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Quadrics Associated with a Möbius Hexad

By H. W. TURNBULL.

§ 1. The following investigation arose out of a reconsideration[1] of a certain invariant of six points in space

$$I = (1234)(3456)(5612) - (2345)(4561)(6123)$$

where (1234) denotes the determinant whose rows are the coordinates of the first four points: and similarly for the other factors. The vanishing of this invariant implies that the sixth point lies upon a certain quadric defined by the other five. The following elementary account brings out several interesting and possibly new theorems upon quadric surfaces.

§ 2. Let (x, y, z, t) be homogeneous coordinates of a point in three dimensional space. Then the equation

$$Q_1 \equiv xz - yt = 0 \tag{1}$$

is that of a quadric surface containing the four edges AB, BC, CD, DA of the tetrahedron of reference, together with a further fixed point G, where the coordinates of A are (1, 0, 0, 0), of B (0, 1, 0, 0), of C (0, 0, 1, 0), of D (0, 0, 0, 1) and of G (1, 1, 1, 1). Such a skew quadrilateral lying on the quadric, together with one further point, will yield exactly enough conditions to determine the quadric uniquely. Evidently the quadric would be equally well specified by

[1] *Cf. Journal of the London Math. Soc.*, 2 (1927), 233-240 (238). Also Blaschke (*Math. Zeitschrift*, 6 (1920), 83-93).

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these five points together with four others, one upon each side of the quadrilateral $ABCD$, a total of nine points in all, nine being the requisite number.

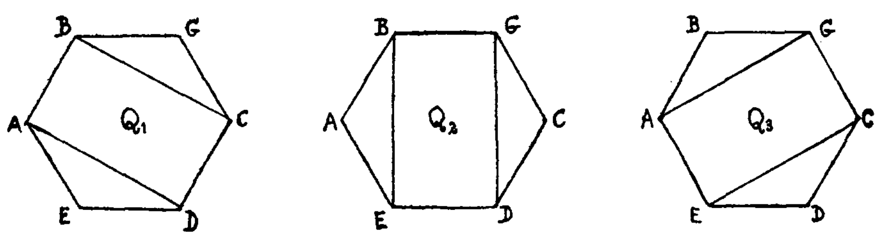
So far the five given points are in general position, no four of them being coplanar. Now let a sixth point $E\,(a, b, c, d)$ be taken on the quadric, such that

$$\cancel{ab - cd} \neq 0, \tag{2}$$

E not lying in the plane of any three of the earlier five points. Then these six points in the cyclic order

$$ABGCDE \tag{3}$$

will be said to form a *Möbius hexad*: and the reason for this nomenclature will appear later. Evidently six such points must satisfy one condition and must not be entirely arbitrary for their configuration to be a Möbius hexad: but there are in fact many different ways of stating the condition.



From the six given vertices of the hexagon $ABGCDE$ a quadric Q_1 has been obtained, passing through each vertex and containing the lines AB, BC, CD, DA. It appears that this property is cyclic and leads to two further circumscribing quadrics as follows:

Theorem 1. *If six points $ABGCDE$ form a Möbius hexad then the three skew quadrilaterals $ABCD$, $BGDE$, $GCEA$ lie upon three different quadrics circumscribing the hexad.*

Proof. The three equations Q_1, Q_2, Q_3: namely

$$\left.\begin{aligned} xz - yt &= 0, \\ (a-d)\,yz + (c-a)\,zx + (d-c)\,xy + (b-c)\,xt + (c-a)\,yt + (a-b)\,zt &= 0, \\ (a-d)\,yz + (d-b)\,zx + (c-d)\,xy + (b-c)\,xt + (d-b)\,yt + (b-a)\,zt &= 0, \end{aligned}\right\} \tag{4}$$

are those of three quadrics circumscribing the tetrahedron of reference, and also containing the points E and G, *provided that $ac = bd$.* These

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facts are easily verified by substitution. To shew that the line CE lies on the quadric Q_3, it is only necessary to verify that, for all values of k, the point

$$(a,\ b,\ c+k,\ d)$$

satisfies the last equation, such a point being on the line CE. Similarly for each line: and this proves the theorem.

Instead of the original condition governing the positions of the six points $ABGCDE$, a second and simpler condition is as follows: *let the planes ABG, GCD, DEA, BCE meet in a point F.* These planes have the respective equations

$$z=t, \quad x=y, \quad cy=bz, \quad dx=at$$

and therefore have a common point if, and only if, $ac=bd$: in which case F the common point is $(b,\ b,\ c,\ c)$. Furthermore *the planes BGC, CDE, EAB, GDA also meet in a point H* $(a,\ b,\ b,\ a)$. Their equations are

$$x=t, \quad bx=ay, \quad dz=ct, \quad y=z.$$

It is easy to verify that both points F and H lie on each of the three quadrics: and hence the eight points $ABCDEFGH$ are common to all three quadrics. But three quadrics in general meet in eight and only eight points, called a system of *associated points*. This accounts therefore for all such common points of Q_1, Q_2, Q_3. Since seven of eight associated points can usually be taken arbitrarily, whereas five only of the present system are independent, it follows that this is a specially restricted system of associated points. It is convenient to collect their coordinates into a table row-wise:

$$\begin{array}{c}A\\B\\C\\D\end{array}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\qquad\begin{array}{c}E\\F\\G\\H\end{array}\begin{bmatrix}a&b&c&d\\b&b&c&c\\b&b&b&b\\a&b&b&a\end{bmatrix}\tag{5}$$

where $ac=bd$. This suggests that the points $EFGH$ have their own properties with regard to the first quadric Q_1, in view of the fact that Q_1 contains the quadrilateral $ABCD$.

THEOREM 2. *The eight points can be grouped in three ways into sets of four, giving rise to pairs of quadrilaterals simultaneously lying upon the same quadric. Namely*

Q_1 contains the quadrilaterals $ABCD$ and $EFGH$,
Q_2 contains the quadrilaterals $BGDE$ and $AFCH$,
Q_3 contains the quadrilaterals $AECG$ and $BFDH$.

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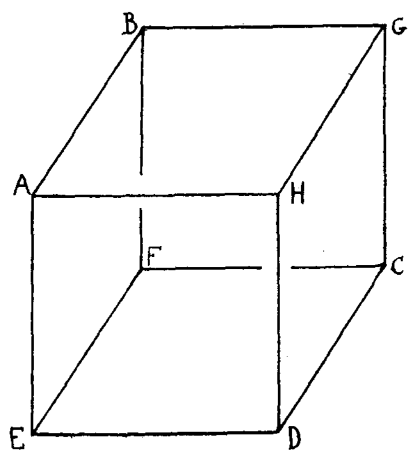
Proof. For all values of k the point

$$(ka + b, kb + b, kc + c, kd + c)$$

lies on the quadric $xz = yt$, that is Q_1. Hence Q_1 contains the line EF: and similarly contains FG, GH and HE. A similar verification applies to the other quadrics Q_2 and Q_3.



The accompanying figure, although not projectively accurate, is convenient to illustrate the essential symmetry of the eight points arising out of an original hexagon (the outline of the figure). Suppress one pair of points AC, BD, EG *or* HF, together with the six lines meeting by threes at the pair, and a hexagon is left. Such is always a Möbius hexad; and there are four possibilities.

Next consider the following four schemes:

$$\begin{matrix} ABGH & ABEF & ADEH & ADGF \\ CDEF\text{'} & CDGH\text{'} & CBGF\text{'} & CBEH\,. \end{matrix} \qquad (6)$$

The columns contain the four pairs of letters always occurring as diagonally opposite in a hexad (and in the cubical illustration). Each such scheme defines a pair of *Möbius tetrads*, namely a pair of tetrahedra each of which is inscribed in the other. Verification shews that if four letters are chosen, *one* from each column and *three* from one row of a scheme, then their four points are always coplanar. For example, B of the second scheme is coplanar with CGH: G with ABF and so on. The corresponding determinant obtained from four suitable rows of (5) vanishes. Hence the tetrahedron $ABEF$

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is inscribed in $CDGH$, while simultaneously $CDGH$ is inscribed in $ABEF$.

It is this property known to Möbius of two interlocking tetrahedra which gives the reason for the preceding nomenclature of the hexad.

When six points form a Möbius hexad the six sides AB, BG, GC, CD, DE, EA belong to a linear complex. This known property is a third alternative way of stating the condition governing the six points. It is proved by shewing that the Plücker coordinates of these six lines arranged in six rows as a determinant cause it to vanish. It can be verified that the determinantal equation obtained from five of these lines together with a sixth row p_{23}, p_{31}, p_{12}, p_{14}, p_{24}, p_{34} is

$$d\,(p_{23} + p_{31} - p_{24}) + cp_{14} = 0. \tag{7}$$

where the six coordinates can be taken as

$$yz' - y'z, \quad zx' - z'x, \quad xy' - x'y, \quad xt' - x't, \quad yt' - y't, \quad zt' - z't.$$

This is the equation of the linear complex, where the line containing points (x, y, z, t) and (x', y', z', t') is a member of the complex. Each of the twelve lines shewn in the last figure belongs to the complex, as may easily be verified. Also if (x', y', z', t') is regarded as a fixed point, equation (7) is that of its null plane. The null plane of each of the eight points is indicated in this figure by the trio of lines meeting at the point.

THEOREM 3. *The linear complex to which these twelve lines of the Möbius tetrads belong reciprocates each of the quadrics Q_1, Q_2, Q_3 into itself.*

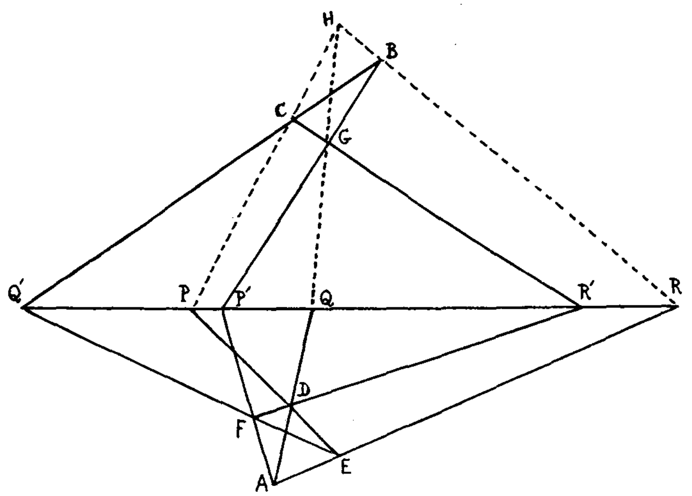
Proof. For the reciprocal of a quadric with regard to a linear complex is the envelope of null planes belonging to points of the quadric. Each line AB is a generator of a quadric Q_1 and of its reciprocal since, AB belongs to the complex. Hence the reciprocal of Q_1 contains both quadrilaterals $ABCD$, $EFGH$ possessed by Q_1 and therefore coincides with Q_1.

THEOREM 4. *Each of these quadrics reciprocates the linear complex into itself.*

Proof. The polar of the line $(p_{23}, p_{31}, p_{12}, p_{14}, p_{24}, p_{34})$ with regard to the quadric $xz - yt = 0$ is the line whose coordinates are

$$(p_{23}, \ -p_{24}, \ -p_{12}, \ p_{14}, \ -p_{31}, \ -p_{34})$$

and this satisfies the condition (7).



The accompanying figure (*Cf.* H. F. Baker, *Principles of Geometry*, **1** (1922), 61) illustrates the construction of such a set of points by means of six given coplanar lines, as shewn, meeting another plane π in points $PP'QQ'RR'$. In the plane π three given lines through $P'Q'R'$ determine a triangle BCG, while the dotted lines complete the figure by meeting at H, the eighth of the required points.

A Series Identity

By J. C. P. Miller.

In connection with some work on the theory of probable errors, the following series identity arose:—

$$\frac{1}{2a} + \left(-\frac{1}{2}\right)^2 \frac{1}{2a+2} + \left(-\frac{1}{2}\cdot\frac{1}{4}\right)^2 \frac{1}{2a+4} + \dots$$
$$+ \left(-\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6}\dots\frac{2r-3}{2r}\right)^2 \frac{1}{2a+2r} + \dots .$$
$$= \frac{1}{2a+1} + \left(-\frac{1}{2}\right)^2 \frac{1}{2a-1} + \left(-\frac{1}{2}\cdot\frac{1}{4}\right)^2 \frac{1}{2a-3} + \dots$$
$$+ \left(-\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6}\dots\frac{2r-3}{2r}\right)^2 \frac{1}{2a-2r+1} + \dots ,$$

where a is a positive integer.