TERM RANK OF THE DIRECT PRODUCT OF MATRICES

RICHARD A. BRUALDI

1. Introduction. Let $A = [a_{ij}]$ be a matrix of 0's and 1's or a (0, 1)-*matrix* of size m by m'. The *term rank* of A is defined as the maximal number of 1's of A with no two of the 1's on the same row or colunn. A theorem due to D. König (**3**, Theorem 5.1, p. 55) asserts that the term rank of A is also equal to the minimal number of rows and columns of A that collectively contain all the 1's. The term rank of A will be denoted by $\rho(A)$. Obviously it is invariant under arbitrary permutations of the rows and columns of A. We assume without loss of generality that all matrices considered have no rows or columns consisting entirely of 0's.

Let $B = [b_{kl}]$ be another (0, 1)-matrix of size n by n'. The *direct product* or *Kronecker product* of A with B is defined by

$$A \times B = \begin{bmatrix} a_{11} B & a_{12} B & \dots & a_{1m'} B \\ a_{21} B & a_{22} B & \dots & a_{2m'} B \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} B & a_{m2} B & \dots & a_{mm'} B \end{bmatrix}. \tag{1.1}$$

It is a (0, 1)-matrix of size mn by $m'n'$. The submatrix of $A \times B$ given by

$$[a_{ij} B] \qquad (1 \leqslant i \leqslant m;\ 1 \leqslant j \leqslant m') \tag{1.2}$$

will be called the (i, j)-*block* of $A \times B$ or simply a *block* of $A \times B$. The blocks of $A \times B$ are of size n by n'. Evidently we have a natural one-to-one correspondence between the elements of A and the blocks of $A \times B$. Thus we say that the block $[a_{ij} B]$ corresponds to the element a_{ij} of A. The submatrix

$$\begin{bmatrix} a_{1i} B \\ a_{2i} B \\ \cdot \\ \cdot \\ \cdot \\ a_{mi} B \end{bmatrix} \qquad (1 \leqslant i \leqslant m') \tag{1.3}$$

of $A \times B$ is called the ith *column block* of $A \times B$. It is of size mn by n'. Similarly the submatrix

$$[a_{j1} B, a_{j2} B, \dots, a_{jm'} B] \qquad (1 \leqslant j \leqslant m) \tag{1.4}$$

of $A \times B$ is called the jth *row block* of $A \times B$. It is of size n by $m'n'$.

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The direct product of matrices has applications in the theory of games and the theory of graphs. It is also known in combinatorial mathematics, where it is shown that the direct product of two Hadamard matrices is a Hadamard matrix (**3**, pp. 104–107). The direct product is known to possess many properties. A good account of these is given by MacDuffee in (**1**, pp. 81–84). Actually we need only two properties here. Given the matrices A and B we may form either $A \times B$ or $B \times A$. In general these two matrices are different. However, one verifies by inspection that the rows and columns of $A \times B$ can be permuted to give $B \times A$. From this it follows that

$$\rho(A \times B) = \rho(B \times A). \tag{1.5}$$

Secondly, permutations of the rows and columns of A or B induce in a natural way permutations of the rows and columns of $A \times B$. This also follows by inspection.

The purpose of this paper is to investigate the term rank $\rho(A \times B)$ of the direct product of A with B. In §2 we first give an elementary upper bound for $\rho(A \times B)$. We then derive a significant lower bound for $\rho(A \times B)$. As a corollary we obtain necessary and sufficient conditions in order that the term rank be multiplicative on $A \times B$. In §3 we introduce the concepts of essential rows and essential columns. We use these to derive a normal form for an arbitrary (0, 1)-matrix. This normal form is then used to find a lower bound on the number of essential rows and columns of $A \times B$. Finally we obtain an upper bound for $\rho(A \times B)$.

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2. A lower bound for $\rho(A \times B)$. We first prove:

THEOREM 2.1. *Let A and B be two* (0, 1)*-matrices of sizes m by m' and n by n' respectively. Then*

$$\rho(A \times B) \leqslant \min\{\rho(A) \max\{n, n'\}, \rho(B) \max\{m, m'\}\}. \tag{2.1}$$

Proof. Let rows $i_1, i_2, \ldots, i_r$ and columns $j_1, j_2, \ldots, j_s$ contain all the 1's of B. Here r and s are non-negative integers with $r + s = \rho(B)$. In each row block of $A \times B$ select rows $i_1, i_2, \ldots, i_r$, and in each column block select columns $j_1, j_2, \ldots, j_s$. It is then clear that the totality of these lines contain all the 1's of $A \times B$. Moreover the number of lines we have used is

$$mr + m's \leqslant (r + s) \max\{m, m'\} = \rho(B) \max\{m, m'\}.$$

Applying König's theorem and (1.5), we obtain (2.1).

Inequality (2.1) is in general not very sharp. This is particularly so if $|m' - m|$ and $|n' - n|$ are both large. At the end of §3 we obtain an upper bound for $\rho(A \times B)$ that will be better under these circumstances.

LEMMA 2.2. $\rho(A \times B) \geqslant \rho(A)\rho(B)$.

Proof. Select $\rho(A)$ 1's of A with no two 1's on the same line and $\rho(B)$ 1's of B with no two 1's on the same line. In each of the blocks of $A \times B$ corresponding to these $\rho(A)$ 1's of A select the prescribed $\rho(B)$ 1's of B. This gives $\rho(A)\rho(B)$ 1's of $A \times B$ with no two of the 1's on the same line.

Now let A be a (0, 1)-matrix of size m by m'. In A select $\rho(A)$ 1's no two of which lie on the same line. (In general this can be done in many ways, but once we have made our choice we do not alter it.) The rows and columns in which these 1's lie must contain all the 1's of A, for otherwise we contradict the definition of the term rank $\rho(A)$ of A. Denote by A_0 that submatrix of A consisting of those elements of A that lie in the intersection of the rows belonging to these $\rho(A)$ 1's with the columns belonging to these $\rho(A)$ 1's. Let A_1 be that submatrix of A consisting of those elements that lie within the rows belonging to the selected $\rho(A)$ 1's of A but not within their columns. Similarly, let A_2 be that submatrix of A consisting of those elements that lie within the columns of the selected $\rho(A)$ 1's but not within their rows. Thus if we permute the rows and columns of A so that these $\rho(A)$ 1's occupy the first $\rho(A)$ positions on the main diagonal of A, we obtain

$$\rho(A)\left\{\overbrace{}^{\rho(A)}\right. \begin{bmatrix} A'_0 & A'_1 \\ A'_2 & 0 \end{bmatrix},$$

where the prime sign on a matrix here denotes a matrix obtained from it by permuting rows and columns, and where 0 denotes all 0's. We proceed by induction. If we have formed $A_{i_1,\ldots,i_k}$ with $i_j = 1$ or 2 for $j = 1, 2, \ldots, k$, then select $\rho(A_{i_1,\ldots,i_k})$ 1's of $A_{i_1,\ldots,i_k}$ with no two of them on the same line. We define $A_{i_1,\ldots,i_k,1}$ to be that submatrix of $A_{i_1,\ldots,i_k}$ consisting of those elements that lie within the rows belonging to the chosen $\rho(A_{i_1,\ldots,i_k})$ 1's but not within their columns. Likewise we define $A_{i_1,\ldots,i_k,2}$ to be that submatrix of $A_{i_1,\ldots,i_k}$ consisting of those elements that lie within the columns belonging to the chosen $\rho(A_{i_1,\ldots,i_k})$ 1's but not within their rows. It is possible that the matrix $A_{i_1,\ldots,i_k,i_{k+1}}$ is vacuous, in which case we define

$$\rho(A_{i_1,\ldots,i_k,i_{k+1}}) = 0.$$

This happens, in particular, if $A_{i_1,\ldots,i_k}$ consists entirely of 0's. The collection of submatrices of A,

$$\{A_0; A_1, A_2; A_{11}, A_{12}, A_{21}, A_{22}; \ldots\}$$

will be called a *decomposition* of A. Different decompositions of A may be obtained by altering the choice of 1's in the different steps. We say that the above decomposition of A *terminates at stage* k where $k \geqslant 0$ provided $A_{i_1,\ldots,i_k,i_{k+1}}$ consists of all 0's or is vacuous for all $i_j = 1$ or $2, j = 1, 2, \ldots, k+1$ and provided $A_{i_1,\ldots,i_k}$ does not have this property for some choice of $i_j = 1$ or 2, $j = 1, 2, \ldots, k$. Now we are assuming that A has no rows or columns consisting entirely of 0's. Therefore a decomposition of A terminates at the stage 0 if and only if A is square with $\rho(A)$ equal to the order of A. It is possible that two different decompositions of A terminate at different stages. However, this is not true if A has a decomposition that terminates at the stage 0. (It is in general true that A_1 contains zero rows and A_2 contains zero columns; in fact, the number of non-zero rows of A_1 plus the number of non-zero columns of A_2 cannot exceed the term rank of A. The zero rows of A_1 and the zero columns of A_2 are of no importance and thus could be excluded from the definition of A_1 and A_2. We do not adopt this convention, however. Similar remarks hold for an arbitrary $A_{i_1,i_2,\ldots i_k}$.)

Two subsets E and F of the elements of A are *row disjoint* (*column disjoint*) provided the rows (columns) of A in which the elements of E lie are distinct from the rows (columns) of A in which the elements of F lie. E and F are *line disjoint* provided they are both row and column disjoint. A collection of subsets of the elements of A is *pairwise line disjoint* provided every pair of subsets is line disjoint.

LEMMA 2.3. *The collection of submatrices* $\{A_{i_1,i_2,\ldots,i_k}\}$ *where* k *is a fixed positive integer and* $i_j = 1$ *or* 2 *for* $j = 1, 2, \ldots, k$ *is pairwise line disjoint.*

Proof. For $k = 1$, the collection consists of two submatrices A_1 and A_2, and these are line disjoint by their definition. Assume the statement of the lemma is true for a $k \geqslant 1$ so that $\{A_{i_1,i_2,\ldots,i_k}\}$ is pairwise line disjoint. But then for $(i_1, i_2, \ldots, i_k)$ fixed $A_{i_1,\ldots,i_k,1}$ and $A_{i_1,\ldots,i_k,2}$ are line disjoint, which shows that the entire collection $\{A_{i_1,i_2,\ldots,i_{k+1}}\}$ is also pairwise line disjoint.

We come now to the main theorem of this section. Let A and B be two matrices of 0's and 1's. Form a decomposition

$$\{A_0; A_1, A_2; A_{11}, A_{12}, A_{21}, A_{22}; \ldots\} \tag{2.2}$$

of A and a decomposition

$$\{B_0; B_1, B_2; B_{11}, B_{12}, B_{21}, B_{22}; \ldots\} \tag{2.3}$$

of B. (Of course, within each of the matrices in these decompositions we have a prescribed set of 1's with no two on the same line.) For $k \geqslant 1$, let i_k denote either the integer 1 or 2. If $i_k = 1$, define $j_k = 2$; while if $i_k = 2$, define $j_k = 1$. We then have

THEOREM 2.4.

$$\rho(A \times B) \geqslant \max\Big\{\rho(A)\rho(B) + \sum_{(i_1)} \rho(A_{i_1})\rho(B_{j_1}) + \ldots + \sum_{(i_1, i_2, \ldots, i_k)} \rho(A_{i_1, i_2, \ldots, i_k})\rho(B_{j_1, j_2, \ldots, j_k}) + \ldots\Big\}$$

where $\sum_{(i_1, i_2, \ldots, i_k)}$ *denotes the sum taken over all ordered k-tuples* $(i_1, i_2, \ldots, i_k)$ *with* $i_s = 1$ *or* 2 *for* $s = 1, 2, \ldots, k$ *and where the maximum is taken over all decompositions of A and all decompositions of B.*

Proof. Let (2.2) and (2.3) be fixed decompositions of A and B respectively. Let

$$\gamma_0 = \rho(A)\rho(B), \tag{2.4}$$

$$\gamma_k = \rho(A)\rho(B) + \ldots + \sum_{(i_1, i_2, \ldots, i_k)} \rho(A_{i_1, i_2, \ldots, i_k})\rho(B_{j_1, j_2, \ldots, j_k}), \quad k = 1, 2, \ldots, \tag{2.5}$$

$$\gamma = \rho(A)\rho(B) + \ldots + \sum_{(i_1, i_2, \ldots, i_k)} \rho(A_{i_1, i_2, \ldots, i_k})\rho(B_{j_1, j_2, \ldots, j_k}) + \ldots. \tag{2.6}$$

We shall show that

$$\rho(A \times B) \geqslant \gamma_k \text{ for } k = 0, 1, 2, \ldots,$$

by an induction argument. In doing so we construct γ_k 1's of $A \times B$ with no two of them on the same line. Since there exists an integer $r \geqslant 0$ such that $\gamma = \gamma_r = \gamma_{r+1} = \ldots$, this will prove the theorem.

In each of the $\rho(A)$ blocks of $A \times B$ corresponding to the prescribed $\rho(A)$ 1's of A select the prescribed $\rho(B)$ 1's of B. Likewise in each of the $\rho(A_{i_1,\ldots,i_s})$ blocks of $A \times B$ corresponding to the prescribed $\rho(A_{i_1,\ldots,i_s})$ 1's of $A_{i_1,\ldots,i_s}$ select the prescribed $\rho(B_{j_1,\ldots,j_s})$ 1's of $B_{j_1,\ldots,j_s}$. Do this for all s-tuples $(i_1, \ldots, i_s)$, $s = 1, 2, \ldots, k$. This gives γ_k 1's of $A \times B$. The induction hypothesis asserts that these γ_k 1's of $A \times B$ are such that no two of them lie on the same line and thus that $\rho(A \times B) \geqslant \gamma_k$. We shall prove a similar statement for γ_{k+1}. By Lemma 2.2 and its proof the induction hypothesis is true for $k = 0$. For each $(k+1)$-tuple $(i_1, \ldots, i_{k+1})$, select in each of the $\rho(A_{i_1,\ldots,i_{k+1}})$ blocks of $A \times B$ corresponding to the prescribed $\rho(A_{i_1,\ldots,i_{k+1}})$ 1's of $A_{i_1,\ldots,i_{k+1}}$ the prescribed $\rho(B_{j_1,\ldots,j_{k+1}})$ 1's of $B_{j_1,\ldots,j_{k+1}}$. Now for $(i_1, \ldots, i_{k+1})$ fixed, these 1's clearly have no two on the same line of $A \times B$. Since by Lemma 2.3 the collection of submatrices $\{A_{i_1,\ldots,i_{k+1}}\}$ taken over all $(k+1)$-tuples is pairwise line disjoint, it follows that the collection of 1's chosen in stage $k+1$ is such that no two lie on the same line. Thus we need only verify that none of the 1's chosen in stage $k+1$ lies on the same line with any of the 1's chosen in stages $0, 1, \ldots, k$.

Consider now a fixed $A_{i_1,\ldots,i_{k+1}}$ and an arbitrary $A_{s_1,\ldots,s_r}$ with $1 \leqslant r \leqslant k$ and $s_i = 1$ or 2, $i = 1, 2, \ldots, r$. Suppose for some integer t, that $s_t \neq i_t$. We may select the smallest such t. Then $A_{i_1,\ldots,i_{k+1}}$ is a submatrix of $A_{i_1,\ldots,i_t}$ and $A_{s_1,\ldots,s_r}$ is a submatrix of $A_{s_1,\ldots,s_t}$. Since $i_t \neq s_t$, $A_{i_1,\ldots,i_t}$ and $A_{s_1,\ldots,s_t}$ are line disjoint, and therefore so are $A_{i_1,\ldots,i_{k+1}}$ and $A_{s_1,\ldots,s_r}$. Thus the 1's of $A \times B$ chosen in the blocks corresponding to the $\rho(A_{s_1,\ldots,s_r})$ 1's of $A_{s_1,\ldots,s_r}$ do not lie on the same line with any of the 1's chosen in the blocks corresponding to the $\rho(A_{i_1,\ldots,i_{k+1}})$ 1's of $A_{i_1,\ldots,i_{k+1}}$. We now have only to verify that for $(i_1, \ldots, i_{k+1})$ fixed, the 1's of $A \times B$ chosen within

$$A_{i_1,\ldots,i_{k+1}} \times B_{j_1,\ldots,j_{k+1}}$$

do not lie on the same line with any of the 1's chosen within

$$A_0 \times B_0, A_{i_1} \times B_{j_1}, \ldots, A_{i_1,\ldots,i_k} \times B_{j_1,\ldots,j_k},$$

in stages $0, 1, \ldots, k$, respectively.

Let r be a fixed integer with $1 \leqslant r \leqslant k$. Depending on whether $i_{r+1} = 1$ or 2, $A_{i_1,\ldots,i_{k+1}}$ and the set of prescribed $\rho(A_{i_1,\ldots,i_r})$ 1's of $A_{i_1,\ldots,i_r}$ are column disjoint or row disjoint. We have a similar statement for $B_{j_1,\ldots,j_{k+1}}$ and $B_{j_1,\ldots,j_r}$. However, since $i_{r+1} \neq j_{r+1}$, if $A_{i_1,\ldots,i_{k+1}}$ and the set of prescribed $\rho(A_{i_1,\ldots,i_r})$ 1's of $A_{i_1,\ldots,i_r}$ are row disjoint (column disjoint), then $B_{j_1,\ldots,j_{k+1}}$ and the set of prescribed $\rho(B_{j_1,\ldots,j_r})$ 1's of $B_{j_1,\ldots,j_r}$ are column disjoint (row disjoint). This implies that the 1's of $A \times B$ chosen within

$$A_{i_1,\ldots,i_{k+1}} \times B_{j_1,\ldots,j_{k+1}}$$

do not lie on the same line with any of the 1's chosen within

$$A_{i_1,\ldots,i_r} \times B_{j_1,\ldots,j_r}$$

for $r = 1, 2, \ldots, k$. A similar argument shows they also cannot lie on the same line with the 1's chosen within $A_0 \times B_0$. Therefore we have succeeded in establishing the induction hypothesis for $k + 1$ and have thus proved the theorem.

COROLLARY 2.5. *Let A and B be two* (0, 1)*-matrices of sizes m by m' and n by n' respectively. Then*

$$\rho(A \times B) \geqslant \rho(A)\rho(B)$$

with equality if and only if one of the following conditions is satisfied:

$$\rho(A) = m = m', \tag{2.7}$$

$$\rho(B) = n = n', \tag{2.8}$$

$$\rho(A) = m \text{ and } \rho(B) = n, \tag{2.9}$$

$$\rho(A) = m' \text{ and } \rho(B) = n'. \tag{2.10}$$

Proof. By the theorem we have

$$\rho(A \times B) \geqslant \rho(A)\rho(B) + \rho(A_1)\rho(B_2) + \rho(A_2)\,\rho(B_1).$$

If $\rho(A \times B) = \rho(A)\rho(B)$, then $\rho(A_1)\rho(B_2) = 0$ and $\rho(A_2)\rho(B_1) = 0$. This implies that one of the above four conditions must hold. The fact that equality occurs under one of these conditions follows from an easy application of König's theorem.

It had been suspected that equality would have to occur in Theorem 2.4. However, the following example shows this need not be the case. Let

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$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

It is not difficult to show that for these two matrices the maximum that can occur on the right side of the inequality in Theorem 2.4 is 11, while one can exhibit 12 (and no more) 1's of $A \times B$ with no two on the same line.

Let (2.2) and (2.3) be fixed decompositions of A and B respectively, and let γ be defined as in (2.6). In the proof of Theorem 2.4 we constructed γ 1's of $A \times B$ with no two of the 1's on the same line. The following theorem implies that if these γ 1's are not a maximal collection of 1's with no two on the same line, i.e. if $\rho(A \times B) > \gamma$, then they cannot be extended to a maximal collection by the addition of more 1's.

THEOREM 2.5. *The rows and columns of the* γ 1's *as constructed in the proof of Theorem* 2.4 *contain all the* 1's *of* $A \times B$.

Proof. We give an inductive proof. Assume the statement of the theorem is true for all matrices A with the decomposition $\{A_0; A_1, A_2; \ldots\}$ terminating at stage k and for an arbitrary B and corresponding decomposition. If the given decomposition of A terminates at the stage 0, then by Corollary 2.5

$$\rho(A \times B) = \gamma$$

and the theorem is obviously true. Thus assume the given decomposition of A terminates at stage $k + 1$. Replace each non-vacuous $A_{i_1,\ldots,i_{k+1}}$ by an appropriate matrix of all 0's. This yields a new matrix A' where the corresponding decomposition of A' terminates at stage k. Denote by γ' the corresponding γ (as defined in (2.6)) for $A' \times B$ using this decomposition of A' and the given decomposition of B. The γ' 1's of $A' \times B$ as constructed in the proof of Theorem 2.4 are a subset of the γ 1's of $A \times B$. Therefore by the induction hypothesis the rows and columns in which these γ' 1's lie contain all the 1's of $A \times B$ except possibly those within $A_{i_1,\ldots,i_{k+1}} \times B$ for each $(k + 1)$-tuple $(i_1, \ldots, i_{k+1})$. Since $A_{i_1,\ldots,i_{k+1}}$ is a submatrix of A, it follows that all but possibly the 1's of the B_{j_1} portion of each block B of $A_{i_1,\ldots,j_{k+1}} \times B$ are contained in the rows and columns of the γ' 1's. Since $A_{i_1,\ldots,i_{k+1}}$ is a submatrix of A_{i_1}, all but possibly the 1's of the B_{j_1,j_2} portion of each block B of $A_{i_1,\ldots,i_{k+1}} \times B$ are contained in the rows and columns of the γ' 1's. Continuing in this way, we see that since $A_{i_1,\ldots,i_{k+1}}$ is a submatrix of $A_{i_1,\ldots,i_k}$, all but possibly the 1's of the $B_{j_1,\ldots,j_{k+1}}$ portion of each block B of $A_{i_1,\ldots,i_{k+1}} \times B$ are included in the rows and columns of the γ' 1's. But if $A_{i_1,\ldots,i_{k+1}}$ is neither a vacuous nor a zero matrix, then upon deletion of its rows and columns that consist entirely of 0's it is square with term rank equal to its order. This is a consequence of the fact that the decomposition of A terminates at stage $k + 1$. From this it follows easily that the rows and columns of the remaining $(\gamma - \gamma')$ 1's include all the 1's of $A \times B$ hitherto unaccounted for. Therefore all the 1's of $A \times B$ are contained in the rows and columns of these γ 1's of $A \times B$.

3. An upper bound for $\rho(A \times B)$. Let A be an arbitrary (0, 1)-matrix with term rank $\rho(A)$. By a *covering* of A we mean a collection of lines of A that together contain all the 1's of A. A *minimal covering* of A is a covering using exactly $\rho(A)$ lines. By König's theorem minimal coverings of A exist, and coverings of A with less than $\rho(A)$ lines do not exist. We define an *essential row* of the matrix A to be a row that is a member of every minimal covering of A. Similarly, an *essential column* of A is a column that belongs to every minimal covering of A.

THEOREM 3.1. *Let A be a* (0, 1)*-matrix of size m by m' with term rank $\rho(A)$. Let A have a_1 essential rows and a_2 essential columns. Choose a set of $\rho(A)$ 1's of A with no two 1's on the same line. Then the rows and columns of A can be permuted to yield*

$$(3.1)\qquad a_1\left\{\begin{array}{c|c|c|c} \begin{smallmatrix}1 & & *\\ & 1 & \\ & & \ddots \\ * & & & 1\end{smallmatrix} & \overbrace{\quad * \quad}^{a_2} & * & A_1 \\ \hline 0 & \begin{smallmatrix}1 & & *\\ & 1 & \\ & & \ddots \\ * & & & 1\end{smallmatrix} & 0 & 0 \\ \hline 0 & * & \begin{smallmatrix}1 & & *\\ & 1 & \\ & & \ddots \\ * & & & 1\end{smallmatrix} & 0 \\ \hline 0 & A_2 & 0 & 0 \end{array}\right.$$

with the chosen set of $\rho(A)$ 1's of A on the main diagonal of (3.1). *The indicated a_1 rows of* (3.1) *correspond to the essential rows of A, while the indicated a_2 columns correspond to the essential columns of A. Here* 0 *denotes all* 0*'s, while* * *denotes arbitrary elements.*

Proof. We may permute the rows and columns of A so that the chosen $\rho(A)$ 1's of A occupy the first positions on the main diagonal. Call this new matrix A'. Let rows $i_1, i_2, \ldots, i_{a_1}$ be the essential rows of A' and columns $j_1, j_2, \ldots, j_{a_2}$ be the essential columns of A'. The two sets of integers $\{i_1, i_2, \ldots, i_{a_1}\}$ and $\{j_1, j_2, \ldots, j_{a_2}\}$ are subsets of the integers $\{1, 2, \ldots, \rho(A)\}$. By the definition of the term rank $\rho(A)$ of A, they can have nothing in common. Thus we may *simultaneously* permute the first $\rho(A)$ rows and columns of A' so that rows $i_1, i_2, \ldots, i_{a_1}$ become rows $1, 2, \ldots, a_1$ and columns $j_1, j_2, \ldots, j_{a_2}$ become

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columns $a_1 + 1, a_1 + 2, \ldots, a_1 + a_2$. Under these simultaneous permutations, we preserve the property of having the chosen set of $\rho(A)$ 1's on the first positions of the main diagonal. We then have

$$(3.2)\qquad \left[\begin{array}{c|c|c|c} \begin{matrix} 1 & & * \\ & \ddots & \\ * & & 1 \end{matrix} & * & * & * \\ \hline E_1 & \begin{matrix} 1 & & * \\ & \ddots & \\ * & & 1 \end{matrix} & E_4 & E_6 \\ \hline E_2 & * & \begin{matrix} 1 & & * \\ & \ddots & \\ * & & 1 \end{matrix} & E_7 \\ \hline E_3 & * & E_5 & E_8 \end{array}\right]$$

Now E_1 and E_8 must be zero matrices, by König's theorem and the definition of $\rho(A)$. E_2, E_6, and E_7 must be zero matrices, since otherwise the rows $1, 2, \ldots, a_1$ would not be the only essential rows of (3.2). Similarly, E_3, E_4, and E_5 must be zero matrices. This establishes the theorem.

The form (3.1) obtained for the matrix A is similar to the "normal form" obtained by Ore in **(2)**; however, we have said more. Theorem 3.1 shows that if we strike out the a_1 essential rows and a_2 essential columns of the matrix A, then the remaining 1's of A are contained in $a_3 \equiv \rho(A) - a_1 - a_2$ rows of A. Likewise they are contained in a_3 columns of A. We call these a_3 rows the *semi-essential rows* of A and these a_3 columns the *semi-essential columns* of A. The degenerate case $a_3 = 0$ may occur, in which case A has precisely one minimal covering, namely that using the essential rows and essential columns of A. If $a_3 \neq 0$, then we have two canonical minimal coverings of A. One consists of the essential rows and columns and the semi-essential rows. The other consists of the essential rows and columns and the semi-essential columns. Finally the preceding theorem implies that the number of minimal coverings of A does not exceed 2^{a_3}.

We now investigate the essential rows and essential columns of the direct product of two (0, 1)-matrices.

THEOREM 3.2. *Let A and B be* (0, 1)*-matrices with term ranks $\rho(A)$ and $\rho(B)$ respectively. Suppose A has a_1 essential rows and a_2 essential columns and B has b_1 essential rows and b_2 essential columns. Then the direct product $A \times B$ has at least*

$$a_1(\rho(B) - b_2) + b_1(\rho(A) - a_2) - a_1 b_1$$

essential rows and at least

$$a_2(\rho(B) - b_1) + b_2(\rho(A) - a_1) - a_2 b_2$$

essential columns.

Proof. We shall show that the direct product of an essential row of A with an essential or semi-essential row of B is an essential row of $A \times B$ and that the direct product of a semi-essential row of A with an essential row of B is an essential row of $A \times B$. We have a similar statement for an essential column of $A \times B$. This will then prove the theorem. Since permutations of the rows and columns of A and B give rise to permutations of the rows and columns of $A \times B$, we may assume that A and B are in the form (3.1) of Theorem 3.1. The a_1 1's on the main diagonal of A in the essential rows of A and the b_1 1's on the main diagonal of B in the essential rows of B give rise to $a_1 b_1$ 1's of $A \times B$ with no two of these 1's on the same line. Any minimal covering of $A \times B$ must use either the row or column of each of these $a_1 b_1$ 1's. The rows and columns of these $a_1 b_1$ 1's contain only those 1's in the direct product of the essential rows of A with the essential rows of B. But the rows of these $a_1 b_1$ 1's themselves contain all the 1's of the direct product of the essential rows of A with the essential rows of B. We shall have shown therefore that these rows are essential rows of $A \times B$ when we establish that any other collection of $a_1 b_1$ rows and columns, one through each of these $a_1 b_1$ 1's, leaves some 1 uncovered which does not lie on the same line with any 1 of the matrix obtained by deleting from $A \times B$ the direct product of the essential rows of A with the essential rows of B. Thus suppose we have such a collection Γ that contains one or more columns. Since the first a_1 rows of A are essential rows of A, then it follows that after suitable permutations of its rows and columns the block of order a_1 in the upper left corner of A may be assumed to have the form

$$\text{(3.3)} \qquad \left[\begin{array}{c|c|c|c} \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & * & * & * \\ \hline E_1 & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & * & * \\ \hline 0 & E_2 & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & * \\ \hline 0 & 0 & \begin{array}{c} E_3 \\ \hline 0 \end{array} & \begin{array}{c} \text{etc.} \\ \hline \text{etc.} \end{array} \end{array}\right],$$

where the first block of rows corresponds to the rows in which A_1 has a 1 and where $E_1, E_2, E_3, \ldots$ are non-vacuous matrices with no rows having all 0's. Consider the first 1 of the a_1 1's on the main diagonal of (3.3) such that in the corresponding block B in the direct product the collection Γ contains the column of one of the b_1 1's of B under consideration. If this 1 is in a row in which A_1 has a 1, then we clearly have a 1 within $A_1 \times B$ satisfying the desired condition. Otherwise let this 1 be in a row of A in which E_j lies. Then we have a 1 within $E_j \times B$ satisfying the desired condition. Therefore the direct product of the a_1 essential rows of A with the b_1 essential rows of B gives $a_1 b_1$ essential rows of $A \times B$. In a similar way we show that the direct product of the a_2 essential columns of A with the b_2 essential columns of B gives $a_2 b_2$ essential columns of $A \times B$. Let $b_3 = \rho(B) - b_1 - b_2$. In the direct product of the a_1 essential rows of A with the b_3 semi-essential rows of B $a_1 b_3$ 1's of $A \times B$ with no two on the same line arise naturally. Any minimal covering of $A \times B$ must use either the row or column of each of these 1's. With that portion of $A \times B$ already covered by the existing essential rows and columns excluded, the rows and columns of these $a_1 b_3$ 1's contain only those 1's in the direct product of the essential rows of A with the semi-essential rows of B. Moreover, the rows of these $a_1 b_3$ 1's contain the 1's of this direct product. Proceeding as above, we may conclude that the direct product of the a_1 essential rows of A with the b_3 semi-essential rows of B gives rise to $a_1 b_3$ essential rows of $A \times B$. Similarly, the direct product of the a_2 essential columns of A with the b_3 semi-essential columns of B gives rise to $a_2 b_3$ essential columns of $A \times B$. Now the rows and columns of $B \times A$ can be permuted to give $A \times B$ with the direct product of the b_1 essential rows of B and the a_3 semi-essential rows of A corresponding to the direct product of the a_3 semi-essential rows of A and the b_1 essential rows of B. Since we have already proved that the former gives rise to $a_3 b_1$ essential rows of $B \times A$, it follows that the latter gives rise to $a_3 b_1$ essential rows of $A \times B$. Likewise the direct product of the a_3 semi-essential columns of A with the b_2 essential columns of B gives rise to $a_3 b_2$ essential columns of $A \times B$. This completes the proof of the theorem.

We wish to clarify a point. For a given selection of $\rho(A)$ 1's of A with no two 1's on the same line, the A_1 as defined in (3.1) differs from the A_1 defined in §2 only in that some of the zero rows of the latter have been deleted and the remaining rows have been permuted amongst themselves. Thus our notation is not inconsistent, at least for our purposes. Similar remarks apply to A_2. We may choose $\rho(A_1)$ 1's of A_1 with no two on a line and $\rho(A_2)$ 1's of A_2 with no two on a line and write A_1 and A_2 in the form of (3.1). In this way we obtain the matrices A_{11}, A_{12}, A_{21}, A_{22}. We repeat our construction on these matrices and inductively define $A_{i_1,\ldots,i_k}$, where each $i_j = 1$ or 2 for $j = 1, 2, \ldots, k$. We let $a_{i_1,\ldots,i_k,1}$ be the number of essential rows of $A_{i_1,\ldots,i_k}$ and $a_{i_1,\ldots,i_k,2}$ be the number of essential columns of $A_{i_1,\ldots,i_k}$.

We now obtain an upper bound for the term rank $\rho(A \times B)$ of the direct product $A \times B$ by constructing a covering Γ of the 1's of $A \times B$. We include in Γ the essential rows and columns of $A \times B$ which were found in Theorem 3.2. If we strike these out from $A \times B$, the remaining 1's of $A \times B$ are contained in

(3.4)
$$a_1\left\{\left[\begin{array}{ccc|c}1 & & * & \\ & \ddots & & A_1\\ * & & 1 & \end{array}\right]\right. \times \overbrace{\left[\begin{array}{ccc}1 & & *\\ & \ddots & \\ * & & 1\\ \hline & B_2 & \end{array}\right]}^{b_2},$$

(3.5)
$$\overbrace{\left[\begin{array}{ccc}1 & & *\\ & \ddots & \\ * & & 1\\ \hline & A_2 & \end{array}\right]}^{a_2} \times \left.\left[\begin{array}{ccc|c}1 & & * & \\ & \ddots & & B_1\\ * & & 1 & \end{array}\right]\right\} b_1,$$

and

(3.6)
$$\overbrace{\begin{bmatrix}1 & & *\\ & \ddots & \\ * & & 1\end{bmatrix}}^{a_3} \times \overbrace{\begin{bmatrix}1 & & *\\ & \ddots & \\ * & & 1\end{bmatrix}}^{b_3}.$$

Moreover, these three matrices are pairwise line disjoint. We note that (3.6) has no essential rows or columns. Thus if $A \times B$ has any other essential rows or columns, they must come from (3.4) or (3.5). The direct product in (3.6) has term rank equal to $a_3 b_3$, and we include in Γ its $a_3 b_3$ columns, which form a (minimal) covering of this matrix. Now consider (3.4). Γ is to contain the first $a_1 b_2$ columns of (3.4). We then write A_1 in the form of (3.1). We add to Γ all the columns of the direct product of the essential and semi-essential columns of A_1 with the matrix on the right in (3.4) and also all the rows of the direct product of the essential rows of A_1 with the first b_2 rows of this matrix on the right in (3.4). We include in Γ analogous rows and columns of the direct product in (3.5). It can now be verified that the number of rows and columns in Γ thus far is

(3.7)
$$\rho(A)\rho(B) + \rho(A_1)b_2 + \rho(A_2)b_1,$$

and that we have left to cover the 1's in

(3.8)
$$a_{11}\left\{\left[\begin{array}{ccc|c}1 & & * & \\ & \ddots & & A_{11}\\ * & & 1 & \end{array}\right]\right. \times B_2$$

and

$$
(3.9)\qquad \overbrace{\left[\begin{array}{c} \begin{matrix} 1 & & * \\ & \ddots & \\ * & & 1 \end{matrix} \\ \hline A_{22} \end{array}\right]}^{a_{22}} \times B_1 .
$$

Moreover, the matrices in (3.8) and (3.9) are line disjoint. We proceed with (3.8) and (3.9) just as we did with $A \times B$. However, the situations here are simpler, since in (3.8) A is replaced by a matrix all of whose rows are essential, while in (3.9) A is replaced by a matrix all of whose columns are essential. Thus corresponding to (3.8) we add

$$
(3.10)\qquad a_{11}\,\rho(B_2) + \rho(A_{11})b_{22}
$$

more lines to Γ, and corresponding to (3.9), we add

$$
(3.11)\qquad a_{22}\,\rho(B_1) + \rho(A_{22})b_{11}
$$

more lines to Γ. We may proceed inductively and obtain a covering Γ of $A \times B$ using the following number of lines:

$$
\begin{aligned}
(3.12)\quad & \rho(A)\rho(B) + \rho(A_1)b_2 + \rho(A_2)b_1 \\
& + a_{11}\,\rho(B_2) + \rho(A_{11})b_{22} + a_{22}\,\rho(B_1) + \rho(A_{22})b_{11} + \dots \\
& + a_{\underbrace{1\dots1}_{k}}\rho(B_{\underbrace{2\dots2}_{k-1}}) + \rho(A_{\underbrace{1\dots1}_{k}})b_{\underbrace{2\dots2}_{k}} + a_{\underbrace{2\dots2}_{k}}\rho(B_{\underbrace{1\dots1}_{k-1}}) + \rho(A_{\underbrace{2\dots2}_{k}})b_{\underbrace{1\dots1}_{k}} \\
& + \dots .
\end{aligned}
$$

Hence by König's theorem, (3.12) furnishes an upper bound for $\rho(A \times B)$. Finally we remark that in order to calculate this upper bound for $\rho(A \times B)$ and the lower bound in Theorem 2.4, it is not necessary to construct the direct product $A \times B$.

Added in proof. It has come to the author's attention that A. L. Dulmage and N. S. Mendelsohn in their joint paper *Coverings of bipartite graphs* (this J., *10* [1958], 517–534) obtained the result in Theorem 3.1 in the language of graphs. They do this by first proving some theorems about minimal covers. Our proof is more direct, and Theorems 5, 6, 7 of Dulmage and Mendelsohn follow easily from our Theorem 3.1. They go on to decompose further the submatrix, which, in our terminology, is the intersection of the semi-essential rows with the semi-essential columns. However, their result is immediate from our approach.

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University of Wisconsin,
Madison, Wis.