ON MR. MAKEHAM'S MODIFICATION OF MR. GOMPERTZ'S THEORY OF THE LAW OF MORTALITY.

To the Editor of the Assurance Magazine.

SIR,—In your eighth and nine volumes (at pages 301 and 361 respectively) are two papers by Mr. W. M. Makeham, describing and illustrating a modification proposed by him of Mr. Gompertz's theory of the law of mortality. I have seen no reference to these papers in subsequent Numbers of the *Magazine*; which, from the estimate I have formed of them, rather surprises me. They seem to me exactly to supply what was wanting to render Mr. Gompertz's theory immediately available for practical purposes; and I shall be glad to be allowed to call the attention of your readers to them anew, that they may not be longer lost sight of, and in the hope also that Mr. M. may be thereby induced to give us some further developments of his method.

The nature of Mr. Gompertz's theory is by this time pretty well known to your readers. He sets out with a postulate having in its favour certainly a high degree of probability—viz., that the power to oppose destruction in the human frame, loses equal proportions in equal times; and he shows that, as a consequence, the logarithmic probabilities of living a year, at successive ages, will form a series in geometrical progression. In seeking to apply this property to represent the results of actual observations however, difficulties present themselves—the presence of a disturbing element makes itself felt. It is found necessary to employ not merely one, but two —perhaps three—conterminous series, in order to obtain a tolerable approximation to the results to be represented. And there still remains divergence enough to induce a suspicion that quantities in the series, treated as constant, ought in reality to have been considered as subject to slow variation.

The discontinuity caused by the breaks in the series is a sore evil. It impedes the free application to it of analytical processes, by introducing a degree of complexity in the resulting formulæ, which it is exceedingly desirable to avoid. It is with a view to this that Mr. Makeham proposes his modification, which appears to me to claim admiration for its simplicity and efficiency. Finding that the *geometrical hypothesis* (as we may call it), as applied to the logarithmic probabilities in their *pure* state, will not *generally* (it may in *some* cases) fulfil the prescribed conditions, he suggests the employment of a series of logarithmic probabilities which, when each increased by a constant quantity, to be determined by reference to the conditions to be fulfilled, shall be in geometrical progression. Mr. M. affords us, in the papers I have indicated, means of judging of the success of his modification, some of which, with additions, I shall reproduce presently. I refer to Mr. M.'s first paper for the very neat manner in which, in applying his method to any specified table, he determines the values of his constants, and forms the series of logarithmic probabilities. On one point here he is, perhaps, less explicit than some might desire him to be. He omits to show how x (or $\log \alpha^n$), the quantity to be added to each of the probabilities a, b, c, in order to bring the results into geometrical progression, is determined. It is easy enough, thus:—

Since $a+x$, $b+x$, and $c+x$ are in geometrical progression, we have

$$(a+x)(c+x)=(b+x)^2;$$

solving which equation, we get

$$x = \frac{b^2 - ac}{a + c - 2b}.$$

Regarding this expression, it is worth while to notice that if $x=0$, which will be the case if $b^2=ac$, Mr. Gompertz's theory will hold, without modification, for the period to which the probabilities a, b, c, have reference.

I now request attention to the following figures:—

Age.	Numbers Living			Mean Durations.		
	C.	G.	M.	C.	G.	M.
20	6090	6090 000	6090·000	41 46	41·47	41·51
30	5642	5591 451	5610·563	34 34	34·73	34 63
40	5075	5074·990	5091 218	27 61	27·74	27·64
50	4397	4470 432	4458 758	21·11	20 79	20 82
60	3643	3642 989	3592 690	14·34	14·31	14·57
70	2401	2424 566	2380 806	9 18	8 84	9·32
80	953	952·991	997·309	5·51	4·82	5 39
90	142	95·195	142 289	3 28	2·34	2·84
100	9	·270	1·629	2·28	1·08	1·41

Annuities, Two Lives, 3 per Cent.							
Difference, 0.				Difference, 10.			
Ages.		C.	G.	Ages.		C.	G.
20	20	17·9923	17·9592	30	20	16·7477	16·9303
30	30	15 7834	16 1711	40	30	14·4488	14·7614
40	40	13·4818	13 7566	50	40	11 9547	11·9709
50	50	10·9423	10·7625	60	50	8 7294	8·7476
60	60	7·2952	7·4810	70	60	5 5650	5·5442
70	70	4 5556	4 4396	80	70	3·2285	2 9212
80	80	2 4589	2·1495	90	80	1·5892	1·2058
90	90	1·0884	·7830	100	90	·9788	·3313
100	100	·9909	·1641				

These tables exhibit a comparison of the original Carlisle table, with adjustments of it by Mr. M. and myself;* and also of results deduced from the former, with the corresponding results deduced from Mr. M.'s adjusted table and my own. The columns marked C. are from the Carlisle table, and those marked M. and G. contain Mr. Makeham's results and mine respectively. I chose for my points of departure ages 20, 40, 60, 80, and it will be observed that at those ages the coincidence in the numbers living is all but complete.† The numbers at the intermediate ages are not very far out;

* I should mention that I had not observed Mr. M.'s adjusted table when I commenced my own.

† The absence of closer coincidence *may* be partly the result of error committed by me. But there are specialties in the operations which disentitle us to expect *perfect* coincidence.

but from age 80 they diminish much more rapidly than in the original. All who have attended to the subject know that they ought to do so, the increase in the value of life after age 90 being one of the weak points of Mr. Milne's table; but whether in my table this error is not over-corrected, may be fairly questioned.

Mr. Makeham mentions that, for facility of computation, he slightly altered the constants in the formation of his table, and therefore in it no coincidences are to be expected. It is, nevertheless, as shown by the columns of mean durations, quite as good a representation of the original as mine. None of the mean durations in Col. G, up to about age 80, differ from those in Col. C by so much as half a year; and the same may be said of those in Col. M up to a yet higher age. Either table, I fancy, furnishes at least as good an approximation as Mr. Milne's to the law of mortality that prevailed at Carlisle during the period of the observations on which the table that takes its name from that city was founded.

The remaining columns (values of annuities), with which I have nothing corresponding of Mr. Makeham's, were constructed for a special purpose, to which I shall refer by-and-bye.

Now what are the advantages to be derived from the adoption of the method of construction arising out of this modification of Mr. Gompertz's theory?

First. It furnishes us with a means of adjusting tables formed from observations, easy of application, and satisfactory in its results. The examples given supply evidence on these points. Four equi-distant values taken from the table to be adjusted afford data for the determination of the constants; the adjusted table will coincide with the original at the points thus selected, and it will, moreover, exhibit no trace of discontinuity.

The method is obviously equally applicable to the formation of a table directly from observation; that is, from an enumeration of the numbers living at the several ages. Four of these numbers, at equi-different ages, say n years apart, when corrected for increase or decrease in the population observed upon, during the last $3n$ years, will, as in the former case, furnish the requisite data.

Secondly. But the preceding are small matters in comparison with that which has now to be brought forward. The great extent of the tables requisite for the correct treatment of problems involving several lives, has hitherto been, and no doubt will continue to be, a bar to the construction of complete sets of tables in which the number of lives involved is more than two.* The inconvenience thence arising it is quite needless to enlarge upon here. Every actuary has felt it, and is constantly feeling it. Now, by the adoption of Mr. Makeham's modification of the mortality table, this state of matters will be at once remedied. The necessity for the construction of those enormous tables will no longer exist. It will suffice to construct, for each rate of interest, and for each number of lives with which it is desired to possess the power of dealing, a single table, in which the ages of the lives involved are equal. From these tables, by an easy process, the values corresponding to any combinations of ages whatsoever can be deduced. And it must be understood that the values so deduced are in no

* This will be apparent to all when it is mentioned that, *for a single rate of interest*, the number of values to be formed for three lives is 161,700, and for four 3,921,225—so rapidly do the numbers increase by the addition of another life.

sense approximations. They are the identical values that would have arisen from the application of the data in the usual manner.

Mr. Makeham's demonstration of this most valuable property, evincing as it does, on the part of Mr. M., the possession of no small amount of analytical skill, is perfectly satisfactory. I have, nevertheless, for my own satisfaction, and to enable me to exhibit here the actual operation, computed the two tables, differences 0 and 10, of which extracts are given above. The former of these is the fundamental table, and, as suggested by Mr. M., I have extended it to tenths of years, two orders of differences being used in the interpolation. When fully written out it has very much the appearance of two pages of a table of logarithms, the arrangement being entirely analogous. A slight preliminary operation is requisite, to determine a number, which, being added to the age of the younger of two lives of different ages, gives the common age of the two lives forming the combination which is equivalent to that formed by the two given lives.* With this age, then, the table being entered, the result is the required value. In the examples now to be given I confine myself to combinations whose difference is 10, as I have the means of exhibiting the verification of no others.

I find the addition to be made to the younger age, when the difference is 10, is 6·152. To find the annuities, therefore, on the combinations in the foregoing table, we have only to take out the results corresponding to 26·152, 36·152, 46·152, &c., as follows:—

26·1	16·9397	66 1	5·5599
pp. 185† × ·52 =	96	300 × ·52 =	156
20·30	16·9301	60·70	5·5443
36·1	14·7743	76·1	2·9326
249† × ·52 =	130	219 × ·52 =	114
30·40	14·7613	70·80	2·9212
46·1	11·9867	86·1	1·2123
306 × ·52 =	159	125 × ·52 =	65
40·50	11·9708	80·90	1·2058
56·1	8·7648	96·1	0·3343
330 × ·52 =	172	53 × ·52 =	28
50·60	8·7476		0·3315

These results (in none of which is there any attempt at *trimming*) may

* It is well worthy of remark that the number thus determined is independent of the *absolute ages* of the lives forming the given combination. It depends only on their *difference*. It is thus the same, for instance, whether the lives be (15·22), (16 23), (20 27), (65·72), &c. It is also independent of the rate of interest.

† 185, 249, &c., are the tabular differences.

now be compared with the table, which was constructed in the usual way.*

I now leave what I have brought forward to receive the consideration to which it is entitled, merely adding, that had the number of lives been three, four, or any greater number, the operation, with the suitable table, would have been just the same, and the results equally exact.

Thirdly. I must advert very briefly to the last of the advantages I have to mention attendant on the adoption of Mr. Makeham's modification. This is chiefly the subject of his second paper, and it is not so fully developed as those forming the subject of the first paper. It well deserves, and I hope it will receive, at the hands of Mr. M. full development. It consists in the power of solving, *accurately*—free from the *hitch* (such as it is) arising from the hypothesis we are at present compelled to use, of a uniform distribution of the deaths of each year—all the ordinary problems involving orders of survivorship amongst two, three, or any number of lives; and this, as in the previous case, by the aid of a single table for each distinct number of lives. The importance of this power cannot be over-rated. The want of it has long been felt, and the shifts to which the actuary is often driven in consequence are almost ludicrous.

I must close. I have occupied more time and space than I expected would have been necessary for the task I imposed on myself. But I will not consider the time wasted if I shall have succeeded in securing for Mr. Makeham's papers the attention they appear to me eminently to deserve; and still less if Mr. M should be thereby induced to extend his investigations, and fully to develop and methodise his results.

I am, Sir,

Yours most obediently,

P. GRAY.

Camden Town,
Nov. 30, 1863.

P.S.—I take this opportunity of stating, with reference to my paper "On the component parts of a terminable annuity," as inserted in the last Number of this *Magazine*, that I now find that the problem was very ably and thoroughly discussed in a work published in January last, by Mr. A. H. Turnbull, of the Standard Life Office. Had I known of this three months ago, I should, of course, have forborne to take up the subject. Mr. T.'s work contains *all* the tables requisite for the practical application of his method, together with the more usual Interest Tables, for quarter and half, as well as entire years; the whole most commodiously arranged and very distinctly printed..

* It is obvious that if it be desired to construct a table for any specified difference of age, a great saving of labour will be effected by the adoption of the process just exemplified.