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EQUIVALENT CONDITIONS OF A TREE MAP WITH ZERO TOPOLOGICAL ENTROPY

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Let $f : T \to T$ be a tree map with n end-points, $SAP(f)$ the set of strongly almost periodic points of f and $CR(f)$ the set of chain recurrent points of f. Write $E(f,T) = \{x : \text{there exists a sequence } \{k_i\} \text{ with } 2 \leqslant k_i \leqslant n \text{ such that } \lim\limits_{i\to\infty} f^{k_1k_2\dots k_i}(x) = x\}$ and $g = f|_{CR(f)}$. In this paper, we show that the following three statements are equivalent:

(1) f has zero topological entropy.

(2) $SAP(f) \subset E(f,T)$.

(3) Map $\omega_g : x \to \omega(x,g)$ is continuous at p for every periodic point p of f.

1. INTRODUCTION

Throughout this paper let $\mathbf{N}$ be the set of all natural numbers. Write $\mathbf{Z}^+ = \mathbf{N} \cup \{0\}$, $\mathbf{N}_n = \{1, 2, \dots, n\}$ and $\mathbf{Z}_n = \{0\} \cup \mathbf{N}_n$ for any $n \in \mathbf{N}$. Let T be a tree (that is, an one-dimensional compact connected branched manifold without cycles). A subtree of T is a subset of T, which is a tree itself. For any $x \in T$, denote by $V(x)$ the number of connected components of $T - \{x\}$. $D(T) = \{x \in T : V(x) \geqslant 3\}$ is called the set of branched points of T and $E(T) = \{x \in T : V(x) = 1\}$ is called the set of end points of T. Let $NE(T)$ denote the number of end points of T. For any $A \subset T$, we use $\overline{A}$, $\dot{A}$ and $[A]$ to denote the closure of A, the interior of A and the smallest subtree of T containing A, respectively. For any $x, y \in T$, we shall use $[x, y]$ to denote $[\{x, y\}]$. Define $(x, y] = [x, y] - \{x\}$ and $(x, y) = (x, y] - \{y\}$. For any $x \in T$ and any $\varepsilon > 0$, write $B(x, \varepsilon) = \{y \in T : d(x, y) < \varepsilon\}$.

Let $C^0(T)$ be the set of all continuous maps from T to T. For any $f \in C^0(T)$ and any $x \in T$, the set of fixed points of f, the set of m-periodic points of f, the ω-limit set of x, the set of recurrent points of f, the set of chain recurrent points of f, the topological entropy of f will be denoted by

$$F(f),\ P_m(f),\ \omega(x,f),\ R(f),\ CR(f),\ h(f),$$

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respectively. Write $O(x,f) = \{f^k(x) : k \in \mathbf{Z}^+\}$, $E(f,T) = \{x :$ there exists a sequence $\{k_i\}$ with $2 \leqslant k_i \leqslant NE(T)$ such that $\lim_{i\to\infty} f^{k_1k_2\dots k_i}(x) = x\}$ and $P(f) = \bigcup_{m=1}^{\infty} P_m(f)$. Let the metric space $K(T) = \{A \subset T : A \text{ is closed}\}$ with the Hausdorff metric H

$$H(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\}.$$

DEFINITION 1: ([**1**]) Let $f \in C^0(T)$, $x, y \in T$ and $\varepsilon > 0$. A finite sequence $\{x_i\}_{i=0}^m (m \geqslant 1)$ in T is called an ε-chain (under f) from x to y if $x_0 = x, x_m = y$ and $d(f(x_{i-1}), x_i) \leqslant \varepsilon$ (for each $i \in \mathbf{N}_m$). We say $x \overset{ch,f}{\longrightarrow} y$ if for any $\varepsilon > 0$ there is an ε-chain from x to y. $CE(x,f) = \{y : x \overset{ch,f}{\longrightarrow} y \text{ and } y \overset{ch,f}{\longrightarrow} x\}$ is called the set of chain equivalent points of x.

The dynamics of a tree map, particularly equivalent conditions which a tree map has zero topological entropy, have been studied intensively in the recent years [**2, 3, 4**]. In [**5, 6**] we obtained the following theorem.

THEOREM A. *Let $f \in C^0(T)$ and $NE(T) = n$. Then the following five statements are equivalent:*

(1) $h(f) > 0$.

(2) *there exist $m \in \mathbf{N}$ and $x \in CR(f)$ such that $CE(x,f) \cap F(f^m) \neq \emptyset$ and $[CE(x,f^m)] - \bigcup_{i=1}^{n} P_i(f^m) \neq \emptyset$.*

(3) *there exist $m \in \mathbf{N}, p \in F(f^m)$ and $s \in CR(f) - \bigcup_{i=1}^{n} P_i(f^m)$ such that $s \overset{ch,f}{\longrightarrow} p$.*

(4) *$f|_{P(f)}$ is not equicontinuous.*

(5) *there exist $x \in CR(f) - P(f)$ such that $\omega(x,f)$ is a finite set.*

In this paper we shall continue to study topological entropy of a tree map. Our aim is to find new equivalent conditions of a tree map with zero topological entropy. Our main result is the following theorem.

THEOREM 1. *Let $f \in C^0(T)$ and $g = f|_{CR(f)}$. Then the following three statements are equivalent:*

(1) $h(f) = 0$.

(2) $SAP(f) \subset E(f,T)$.

(3) *Map $\omega_g : x \to \omega(x,g)$ is continuous at p for every $p \in P(f)$.*

2. Proof of Theorem 1

In this section we shall give the proof of Theorem 1. To do this we need the following definition and known results.

DEFINITION 2: ([**6**]) Let $f \in C^0(T)$ and M is a closed subset of T with $f(M) \subset M$. M is called divisible if there exist a non-degenerate proper subtree A of T and mutually disjoint subsets

$$B_1(=B_{l+1}), B_2, \dots, B_l\big(2 \leqslant l \leqslant NE(T)\big)$$

of $\overline{T-A}$ with $M \subset \bigcup_{i=1}^{l} B_i$ such that

(i) For every $i \in \mathbf{N}_l$, B_i is the union of some connected components of $\overline{T-A}$.

(ii) For every $i \in \mathbf{N}_l$, $f(B_i \cap M) = B_{i+1} \cap M$.

Lemma 1. ([**3, 6, 7**]) *Let $f \in C^0(T)$, $p \in P(f)$ and $NE(T) = n$. Then*

(1) $\overline{P(f)} = \overline{R(f)}$.

(2) *If $h(f) = 0$, then $\omega(x, f)$ is divisible for each $x \in T$.*

(3) *If $h(f) = 0$, then there exist $2 \leqslant i_1, i_2, \dots, i_k \leqslant n$ such that $f^{i_1 i_2 \dots i_k}(p) = p$.*

For $f \in C^0(T)$ and any subtree S of T, let $r_S : T \to S$ denote the natural retraction from T to S and $g_S = r_S \circ f|_S$.

Lemma 2. ([**8**]) *If $h(f) = 0$, then $h(g_S) = 0$.*

Proposition 1. *Let $f \in C^0(T)$. If $h(f) = 0$, then $SAP(f) \subset E(f, T)$.*

Proof: Suppose $h(f) = 0$ and $x \in SAP(f)$.

If $x \in P(f)$, then it follows from Lemma 1 that there exist $2 \leqslant i_1, i_2, \dots, i_k \leqslant n$ such that $f^{i_1 i_2 \dots i_k}(x) = x$. Hence $\lim_{s\to\infty} f^{i_1 i_2 \dots i_k 2^s}(x) = x$, which implies $x \in E(f, T)$.

If $x \notin P(f)$, then by theorem A, we know that $\omega(x, f)$ is an infinite set. Let $S = \big[\omega(x, f)\big]$, $m = NE(S)$ and $g = g_S$, then $\omega(x, g) = \omega(x, f)$ and $h(g) = 0$. Now we show the proposition by induction on m.

(1) If $m = 2$. As g is an interval map, it follows from [**9**, Proposition VI.19] that $x = \lim_{s\to\infty} g^{2^s}(x) = \lim_{s\to\infty} f^{2^s}(x)$. Hence $x \in E(f, T)$.

(2) Suppose that the proposition holds for $2 \leqslant m \leqslant k$. We need only to prove the proposition still holds for $m = k + 1$.

Since $h(g) = 0$, it follows from Lemma 1 that there exist a non-degenerate proper subtree A of S and mutually disjoint subsets

$$B_1(=B_{l+1}), B_2, \dots, B_l(2 \leqslant l \leqslant k+1)$$

of S such that

(i) $x \in B_1$ and $\omega(x, f) \subset \overline{S-A} = \bigcup_{i=1}^{l} B_i$.

(ii) For every $i \in \mathbf{N}_l$, B_i is the union of some connected components of $\overline{S-A}$.

(iii) For every $i \in \mathbf{N}_l$, $f\big(B_i \cap \omega(x, f)\big) = B_{i+1} \cap \omega(x, f)$. □

Case 1. If

$$NE\Big(\big[B_1 \cap \omega(x, f)\big]\Big) \leqslant k,$$

then

$$B_1 \cap \omega(x, f) = \omega(x, f^l)$$

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and $h(f^l) = 0$. By inductive hypothesis, we know that there exists a sequence $\{k_i\}$ with $2 \leqslant k_i \leqslant k$ such that $\lim_{s\to\infty} f^{lk_1k_2\dots k_s}(x) = x$. Thus $x \in E(f,T)$.

CASE 2. If

$$NE\Big([B_1 \cap \omega(x,f)]\Big) = k+1,$$

then $l = 2$,

$$NE\Big([B_2 \cap \omega(x,f)]\Big) = 2$$

and $B_2 \cap \omega(x,f) = \omega(f(x), f^2)$. Let $S_1 = \Big[\omega\big(f(x), f^2\big)\Big]$ and $g_1 = g_{S_1}$, then

$$\omega(f(x), g_1) = \omega(f(x), f^2)$$

and g_1 is an interval map. By [**9**, Lemma VI.14], we know that for any $t \in \mathbf{N}$, there exist mutually disjoint closed subintervals $C_1^t, C_2^t, \dots, C_{2^t}^t$ of S_1 such that

(i) $\omega(f(x), g_1) \subset \bigcup_{i=1}^{2^t} C_i^t$.

(ii) $C_{i+1}^t \subset [C_i^t \cup C_{i+2}^t]$ for any $i \in \mathbf{N}_{2^t-2}$.

(iii) $g_1^{2^t}(C_{2j-1}^t \cap \omega\big(f(x), g_1)\big) = C_{2j}^t \cap \omega(f(x), g_1)$ and $g_1^{2^t}(C_{2j}^t \cap \omega\big(f(x), g_1)\big) = C_{2j-1}^t \cap \omega(f(x), g_1)$ for any $j \in \mathbf{N}_{2^{t-1}}$.

(iv) $C_{2j-1}^t \cup C_{2j}^t \subset C_j^{t-1}$ for any $j \in \mathbf{N}_{2^{t-1}}$.

It is easy to see that for any $t \in \mathbf{N}$,

$$B_1 \cap \omega(x,f) = \bigcup_{i=1}^{2^t} f(C_i^t \cap \omega\big(f(x), f^2)\big),$$

and for any $i, j \in \mathbf{N}_{2^t}$ ($i \neq j$),

$$f(C_i^t \cap \omega\big(f(x), f^2)\big) \cap f(C_j^t \cap \omega\big(f(x), f^2)\big) = \emptyset.$$

Choose $i_t \in \mathbf{N}_{2^t}$ such that

$$x \in M_t(x) = f(C_{i_t}^t \cap \omega\big(f(x), f^2)\big).$$

Put $M(x) = \bigcap_{t=1}^{\infty} M_t(x)$, then $M_1(x) \supset M_2(x) \supset \cdots$ and $x \in M(x)$.

CLAIM 1. $z \xrightarrow{ch,f} u$ for any $u \in \big[M(x)\big]$ and any $z \in \omega(x,f)$.

PROOF OF CLAIM 1: Take $B_2 = C_{j_0}^0 \supset C_{j_1}^1 \supset C_{j_2}^2 \supset \cdots$ such that

$$\operatorname{diam}(C_{j_t}^t) \leqslant \operatorname{diam} \frac{(C_{j_{t-1}}^{t-1})}{2} (t \in \mathbf{N}),$$

then $\bigcap_{t=1}^{\infty} C_{j_t}^t = \{w\} \in \omega(x,f)$.

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For any $\varepsilon > 0$, there exists $\delta > 0$ such that $d(f(x), f(y)) < \varepsilon$ whenever $d(x,y) < \delta$ with $x, y \in T$. Since $M(x) = \bigcap_{t=1}^{\infty} M_t(x)$ and $\bigcap_{t=1}^{\infty} C_{j_t}^t = \{w\}$, we may choose $s, t \in \mathbf{N}$ such that $C_{j_t}^t \subset B(w,\delta)$ and $f^s(C_{j_t}^t) \supset [M(x)]$. Thus, for any $u \in [M(x)]$ there exists $v \in C_{j_t}^t$ such that $f^s(v) = u$, therefore $w, f(v), f^2(v), \dots, f^s(v)$ is an ε chain from w to u, which implies $z \overset{ch,f}{\longrightarrow} w \overset{ch,f}{\longrightarrow} u$ for any $z \in \omega(x,f)$. Claim 1 is proven. ∎

By Claim 1 and theorem A, we obtain the following.

CLAIM 2. $[M(x)] \cap P(f) = \emptyset$.

CLAIM 3. $M(x) = \{x\}$.

PROOF OF CLAIM 3: Assume on that contrary that $M(x) - \{x\} \neq \emptyset$. Let $y \in M(x) - \{x\}$, then $y \in R(f)$ since $M(x)$ is a closed subset and $\omega(x,f)$ is a minimal set, it follows from Lemma 1 and Claim 2 that $x, y \in \partial\big([M(x)]\big)$.

Let
$$\varepsilon = \min\Big\{d(u,v)/3 : u, v \in \partial\big([M(x)]\big), u \neq v\Big\},$$
then $M(x) \subset M_t(x) \subset B(M(x), \varepsilon)$ and $f^{2^{t+1}}(M_t(x)) = M_t(x)$ for some $t \in \mathbf{N}$. Since $x \in SAP(f) = SAP(f^{2^{t+1}})$, there exists $N \in \mathbf{N}$ such that $f^{2^{t+1}Nj}(x) \in B(x,\varepsilon) - [M(x)]$ for all $j \in \mathbf{N}$. Again since $y \in R(f) = R(f^{2^{t+1}N}))$, there exists $j_1 \in \mathbf{N}$ such that $f^{2^{t+1}Nj_1}(y) \in B(y,\varepsilon) - [M(x)]$. Thus $x \in [f^{2^{t+1}Nj_1}(x), y]$ and $y \in [x, f^{2^{t+1}Nj_1}(y)]$. By [**10**, Lemma 1] we get $[M(x)] \cap P(f) \neq \emptyset$, which contradicts Claim 2. Claim 3 is proven. ∎

Since $M(x) = \{x\}$, for any $\varepsilon > 0$, we may choose $N \in \mathbf{N}$ such that $M_t(x) \subset B(x,\varepsilon)$ for all $t \geqslant N$. On the other hand, $d(f^{2^{t+1}}(x), x) < \varepsilon$ since $f^{2^{t+1}}(M_t(x)) = M_t(x)$. Thus $\lim_{s\to\infty} f^{2^s}(x) = x$. Thus $x \in E(f,T)$.

PROPOSITION 2. *Let $f \in C^0(T)$ and $g = f|_{CR(f)}$. If $h(f) = 0$, then for every $p \in P(f)$, map $\omega_g : x \to \omega(x,g)$ is continuous at p.*

PROOF: Assume on the contrary that there exists $p \in P_m(f)$ such that ω_g is not continuous at p. Then there exist $\varepsilon_0 > 0$, sequence $\{x_n\}$ in $CR(f)$ with $x_n \to p$ such that
$$H\big(\omega(x_n,g), \omega(p,g)\big) \geqslant 2\varepsilon_0.$$

Without loss of generality we may suppose (by taking a subsequence) that either

(i) $\sup_{a\in\omega(x_n,g)} d\big(a, O(p,g)\big) \geqslant 2\varepsilon_0$ for each $n \in \mathbf{N}$ or

(ii) $\sup_{b\in O(p,g)} d\big(b, \omega(x_n,g)\big) \geqslant 2\varepsilon_0$ for each $n \in \mathbf{N}$.

CLAIM 4. We can suppose that there exist $r \in \mathbf{Z}_{m-1}$ and $k_n \to \infty$ such that
$$d\big(g^{mk_n+r}(x_n), f^r(p)\big) \geqslant \varepsilon_0.$$

PROOF OF CLAIM 4: If (i) holds, then for each $n \in \mathbf{N}$ we can choose $l_n \in \mathbf{N}$ such that $d\big(g^{l_n}(x_n), O(p,g)\big) \geqslant \varepsilon_0$ with $l_n > l_{n-1}$. By taking a subsequence, we may assume that $l_n = mk_n + r$ for some $r \in \mathbf{Z}_{m-1}$ and all $n \in \mathbf{N}$. Thus we have

$$d\big(g^{mk_n+r}(x_n), f^r(p)\big) \geqslant \varepsilon_0.$$

If (ii) holds, then by taking a subsequence, we may assume that there exists $r \in \mathbf{Z}_{m-1}$ such that

$$d\big(g^r(p), \omega(x_n, g)\big) \geqslant 2\varepsilon_0 \text{ for all } \ n \in \mathbf{N}.$$

Thus, for each $n \in \mathbf{N}$, there exists $k_n > k_{n-1}$ such that $d\big(g^k(x_n), g^r(p)\big) \geqslant \varepsilon_0$ whenever $k > k_n$. Hence we have

$$d\big(g^{mk_n+r}(x_n), f^r(p)\big) \geqslant \varepsilon_0.$$

Claim 4 is proven. ∎

Since $CR(f) = CR(f^m)$ and $f\big(CR(f)\big) \subset CR(f)$, we may assume that $r = 0$ and $G = g^m$. Then $p \in \mathrm{Fix}(G)$, $x_n \in CR(G)$, $x_n \to p$ and

$$d\big(p, G^{k_n}(x_n)\big) \geqslant \varepsilon_0.$$

For convenience, we may assume that $G^{k_n}(x_n) \to b$. It is easy to see that $b \in CE(p, G)$. Let $J = \big[CE(p,G)\big]$. Since $h(G) = 0$, by theorem A, we may assume that for each $n \in \mathbf{N}$, $O(x_n, G)$ is an infinite set and $\left(\bigcup_{n=1}^{\infty} O(x_n, G)\right) \cap J = \emptyset$.

Let T_1 and T_2 are the connected components of $\overline{T - J}$ containing p and b, respectively. By taking a subsequence, we may assume that $\{x_n\} \subset T_1$, $\big\{G^{k_n}(x_n)\big\} \subset T_2$. Put

$$s_n = \min\big\{k \geqslant 0 : G^k(x_n) \notin T_1\big\},$$

then there exist a connected component T_3 of $\overline{T - J}$ and a subsequence $\{s_{i_n}\}$ of $\{s_n\}$ such $T_3 \cap T_1 = \emptyset$ and $G^{s_{i_n}}(x_{i_n}) \in T_3$. For convenience, we may assume that $s_{i_n} = s_n$, $G^{s_n}(x_n) \to y$ and $G^{s_n - 1}(x_n) \to z$. It is easy to show that $y, z \in CE(p, G)$. Hence $z = p = G(z) = y$, which contradicts $T_3 \cap T_1 = \emptyset$. Proposition 2 is proven.

Proposition 3. *Let $f \in C^0(T)$ and $g = f|_{CR(f)}$. If $h(f) > 0$, then*

(1) *there exists $x \in SAP(f)$ such that $x \notin E(f, T)$.*

(2) *there exists $p \in P(f)$ such that $\omega_g : x \to \omega(x, g)$ is not continuous at p.*

Proof: Since $h(f) > 0$, it follows from [**11**] that there is the closure J of a connected component of $T - D(T)$, $s \in \mathbf{N}$ and $[a, b], [c, t] \subset J$ such that:

(i) $(a, b] \cap [c, t) = \emptyset$ and $f^s\big([a, b]\big) \cap f^s\big([c, t]\big) \supset [a, b] \cup [c, t]$.

(ii) $f^s(a) = f^s(t) = a$ and $f^s(b) = f^s(c) = t$.

(iii) $f^s(x) \in (x, t)$ if $x \in (a, b)$ and $f^s(x) \in (a, t)$ if $x \in (c, t)$.

(1) it is easy to see that f^s has a periodic point x with period l for each prime number $l > NE(T)$. Obviously $x \notin E(f, T)$.

(2) Choose points $\{b_k\}_{k=1}^{\infty}$ such that

$$f^s(b_k) = b_{k-1},\ f^s(b_1) = b\ b_k \in (a, b_{k-1}),\ f^s[b_k, b_1] \subset [b_{k-1}, b]$$

for each $k \geqslant 2$, then $\lim\limits_{k\to\infty} b_k = a$. Since

$$f^{s(k+2)}(b_k) = a,\ f^{s(k+2)}(b_{k+1}) = t,$$

it follows from [**10**, Lemma 1] that there exists $a_{k+1} \in (b_{k+1}, b_k)$ such that $a_{k+1} \in \mathrm{Fix}(f^{s(k+2)})$ and $\lim\limits_{k\to\infty} a_{k+1} = a$. For any $k \in \mathbf{N}$, there exists $i_k \in \mathbf{N}_k$ such that $f^{si_k}(a_k) \in [b_1, b]$. Thus

$$H\big(\omega(a_n, g), \omega(a, g)\big) \geqslant \min\big\{d(b_1, a), d(b, b_1)\big\}.$$

Which implies that ω_g is not continuous at a. ∎

PROOF OF THEOREM 1: (1)⇔(2) follows from Proposition 1 and Proposition 3.
(1)⇔(3) is from Proposition 2 and Proposition 3. ∎

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