mathbb{R}^k$ and $\delta\geq 0$ the following sets:

$$\begin{aligned}\mathcal{P}_+&:=\{\alpha\in\mathbb{R}^k:g_\alpha(x)\geq 0,\forall x\in U\},\\ \mathcal{P}_+(c)&:=\{\alpha\in\mathcal{P}_+,\langle\alpha,c\rangle+1=0\},\\ R_{\alpha,\delta}&:=\{x\in U:g_\alpha(x)\leq\delta\}.\end{aligned}$$

It is well known (see [15], Chapter III) that a point $c\in\mathbb{R}^k$ is weakly determined if and only if for all $\alpha\in\mathcal{P}_+$, $\langle\alpha,c\rangle+1$ is nonnegative and $\mathcal{P}_+(c)$ is non empty. An other characterization of weakly determined points is developed in [9] and [16].

2.2. *Weak superconcentration theorem*

LEMMA 2.1. *If c is weakly determined then for any $\alpha\in\mathcal{P}_+(c)$ and $\delta>0$*

$$\sup_{\mu\in\mathcal{S}(c,\epsilon)}\mu(R^c_{\alpha,\delta})\leq\frac{\epsilon}{\delta}\|\alpha\|,$$

where $R^c_{\alpha,\delta}$ denotes the complement of $R_{\alpha,\delta}$ relative to U.

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PROOF OF LEMMA 2.1. Let $d \in B(c,\epsilon)$ and $\alpha \in \mathcal{P}_+(c)$, then

$$1 + \langle \alpha, d\rangle = \langle \alpha, d-c\rangle \leq \|\alpha\|\,\|d-c\| \leq \epsilon\|\alpha\|,$$

so that for $\mu \in \mathcal{S}(c,\epsilon)$, $\int_U g_\alpha(x)\mu(dx) \leq \epsilon\|\alpha\|$. This integral of a nonnegative function yields the result using the Markov inequality on the set $R^c_{\alpha,\delta} = \{x \in U : g_\alpha(x) > \delta\}$. ∎

Define for $\delta > 0$, $\epsilon > 0$ and $\alpha \in \mathbb{R}^k$

$$F_\alpha(\delta) := P(R_{\alpha,\delta}),$$

$$h_F(\alpha,\epsilon) := \inf_{\delta>0}\Big\{\frac{\epsilon}{\delta}\|\alpha\| + F_\alpha(\delta)\Big\}. \tag{5}$$

LEMMA 2.2. *For each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $h_F(\alpha,\epsilon) = \big\{\frac{\epsilon}{\delta(\epsilon)}\|\alpha\| + F_\alpha\big(\delta(\epsilon)\big)\big\}$, moreover $\lim_{\epsilon\to 0^+} h_F(\alpha,\epsilon) = 0$.*

PROOF OF LEMMA 2.2. As $(1,\Phi)$ is pseudo-Haar, $F_\alpha(\delta)$ decreases to 0 with δ. Since $\frac{\epsilon}{\delta}$ increases to infinity as δ decreases to 0, we obtain the first point. Choose now $\tilde\delta(\epsilon)$ with $\tilde\delta(\epsilon)F_\alpha\big(\tilde\delta(\epsilon)\big) = \epsilon\|\alpha\|$, then $h_F(\alpha,\epsilon) \leq 2F_\alpha\big(\tilde\delta(\epsilon)\big)$, moreover $\tilde\delta(\epsilon) \geq \epsilon\|\alpha\|$ since $F_\alpha\big(\tilde\delta(\epsilon)\big) \leq 1$. Assume now that $\delta(\epsilon)$ is bounded below by some constant $q > 0$, uniformly with respect to ϵ: the monotonicity of F_α implies $\tilde\delta(\epsilon)F_\alpha(q) \leq \epsilon\|\alpha\|$; now $F_\alpha(q) > 0$ yields a contradiction. So $\tilde\delta(\epsilon)$ converges to 0 with ϵ, and the lemma follows. ∎

The following weak superconcentration inequality is now a consequence of the previous Lemmas 2.1 and 2.2:

THEOREM 2.3. *Assume that c is weakly determined and let $\alpha \in \mathcal{P}_+(c)$, then*

$$P(R_{\alpha,\delta(\epsilon)}) + \sup_{\mu\in\mathcal{S}(c,\epsilon)} \mu(R^c_{\alpha,\delta(\epsilon)}) \leq h_F(\alpha,\epsilon). \tag{6}$$

REMARK. Note that the previous result implies in turn,

$$\limsup_{\epsilon\to 0^+} R_{\alpha,\delta(\epsilon)} = R_{\alpha,0}.$$

2.3. *Rates of weak superconcentration*

DEFINITION 2.4. Let $\theta, C_0 > 0$. A nonnegative function f on U is called (C_0,θ)-weakly concentrated, if for $t > 0$ small enough:

$$P\big(A_t(f)\big) \leq C_0 t^{\frac{1}{\theta}}. \tag{7}$$

with $A_t(f) := \{x \in U : f(x) \leq t\}$. If C_0 is not explicit, we will only say that the function f is θ-weakly concentrated.

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EXAMPLES. 1) Denote $\nu = \frac{\varrho(B(0,1))}{\varrho(U)}$, where ϱ is the Lebesgue measure on $\mathbb{R}^m$. Let h be a positive continuous function on U and $x_0 \in \operatorname{int}(U)$. Then the function $f(x) := \|x - x_0\|^{m\theta} h(x)$, is (C_0, θ)-weakly concentrated for any $C_0 > \nu[h(x_0)]^{-\frac{1}{\theta}}$.

2) Let f_i be (C_i, θ_i)-weakly concentrated on U_i in $\mathbb{R}^{m_i}$ $(i = 1, 2)$. Define f on $U_1 \times U_2$ by $f(x_1, x_2) := f_1(x_1) + f_2(x_2)$ if $x_i \in U_i$, $i = 1, 2$, then, setting $\theta^{-1} := \theta_1^{-1} + \theta_2^{-1}$, f is (C_1C_2, θ)-weakly concentrated on $U_1 \times U_2$ (indeed, $A_t(f) \subset A_t(f_1) \times A_t(f_2)$); *e.g.*, if $(0, 0) \in \operatorname{Int} U \subset \mathbb{R}^2$, then $f(x_1, x_2) := |x_1|^{\theta_1} + |x_2|^{\theta_2}$ is $(4, \theta)$-weakly concentrated on U.

3) Let f be (C_0, θ)-weakly concentrated on U. Then, $f_1(x, y) := f(x)$, $x \in U, y \in V$ is also (C_0, θ)-weakly concentrated on $U \times V$ whatever V is.

4) On $U = [-1, 1]$, set $f(x) := (-x)^\omega \mathbf{1}_{x \leq 0} + (x)^{\omega'} \mathbf{1}_{x \geq 0}$ with $0 < \omega < \omega' < 1$, then, for $C_0 > 1$, f is (C_0, ω')-weakly concentrated and ω-Hölderian. The ω-weak concentration condition is not identical to the ω-Hölderian condition.

The main example we shall use in the sequel is given in the frame of the following lemma.

LEMMA 2.5. *Let f be a nonnegative C^2-function on U such that the zero set of f,*

$$Z(f) := \{x \in U : f(x) = 0\},$$

has a finite cardinality p, and is not included in ∂U (boundary of U for the usual topology). Assume that the $m \times m$ symmetric matrix $D^2 f(x)$ is nondegenerated for all x in $Z(f)$ and let η be the minimum on $Z(f)$ of the smallest eigenvalue of $D^2 f$. Then, if $C_0 > \nu p(\frac{2}{\eta})^{\frac{m}{2}}$, f is $(C_0, \frac{2}{m})$-weakly concentrated.

PROOF OF LEMMA 2.5. A Taylor expansion writes, for $y \in U$ and $x \in Z(f)$

$$f(y) = Df(x)(y - x) + \frac{1}{2} D^2 f(x)(y - x, y - x) + o(\|x - y\|^2). \tag{8}$$

Any x in $Z(f)$ is a minimum of f on U hence $Df(x)(y - x) \geq 0$ and now as $D^2 f(x)$ is positive, there exists $s(x) > 0$ such that if $\chi > \frac{2}{\eta}$

$$\forall y \in U \cap B\big(x, s(x)\big), \quad f(y) \geq \chi \|x\|^2. \tag{9}$$

$Z(f)$ is a finite subset $\{x_1, \ldots, x_p\}$ of U, so

$$Z(f) \subset V := \bigcup_{1 \leq i \leq p} B\big(x_i, s(x_i)\big). \tag{10}$$

$\big(A_t(f)\big)$ is a family of compact subsets of U such that $\bigcap_{t>0} A_t(f) = Z(f)$ and increasing with $t > 0$. Now a compactness argument proves that for t small enough $A_t(f) \subset V$. Use the previous points (9) and (10) yields

$$A_t(f) \subset \bigcup_{1 \leq i \leq p} B\left(x_i, \sqrt{\frac{t}{\chi}}\right) \tag{11}$$

which implies setting $C_0 = \nu \chi^{\frac{m}{2}}$ the $(C_0, \frac{2}{m})$-weak concentration of f. ∎

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We are now in position to precise the rate of weak superconcentration, through trivial calculations, according to the weak concentration of g_α (where $\alpha \in \mathcal{P}_+(c)$), this is the aim of the next Theorem.

THEOREM 2.6. *Assume that*

1) c is weakly determined.

2) There exists $\alpha \in \mathcal{P}_+(c)$ and positive real constants θ, C_0 such that g_α is (C_0, θ)-weakly concentrated.

Let $C := (\theta\|\alpha\|C_0^\theta)^{\frac{1}{\theta+1}}(1+\frac{1}{\theta})$, then setting $\delta_F(\epsilon) := (\frac{\epsilon\theta\|\alpha\|}{C_0})^{\frac{\theta}{\theta+1}}$ yields, for ϵ small enough, the following bound

$$P(R_{\alpha,\delta_F(\epsilon)}) + \sup_{\mu \in \mathcal{S}(c,\epsilon)} \mu(R^c_{\alpha,\delta_F(\epsilon)}) \leq C\epsilon^{\frac{1}{\theta+1}}.$$

EXAMPLES. 1) Set $U = [0, 1]$ and let c be a weakly determined point. Assume that there exists $\alpha \in \mathcal{P}_+(c)$ with g_α satisfying assumptions of Lemma 2.5. This is true for example if Φ is analytic and g''_α does not vanish on $R_{\alpha,0} \not\subset \{0, 1\}$. Then, g_α is 2-weakly concentrated and the weak superconcentration rate has order $\epsilon^{\frac{1}{3}}$. Note, that this improves on previous results of Gamboa and Gassiat ([10]) where the rate obtained was ϵ^λ for any $\lambda < \frac{1}{3}$.

2) Let $U := [0, 1]$ and $\Phi(x) := (x, x^2)$. Here, the sets of weakly and strongly determined points coincide (see Section 3.2) and are given by:

$$\partial\mathcal{K} = \{(c_1, c_2) \in \mathbb{R}^2 : 0 \leq c_1 \leq 1, c_2 = c_1^2, \text{ or } c_2 = c_1\},$$

if $c_1 \neq 0, 1$ and $c_2 \neq c_1$ the previous theorem provides the weak superconcentration rate $\epsilon^{\frac{1}{3}}$. For $c_1 = 0$, note that $\alpha := (1, 0)$ lies in $\mathcal{P}_+(c)$, for this choice of α, g_α is 1-weakly concentrated, so that the weak superconcentration rate becomes $\epsilon^{\frac{1}{2}}$.

REMARK. Notice that if we know some $t_0 > 0$ such that inequality (7) holds for any $t < t_0$, then the conclusions of Theorem 2.6 hold for $\delta_F(\epsilon) < t_0$. For example, that is true if for some $\alpha \in \mathcal{P}_+(c)$ a uniform bound on the third derivative of g_α is known.

3. Upper bounds for the Prokhorov radius.

3.1. *The Prokhorov metric* The Prokhorov distance is the metric of the weak topology on $\mathbb{P}(U)$, we refer to [20] for the general properties of this distance. When $U \subset \mathbb{R}^k$ is compact and for a single delta measure δ_ξ, $\xi \in U$, we have that for any $\mu \in \mathbb{P}(U)$,

$$\pi(\mu, \delta_\xi) = \inf\{r > 0 : \mu\big(B(\xi, r) \cap U\big) \geq 1 - r\}. \tag{12}$$

The following lemma will be useful in the sequel.

LEMMA 3.1. *Let $\mu_i, \sigma_i \in \mathbb{P}(U)$ be probability measures on U and $\xi_i \in [0, 1]$ for $i = 1, \ldots, p$ with $\xi_1 + \cdots + \xi_p = 1$ then,*

a) $\pi(\xi_1\mu_1 + (1 - \xi_1)\mu_2, \sigma_1) \leq \xi_1 + \pi(\mu_2, \sigma_1)$.

b) $\pi(\xi_1\mu_1 + \cdots + \xi_p\mu_p, \xi_1\sigma_1 + \cdots + \xi_p\sigma_p) \leq \max\big(\pi(\mu_1, \sigma_1), \ldots, \pi(\mu_p, \sigma_p)\big)$.

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c) *Assume that σ_1, σ_2 are discrete probability measures:*

$$\sigma_1 := p_1\delta_{x_1} + \cdots + p_l\delta_{x_l}$$
$$\sigma_2 := q_1\delta_{x_1} + \cdots + q_l\delta_{x_l},$$

where $x_1, \ldots x_l \in U$, $x_i \neq x_j$ if $i \neq j$. Then,

$$\pi(\sigma_1, \sigma_2) \leq e := \frac{1}{2}(|p_1 - q_1| + \cdots + |p_l - q_l|).$$

Moreover, equality holds if $e \leq \min_{i \neq j} d(x_i, x_j)$.

PROOF OF LEMMA 3.1. a) If $\gamma > \pi(\mu_2, \sigma_1)$, any closed subset A of U satisfies $\mu_2(A) \leq \gamma + \sigma_1(A^\gamma)$ hence the result follows by noticing that

$$[\xi_1\mu_1 + (1 - \xi_1)\mu_2](A) \leq \xi_1 + \mu_2(A) \leq \xi_1 + \gamma + \sigma_1(A^\gamma).$$

b) Let $\gamma > \max\big(\pi(\mu_1, \sigma_1), \ldots, \pi(\mu_p, \sigma_p)\big)$, any closed subset A of U satisfies

$$\mu_i(A) \leq \gamma + \sigma_i(A^\gamma), \quad i = 1, \ldots, p.$$

The result follows by considering the adequate convex combination of the previous inequalities.

c) First note that $e = \max_{J \subset \{1,\ldots,l\}} \sum_J (p_i - q_i)$. The first point follows from the definition of π. For equality, consider J attaining e in the previous equality then the closed set $A := \{x_i : i \in J\}$ has the property $\sigma_2(A^e) = \sigma_2(A)$ if $e \leq \min_{i \neq j} d(x_i, x_j)$. This allows to conclude to the reverse inequality. ∎

3.2. *Pointwise T-systems* Assume first that $U = [0, 1]$. Following [14] and [15], we say that Φ is a T-system, if any nonzero polynomial constructed with $(1, \Phi)$ has at most k roots. An equivalent condition is that the determinant

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \Phi_1(x_0) & \Phi_1(x_1) & \cdots & \Phi_1(x_k) \\ \vdots & \vdots & \vdots & \vdots \\ \Phi_k(x_0) & \Phi_k(x_1) & \cdots & \Phi_k(x_k) \end{pmatrix} \tag{13}$$

does not vanish for any pairwise distinct $x_0, \ldots, x_k$. A T-system possesses the nice property that any weakly determined point c is also strongly determined.

More generally, we say that Φ is a *pointwise T-system* at some weakly determined point $c \in \mathbb{R}^k$ if

1) There exist an integer $0 < l \leq k$, pairwise distinct points $x_1, \ldots, x_l \in U$ and constants $p_1, \ldots, p_l > 0$ with $p_1 + \cdots + p_l = 1$ such that

$$\sigma_c = p_1\delta_{x_1} + \cdots + p_l\delta_{x_l} \in \mathcal{S}(c, 0). \tag{14}$$

2) There exists $\alpha \in \mathcal{P}_+(c)$ such that $R_{\alpha,0} = \{x_1, \ldots, x_l\}$.

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In the sequel, the set of vectors α such that 2) holds is denoted by $\tilde{\mathcal{P}}_+(c)$.

3) The following matrix has rank l (full rank)

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \Phi_1(x_1) & \Phi_1(x_2) & \cdots & \Phi_1(x_l) \\ \vdots & \vdots & \vdots & \vdots \\ \Phi_k(x_1) & \Phi_k(x_2) & \cdots & \Phi_k(x_l) \end{pmatrix}. \tag{15}$$

It is clear that a T-system is a pointwise T-system at any determined point. As we shall see in Section 4.2 the pointwise T-system assumption is sufficient for a weakly determined point to be strongly determined. Notice that condition 2) implies the necessary and sufficient assumption in order that the Korovkin Theorem for measures holds (see [1] Theorem 2.2.4. p. 92).

3.3. *Strong superconcentration* An alternative to the function defined in (5) may be worked out under the pointwise T-system assumption. Define for $\delta > 0$, $\epsilon > 0$ and $\alpha \in \mathbb{R}^k$

$$G_\alpha(\delta) := \sup_{x \in R_{\alpha,\delta}} \inf_{y \in R_{\alpha,0}} \|x-y\|.$$

$$h_G(\alpha,\epsilon) := \inf_{\delta>0}\left\{\frac{\epsilon}{\delta}\|\alpha\| + G_\alpha(\delta)\right\}.$$

LEMMA 3.2. *Assume that Φ is a pointwise T-system at c. Let $\alpha \in \tilde{\mathcal{P}}_+(c)$ and $0 < \xi < \frac{1}{2}\min_{i \neq j, 1 \leq i,j \leq l} \|x_i - x_j\|$. Let $B_{\delta,j} := B(x_j,\xi) \cap R_{\alpha,\delta}$, ($j = 1,\ldots,l$).*

1) For $\delta > 0$ small enough, $(B_{\delta,j})_{j=1\cdots l}$ is a partition of $R_{\alpha,\delta}$.

2) For $\delta > 0$ small enough,

$$G_\alpha(\delta) = \max_{j=1,\ldots,l} \sup_{x \in B_{\delta,j}} \|x - x_j\|.$$

3) $\lim_{\delta \to 0^+} G_\alpha(\delta) = 0$.

4) For $j = 1,\ldots,l$, x_j lies in the interior of $B_{\delta,j}$.

Thus, $G_\alpha(\delta)$ is the maximal radius of the *clusters* of the set $R_{\alpha,\delta}$. The previous lemma may be seen as a Morse Theorem (see for example [18]). It is global while the classical Morse Theorem is local and it is general in the sense that we do not require second order differentiability.

PROOF OF LEMMA 3.2. 1) The set $R_{\alpha,\delta} \cap \left(\bigcup_{j=1}^l B^c(x_j,\xi)\right)$ is compact and decreases to the empty set as δ decreases to 0. So that, there exists $\delta(\xi) > 0$ such that, $R_{\alpha,\delta} \subset \bigcup_{j=1}^l B(x_j,\xi)$ as soon as $\delta \leq \delta(\xi)$.

2) Let $x \in R_{\alpha,\delta}$ then $\inf_{y \in R_{\alpha,0}} \|x-y\| = \min_{y \in R_{\alpha,0}} \|x-y\| = \|x - x_j\|$ for some $j \in \{1,\ldots,l\}$. Therefore,

$$G_\alpha(\delta) = \sup_{x \in R_{\alpha,\delta}} \inf_{y \in R_{\alpha,0}} \|x-y\| \leq \max_{j=1,\ldots,l} \sup_{x \in B_{\delta,j}} \|x - x_j\|.$$

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From previous inequality, $G_\alpha(\delta) \le \xi$ whenever $\delta \le \delta(\xi)$, so that we may write $G_\alpha(\delta) = \|x - x_j\|$ for some $j \in \{1, \ldots, l\}$ and $x \in B_{\delta,j}$, which implies the reverse inequality.

3) Follows from 2).

4) For $\delta > 0$ and $j = 1 \cdots l$, x_j lies in the set $\{x \in U : g(x) < \delta, \|x - x_j\| < \xi\}$ which is open and contained in $B_{\delta,j}$. ■

The previous set function yields a natural counterpart of Lemma 2.2.

LEMMA 3.3. *Assume that Φ is a pointwise T-system at c and let $\alpha \in \tilde{\mathcal{P}}_+(c)$. Then, for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $h_G(\alpha, \epsilon) = \{\frac{\epsilon}{\delta(\epsilon)}\|\alpha\| + G_\alpha(\delta(\epsilon))\}$, moreover* $\lim_{\epsilon \to 0^+} h_G(\alpha, \epsilon) = 0$

In order to precise $\delta(\epsilon)$, we first define an analogue of weak concentration adapted to the present framework.

DEFINITION 3.4. Let $\theta > 0$ and $C_0 > 0$. A nonnegative function f on U is called (C_0, θ)-strongly concentrated, if for $t > 0$ small enough:

$$G_f\big(A_t(f)\big) \le C_0 t^{\frac{1}{\theta}} \tag{16}$$

where we set $G_f(A) = \sup_{x \in A} \inf_{y \in Z(f)} \|x - y\|$. If C_0 is not explicit, we say that f is θ-strongly concentrated.

EXAMPLES. 1) Going back to the examples of Section 2.2. In example 1), f is $(C_0, m\theta)$-strongly concentrated for any $C_0 > \frac{1}{h(x_0)}$. Whereas, in example 2), for strongly concentrated functions f_1, f_2, f is $\big(\max(C_1, C_2), \min(\theta_1, \theta_2)\big)$-strongly concentrated on $U_1 \times U_2$, *e.g.*, $f(x_1, x_2) := |x_1|^{\theta_1} + |x_2|^{\theta_2}$ is $\big(1, \min(\theta_1, \theta_2)\big)$-strongly concentrated on U whenever $(0,0) \in \operatorname{Int} U \subset \mathbb{R}^2$.

2) Let f be a nonnegative function on $U = [0,1]$ such that $Z(f)$ is finite and is not a subset of $\{0, 1\}$, then f is θ-strongly concentrated if and only if it is θ-weakly concentrated. Indeed, the Lebesgue measure of an interval is its length.

The following lemma is simple to prove (see Lemma 2.5). It leads to our main example in the sequel.

LEMMA 3.5. *Under the assumptions of Lemma 2.5, let $C_0 > \sqrt{\frac{2}{\eta}}$ (η is defined in Lemma 2.5). Then, f is $(C_0, 2)$-strongly concentrated.*

Now, we provide the strong counterpart of Theorem 2.6.

THEOREM 3.6. *Assume that*

1) Φ is a pointwise T-system at $c \in \partial \mathcal{K}$.

2) There exists $\alpha \in \tilde{\mathcal{P}}_+(c)$ and positive real constants θ and C_0 such that g_α is (C_0, θ)-strongly concentrated.

Let $C := (\theta\|\alpha\|C_0^\theta)^{\frac{1}{\theta+1}}(1 + \frac{1}{\theta})$, then setting $\delta_G(\epsilon) := (\frac{\epsilon\theta\|\alpha\|}{C_0})^{\frac{\theta}{\theta+1}}$ yields, for ϵ small enough, the following bound

$$G_\alpha\big(\delta_G(\epsilon)\big) + \sup_{\mu \in \mathcal{S}(c,\epsilon)} \mu(R^c_{\alpha,\delta_G(\epsilon)}) \le C \cdot \epsilon^{\frac{1}{\theta+1}}.$$

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3.4. *Main theorems* Our main result follows.

THEOREM 3.7. *Let Φ be an ω-Hölderian ($0 < \omega \le 1$) pointwise T-system at some point $c \in \partial \mathcal{K}$. Assume that g_α is θ-strongly concentrated for some $\alpha \in \tilde{\mathcal{P}}_+(c)$. Then, c is strongly determined, and there exists $a > 0$ such that for $\epsilon > \epsilon_0 > 0$ small enough*

$$\pi_\epsilon \le a \cdot \epsilon^{\frac{\omega}{\theta+1}}. \tag{17}$$

REMARKS. 1) In this result it is enough for Φ to be ω-Hölderian at the neighborhood of $R_{\alpha,0}$.

2) The value of ϵ_0 involved in the conclusion of Theorem 3.7 is not simple to exhibit in our general frame. Indeed, the proof of Lemma 3.2 uses a compactness argument which is not constructive. In order to precise ϵ_0, we should first obtain an alternative explicit Lemma. Assumptions for such a result imply a very strong knowledge of the level surfaces of g_α for some $\alpha \in \tilde{\mathcal{P}}_+(c)$. However, explicit superconcentration constants may be obtained; for example, on the 1-dimensional torus let $\Phi(x) = (\cos x, \sin x, \cos 2x, \sin 2x)$. Let $\xi \in]0, \pi]$, we set $c = \frac{1}{2}(1 + \cos\xi, \sin\xi, 1 + \cos 2\xi, \sin 2\xi)$ then $\mathcal{S}(c, 0) = \{\frac{1}{2}(\delta_0 + \delta_\xi)\}$. Here, $\tilde{\mathcal{P}}_+(c)$ reduces to a singleton α_ξ such that $\frac{1}{4} \le \|\alpha_\xi\| \le 3$. After a few calculations we obtain that for $\epsilon \le \epsilon_0 = \frac{\sqrt{\xi}}{6}$

$$G_\alpha\big(\delta_G(\epsilon)\big) + \sup_{\mu \in \mathcal{S}(c,\epsilon)} \mu(R^c_{\alpha,\delta_G(\epsilon)}) \le \frac{7}{\xi^{\frac{2}{3}}} \cdot \epsilon^{\frac{1}{3}}.$$

Following the lines of the proof of Theorem 3.7 yields $\pi_\epsilon \le 28(\xi^{\frac{1}{3}} + \xi^{\frac{-2}{3}})\epsilon^{\frac{1}{3}}$ for $\epsilon \le \epsilon_0$.

A forthcoming work will be devoted to evaluate explicitly the constants ϵ_0 and a in (17) for the special case of trigonometric functions.

PROOF OF THEOREM 3.7. From Lemma 3.2, $R_{\alpha,\delta_G(\epsilon)} = \bigcup_{j=1}^{l} B_{\delta_G(\epsilon),j}$ (for small ϵ) and $x_j \in \mathrm{Int}(B_{\delta_G(\epsilon),j})$, $j = 1, \ldots, l$. Theorem 3.6 states that for some constant $C_0 > 0$ only depending on Φ and c:

$$\sup_{x \in B_{\delta_G(\epsilon),j}} \|x - x_j\| \le G_\alpha\big(\delta_G(\epsilon)\big) \le C_0 \cdot \epsilon^{\frac{1}{\theta+1}}. \tag{18}$$

$$\mu(R^c_{\alpha,\delta_G(\epsilon)}) \le C_0 \cdot \epsilon^{\frac{1}{\theta+1}} \quad \text{for } \mu \in \mathcal{S}(c,\epsilon). \tag{19}$$

Any probability measure $\mu \in \mathcal{S}(c,\epsilon)$ may be decomposed as a convex combination of probability measures ν and μ' with respective supports included in $R^c_{\alpha,\delta_G(\epsilon)}, R_{\alpha,\delta_G(\epsilon)}$

$$\mu = \lambda\nu + (1-\lambda)\mu', \tag{20}$$

with $\lambda := \mu(R^c_{\alpha,\delta_G(\epsilon)})$.

Using Lemma 3.1.a) yields

$$\pi(\mu, \sigma_c) \le \lambda + \pi(\mu', \sigma_c) \le C_0 \cdot \epsilon^{\frac{1}{\theta+1}} + \pi(\mu', \sigma_c). \tag{21}$$

Set

$$\mu' = \mu'(B_{\delta_G(\epsilon),1})\mu_1 + \cdots + \mu'(B_{\delta_G(\epsilon),l})\mu_l,$$

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and let

$$\sigma' := \mu'(B_{\delta_G(\epsilon),1})\delta_{x_1} + \cdots + \mu'(B_{\delta_G(\epsilon),l})\delta_{x_l},$$

where μ_j is a probability measure with support in $B_{\delta_G(\epsilon),j}$. Triangular inequality gives

$$\pi(\mu', \sigma_c) \leq \pi(\mu', \sigma') + \pi(\sigma', \sigma_c). \tag{22}$$

Now, it follows from Lemma 3.1. b), and equations (18), (12) that for some constant C_1:

$$\pi(\mu', \sigma') \leq \max_{1 \leq j \leq l} \pi(\mu_j, \delta_{x_j}) \leq C_1 \cdot \epsilon^{\frac{1}{\theta+1}}. \tag{23}$$

Uniformly with respect to $\mu \in \mathcal{S}(c, \epsilon)$, from relation (19) we obtain, for small enough $\epsilon > 0$, $\mu(R^c_{\alpha,\delta_G(\epsilon)}) \leq \frac{1}{2}$. Hence,

$$|\mu'(B_{\delta_G(\epsilon),j}) - \mu(B_{\delta_G(\epsilon),j})| \leq 2\mu(R^c_{\alpha,\delta_G(\epsilon)}) \leq 2C_0 \cdot \epsilon^{\frac{1}{\theta+1}}. \tag{24}$$

Use now Lemma 3.1. c) leads with (24) to

$$\pi(\sigma', \sigma_c) \leq \frac{1}{2}\sum_{j=1}^{l} |\mu'(B_{\delta_G(\epsilon),j}) - p_j| \leq \frac{1}{2}\sum_{j=1}^{l} |\mu(B_{\delta_G(\epsilon),j}) - p_j| + l \cdot C_0 \cdot \epsilon^{\frac{1}{\theta+1}}. \tag{25}$$

Let now L be the linear operator $\mathbb{R}^l \longrightarrow \mathbb{R}^k$ defined for $r \in \mathbb{R}^l$ by $Lr = \sum_{j=1}^{l} r_j \Phi(x_j)$, from the pointwise T-system assumption (see Section 3.2 matrix (15)), there exists a constant $C_2 > 0$ with

$$\forall r \in \mathbb{R}^l, \|r\| \leq C_2 \|Lr\|. \tag{26}$$

Set now $p = (p_j)$, $q = \big(\mu(B_{\delta_G(\epsilon),j})\big)$. Then, $c = Lp$ and since $\mu \in \mathcal{S}(c, \epsilon)$, $\|\int \Phi\, d\mu - c\| \leq \epsilon$. Now, by the triangular inequality

$$\Big\|\int \Phi\, d\mu - Lq\Big\| \leq \Big\|\int_{R^c_{\alpha,\delta_G(\epsilon)}} \Phi\, d\mu\Big\| + \sum_{j=1}^{l} \Big\|\int_{B_{\delta_G(\epsilon),j}} \big(\Phi(x) - \Phi(x_j)\big)\, d\mu(x)\Big\|. \tag{27}$$

As Φ is ω-Hölderian, using relation (19) there exists $C_3 > 0$ with

$$\Big\|\int_{R^c_{\alpha,\delta_G(\epsilon)}} \Phi\, d\mu\Big\| \leq C_3 \cdot \epsilon^{\frac{1}{\theta+1}}, \tag{28}$$

$$\Big\|\int_{B_{\delta_G(\epsilon),j}} \big(\Phi(x) - \Phi(x_j)\big)\, d\mu(x)\Big\| \leq C_3 \mu(B_{\delta_G(\epsilon),j})\big(\sup_{x \in B_{\delta_G(\epsilon),j}} \|x - x_j\|\big)^{\omega} \tag{29}$$

Collecting inequalities (18), (27), (28) and (29) leads with (25) and (26) to the existence of a constant C_4 with:

$$\pi(\sigma', \sigma_c) \leq C_4 \cdot \epsilon^{\frac{\omega}{\theta+1}}. \tag{30}$$

Now, using (21), (22), (23) and (30) leads, for a constant $a > 0$, to

$$\pi(\mu, \sigma_c) \leq a \cdot \epsilon^{\frac{\omega}{\theta+1}}. \tag{31}$$

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COROLLARY 3.8. *Under the assumptions of Theorem 3.7, if σ_c is a Dirac mass at point x_1 then*

$$\pi_\epsilon \leq a \cdot \epsilon^{\frac{1}{\theta+1}}. \tag{32}$$

COROLLARY 3.9. *Assume that Φ is a T-system on $U = [0,1]$. Let c be a determined point. Assume that there exist $\alpha \in \mathcal{P}_+(c)$ and $\theta > 0$ such that g_α is θ-strongly concentrated. Then, if Φ is ω-Hölderian there exists a constant $a > 0$ such that for $\epsilon > 0$ small enough (17) holds.*

PROOF OF COROLLARY 3.8. Only notice that here $\sigma' = \sigma_c = \delta_{x_1}$ and use inequalities (18)-(24) to conclude.

REMARKS. 1) Lemma 3.5 provides a sufficient condition for 2-strong concentration.

2) Anastassiou ([3]) considers the set up of Corollary 3.8 in the special case $\Phi(x) := (x, x^2)$, hence $\theta = 2$. He obtains the exact asymptotic $C \cdot \epsilon^{\frac{1}{3}}$ for the Prokhorov radius. Now, Corollary 3.8 gives the decay rate for this particular case.

3) Theorem 3.7 applies for the following classical T-systems:

$$\Phi(x) = (\cos x, \sin x, \dots, \cos px, \sin px), \quad (k = 2l), \; U = [0, 2\pi] \tag{33}$$

$$\Phi(x) = (x, x^2, \dots, x^k), \quad U = [0,1]. \tag{34}$$

In the case (33) $\theta = 2$, $\omega = 1$ and the Prokhorov rate bound is $O(\epsilon^{\frac{1}{3}})$. Whereas in the case (34), the Prokhorov rate bound can be $O(\epsilon^{\frac{1}{3}})$ or $O(\epsilon^{\frac{1}{2}})$ (the second bound holds whether the support of σ_c is a subset of $\{0, 1\}$).

4) A non trivial example in m dimensions is the following. Set $U = [-\pi, \pi]^m$ and $\Phi(x^1, \dots, x^m) := (\cos x^1, \dots, \cos x^m)$, then the point $c = (1, \dots, 1)$ is such that $\mathcal{S}(c, 0) = \{\delta_0\}$ and now $\alpha = (\frac{1}{m}, \dots, \frac{1}{m}) \in \mathcal{P}_+(c)$ yields an upper bound for the Prokhorov radius with the rate $O(\epsilon^{\frac{1}{3}})$.

4. **Lower bounds for the Prokhorov radius.**

The main theorem of this section involves the following assumption:

$$A(x,\omega) \quad \exists u_0,\, K > 0,\; \forall u, |u| < u_0 \Rightarrow \|\Phi(x+u) + \Phi(x-u) - 2\Phi(x)\| \leq 2K|u|^\omega.$$

Notice that this assumption holds for $0 < \omega \leq 1$ if Φ is ω-Hölderian and for $1 < \omega \leq 2$ whenever Φ' is $(\omega - 1)$-Hölderian. This condition is usually called a Zygmund condition (see [4], p. 52).

THEOREM 4.1. *Let $c \in \mathcal{K}$ such that:*

1) $\sigma_c := \sum_{j=1}^q p_j \delta_{x_j}$ lies in $\mathcal{S}(c, 0)$ with $p_1 \neq 0$,

2) $A(x_1, \omega)$ holds for some $\omega > 0$,

3) x_1 lies in the interior of U (for the standard topology on $\mathbb{R}^m$).

Then, there exist $a > 0$, $\epsilon_0 > 0$ such that $\epsilon < \epsilon_0$ implies:

$$\pi(\mu, \sigma_c) \geq a \cdot \epsilon^{\frac{1}{1+\omega}}, \quad \textit{for some } \mu \in \mathcal{S}(c, \epsilon).$$

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Using Theorems 3.7, 4.1 and Lemma 2.5 yields:

COROLLARY 4.2. *Assume that Φ is C^2 on U. Assume that Φ is a pointwise T-system at c and that the support of σ_c is not a subset of ∂U. If there exists $\alpha \in \tilde{\mathcal{P}}_+(c)$ such that $D^2 g_\alpha$ has full rank on $R_{\alpha,0}$. Then, there exist $\epsilon_0, A > 0$ such that:*

$$A^{-1} \cdot \epsilon^{\frac{1}{3}} \leq \pi_\epsilon \leq A \cdot \epsilon^{\frac{1}{3}} \quad \text{if } \epsilon < \epsilon_0.$$

Replacing now Lemma 2.5 by a general assumption implies:

COROLLARY 4.3. *Assume that Φ is ω-Hölderian. Moreover, assume that Φ fulfills assumptions of Corollary 3.8 and Theorem 3.7, with $\theta = \omega$. Then, there exist $\epsilon_0, A > 0$ such that:*

$$A^{-1} \cdot \epsilon^{\frac{1}{\omega+1}} \leq \pi_\epsilon \leq A \cdot \epsilon^{\frac{1}{\omega+1}} \text{ if } \epsilon < \epsilon_0.$$

PROOF OF THEOREM 4.1. Set $\mu := \lambda\mu_1 + (1-\lambda)\sigma_c$ with $\mu_1 := \frac{\delta_{x^+} + \delta_{x^-}}{2}$, $x^+ := x_1 + u\epsilon^{\frac{1}{\omega+1}}$, $x^- := x_1 - u\epsilon^{\frac{1}{\omega+1}}$, $u \in \mathbb{R}^m$ $\|u\| = 1$ and $\lambda := K^{-1} \cdot \epsilon^{\frac{1}{\omega+1}}$. For ϵ small enough, x^+, x^- both lie in U. Now, assumption $A(x_1, \omega)$ ensures that μ lies in $\mathcal{S}(c, \epsilon)$ as soon as ϵ is small enough. Therefore, $\pi(\mu, \sigma_c) \geq \inf E_\epsilon$ with

$$E_\epsilon := \left\{ t > 0 : \sigma_c(\{x_1\}) \leq \mu(\{x_1\}^t) + t \right\}.$$

But, as $\pi(\mu, \sigma_c)$ vanishes with ϵ, we may assume that $t < \min_{i \neq 1} |x_1 - x_i|$. Thus

$$E_\epsilon = \left\{ t > 0 : \sigma_c(\{x_1\}) \leq \mu_1(\{x_1\}^t) + \frac{t}{\lambda} \right\}.$$

This leads to the conclusion with $a = \min(\frac{p_1}{K}, 1)$. ■

EXAMPLE AND REMARK. 1) Any smooth T-system on $U = [0, 1]$ such as trigonometric or polynomial system provides the exact rate $\epsilon^{\frac{1}{3}}$ (as soon as the support of σ_c is not a subset of $\{0, 1\}$).

2) In the case of an ω-Hölderian function Φ with $\omega < 1$ the previous lower bound and the upper bound of Section 3 do not have the same rates. It seems reasonable to think that the main hole is on the upper bound but the method used here does not seem to be improvable up to such rates.

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