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Abstract

An abelian cover is a finite morphism $X \to Y$ of varieties which is the quotient map for a generically faithful action of a finite abelian group G. Abelian covers with Y smooth and X normal were studied in [R. Pardini, *Abelian covers of algebraic varieties*, J. Reine Angew. Math. **417** (1991), 191–213; MR 1103912(92g:14012)]. Here we study the non-normal case, assuming that X and Y are S_2 varieties that have at worst normal crossings outside a subset of codimension greater than or equal to two. Special attention is paid to the case of $\mathbb{Z}_2^r$-covers of surfaces, which is used in [V. Alexeev and R. Pardini, *Explicit compactifications of moduli spaces of Campedelli and Burniat surfaces*, Preprint (2009), math.AG/arXiv:0901.4431] to construct explicitly compactifications of some components of the moduli space of surfaces of general type.

Contents

Introduction

An abelian cover is a finite morphism $X \to Y$ of varieties which is the quotient map for a generically faithful action of a finite abelian group G. This means that for every component Y_i of Y the G-action on the restricted cover $X \times_Y Y_i \to Y_i$ is faithful. The paper [Par91] contains a comprehensive theory of such covers in the case when Y is smooth and X is normal. The covers are described in terms of the *building data* consisting of branch divisors D_{H_i,ψ_i} ranging over cyclic subgroups $H_i \subset G$, and line bundles L_χ with χ ranging over the character group of G. This collection must satisfy the *fundamental relations*.

Here, we extend this theory to the case of singular varieties. Namely, we allow X and Y to be varieties satisfying Serre's condition S_2 and having double crossing singularities in codimension 1, which we abbreviate to gdc for 'generically double crossings' (see § 1.3 for the precise definition). Our interest in this case lies in applications to the moduli theory. Such non-normal abelian covers appear in our work [AP09] where we explicitly construct compactifications of moduli spaces of some Campedelli and Burniat surfaces by adding stable surfaces on the boundary. 'Stable surfaces' here are in the sense of [KS88]: they have slc (semi log canonical) singularities and an ample canonical class.

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In this paper, we give a comprehensive treatment of the situation. In § 1.3 we show that the theory of standard covers of [Par91] has a very natural extension to the case when Y is still smooth but X is possibly gdc. In § 1.4 we extend it to the case of normal base by an S_2-fication trick. In § 1.5 we prove that a cover with non-normal Y can be obtained by gluing a cover over the normalization $\widetilde{Y}$, and we spell out which additional data must be specified.

In § 2 we study the singularities of covers. We determine the conditions for X to have slc singularities, to be Cohen–Macaulay, and we determine the index of the canonical divisor in the situations appearing in common applications.

In § 3 we treat in detail the special case when the group G is $\mathbb{Z}_2^r$ and $\dim X = \dim Y = 2$, as in [AP09]. We restrict ourselves to the situation where the base Y is smooth or has two smooth branches meeting transversally, and the components of branch divisors and the double locus are smooth and have distinct tangent directions at the points of intersection, i.e. locally they look like a collection of lines in the plane. In this situation, we give a complete classification of the covers and the singularities of X. The answer is contained in nine tables. Some of these covers appear on the boundary of moduli of Campedelli and Burniat surfaces, but the full list is longer.

Notations. G denotes a finite abelian group. We work with equidimensional varieties defined over an algebraically closed field $\mathbb{K}$ whose characteristic does not divide the order of G. We denote by G^* the group $\mathrm{Hom}(G, \mathbb{K}^*)$ of characters of G, and we write it multiplicatively. The abbreviations *lc* and slc stand for *log canonical* and *semi log canonical* (cf. § 2 for the definitions). Also, $\widetilde{X}, \widetilde{C}$, etc. denote the normalization of X, C, etc. We use the additive and multiplicative notation for line bundles and divisors interchangeably. Linear equivalence will be denoted by $\sim$.

1. General structure of abelian covers

1.1 Setup

We recall some basic facts about Serre's condition S_2 and the S_2-fication of a coherent sheaf. For a comprehensive treatment, the reader may consult [Gro65, 5.9–11], where the latter appears under the name '$Z^{(2)}$-closure'.

All varieties below are assumed to be reduced, equidimensional, but possibly reducible. Let $\mathcal{F}$ be a coherent sheaf on X all of whose associated components are irreducible components of X. Then there exists a unique S_2-*fication*, or *saturation in codimension* 2, a coherent sheaf defined by

$$S_2(\mathcal{F})(V) = \varinjlim_{U \subset X, \operatorname{codim}(X \setminus U) \geqslant 2} \mathcal{F}(V \cap U).$$

The sheaf $S_2(\mathcal{F})$ is S_2, and $\mathcal{F}$ is S_2 if and only if the map $\mathcal{F} \to S_2(\mathcal{F})$ is an isomorphism. In particular, for $\mathcal{F} = \mathcal{O}_X$ one obtains the S_2-fication $S_2(X) \to X$, which is dominated by the normalization of X.

On a normal variety X, an S_2-sheaf is the same as a *reflexive sheaf*, satisfying $\mathcal{F}^{**} = \mathcal{F}$, see [Bou65]. Further, reflexive sheaves of rank one are the same as *divisorial sheaves*, isomorphic to $\mathcal{O}_X(D)$ for some Weil divisor D (see e.g. [Rei80, Appendix to § 1]). On a smooth (or factorial) variety Weil divisors are the same as Cartier divisors, and rank-one S_2 sheaves are the same as invertible sheaves.

Let G be a finite abelian group. An *abelian cover* with Galois group G, or G-*cover*, is a finite morphism $X \to Y$ of varieties which is the quotient map for a generically faithful action of a finite abelian group G. This means that for every component Y_i of Y the G-action on the restricted cover $X \times_Y Y_i \to Y_i$ is faithful. An *isomorphism* of G-covers $\pi_1 : X_1 \to Y$, and $\pi_2 : X_2 \to Y$ is an isomorphism $\phi : X_1 \to X_2$ such that $\pi_1 = \pi_2 \circ \phi$.

The G-action on X with $X/G = Y$ is equivalent to a decomposition:

$$\pi_* \mathcal{O}_X = \bigoplus_{\chi \in G^*} \mathcal{F}_\chi, \quad \mathcal{F}_1 = \mathcal{O}_Y \tag{1}$$

where G acts on $\mathcal{F}_\chi$ via the character χ. If π is Galois then each $\mathcal{F}_\chi$ has rank one: if $y \in Y$ is a general closed point, then G acts freely on $\pi^{-1}(y)$, so it acts on $\mathcal{O}_{\pi^{-1}(y)} = \bigoplus_\chi (\mathcal{F}_\chi \otimes \mathbb{K}(y))$ as the regular representation. Thus, $\mathcal{F}_\chi \otimes \mathbb{K}(y)$ is one-dimensional for every χ. When the sheaves $\mathcal{F}_\chi$ are locally free, it is customary to write $\mathcal{F}_\chi = L_\chi^{-1}$, with L_χ a line bundle.

LEMMA 1.1. (i) *The sheaf $\mathcal{O}_X$ is S_n for some n if and only if every $\mathcal{F}_\chi$ is S_n.*

(ii) *If $\pi : X \to Y$ is flat then X is CM (Cohen–Macaulay) if and only if Y is CM.*

(iii) *If Y is smooth and X is S_2 then π is flat and X is CM.*

Proof. (i) Part (i) is clear from the definition of depth.

(ii) The morphism π is flat if and only if every $\mathcal{O}_Y$-module $\mathcal{F}_\chi$ is invertible. Then each $\mathcal{F}_\chi$ is CM if and only if $\mathcal{O}_Y$ is CM.

(iii) On a smooth variety every divisorial sheaf is invertible, and so flat. Now part (ii) applies. □

A G-cover $\pi : X \to Y$, where X and Y are S_2 varieties, is determined by its restriction to the complement of a closed subset of codimension greater than or equal to two.

LEMMA 1.2. *Let Y be an S_2 variety, $Y_0 \subseteq Y$ an open subset with* $\operatorname{codim}(Y \setminus Y_0) \geqslant 2$, *and $\pi_0 : X_0 \to Y_0$ a G-cover with X_0 an S_2 variety. Then there exist a unique S_2 variety X and a G-cover $\pi : X \to Y$ whose restriction to Y_0 is π_0.*

Proof. For the existence, we take $\mathcal{O}_X := i_* \mathcal{O}_{X_0}$, where $i : Y_0 \to Y$ is the inclusion. Then $\mathcal{O}_X = \bigoplus_{\chi \in G^*} \mathcal{F}_\chi$, where each $\mathcal{F}_\chi$ is a rank-one S_2-sheaf. The algebra structure on $\mathcal{O}_X$ is defined as follows. For an open set $U \subset X$ and sections $s \in \mathcal{F}_\chi(U)$, $s' \in \mathcal{F}_{\chi'}(U)$, their product is

$$s|_{U \cap X_0} \cdot s'|_{U \cap X_0} \in \mathcal{F}_{\chi\chi'}(U \cap X_0) = \mathcal{F}_{\chi\chi'}(U),$$

since $\operatorname{codim}_U(U \setminus U \cap X_0) \geqslant 2$ and $\mathcal{F}_\chi$ is saturated in codimension 2. Thus, $X := \operatorname{Spec}_{\mathcal{O}_Y} \mathcal{O}_X$ is an S_2 variety with a finite morphism to Y. The G^*-grading on $\mathcal{O}_X$ defines the G-action on X. By construction, the eigenspace $\mathcal{F}_1$ for the trivial character is $i_* \mathcal{O}_{Y_0} = \mathcal{O}_Y$. Therefore, $X/G = Y$.

Uniqueness follows from the uniqueness of the S_2-fication. □

Given a G-cover $\pi : X \to Y$ and an irreducible subset $S \subset Y$, we define the *inertia subgroup* H_S of S to be the subgroup of G consisting of the elements that fix $\pi^{-1}(S)$ pointwise, or, equivalently since G is abelian, that fix an irreducible component of $\pi^{-1}(S)$ pointwise. The *branch locus* D_π of π is the set of points of Y whose inertia subgroup is not trivial (notice that we regard D_π simply as a set, without giving it a scheme structure). If π is flat, then D_π is a divisor by [AK70, Theorem 6.8]. If F is an irreducible divisor of Y such that X is generically smooth along $\pi^{-1}(F)$, then the natural representation ψ of H_F on the tangent space $T_{X,R}$ at the generic point of an irreducible component R of $\pi^{-1}(F)$ is faithful, and hence H_F is cyclic (cf. [Par91, § 1]). Notice that ψ does not depend on the choice of the component R of $\pi^{-1}(F)$, since G is abelian.

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1.2 Standard covers

In this section we recall, in a form which is convenient for our later applications, the definition of standard abelian covers, a class of flat abelian covers that can be constructed from a collection of line bundles and effective divisors on the target variety (cf. [Par91, FP97]). The prototypical example is the classical construction of a double cover of a variety Y from the data of an effective divisor D on Y and a line bundle L such that $2L \sim D$.

Let Y be a variety. A set of *building data for a standard G-cover $\pi : X \to Y$* consists of the following:

- irreducible effective Cartier divisors $D_1, \ldots, D_k$ (possibly not distinct);
- for each D_i a pair (H_i, ψ_i), where H_i is a cyclic subgroup of G of order m_i and ψ_i is a generator of the group of characters H_i^*;
- line bundles L_χ, for $\chi \in G^* \setminus \{1\}$.

Moreover we assume that these data satisfy the so called *fundamental relations*:

$$\forall \chi, \chi', \quad L_\chi + L_{\chi'} \sim L_{\chi\chi'} + \sum_i \varepsilon^i_{\chi,\chi'} D_i, \tag{2}$$

where for a character χ we write $\chi|_{H_i} = \psi_i^{a^i_\chi}$, with $0 \leqslant a^i_\chi < m_i$, and we define $\varepsilon^i_{\chi,\chi'} := [(a^i_\chi + a^i_{\chi'})/m_i]$. Observe that $\varepsilon^i_{\chi,\chi'}$ is equal either to 0 or to 1.

We call the divisors D_i, together with the pairs (H_i, ψ_i), the *branch data* of the cover. An equivalent way of describing the branch data, and therefore the building data, is to give for each pair (H, ψ), with $H \subset G$ a cyclic subgroup and $\psi \in H^*$ a generator, the divisor $D_{H,\psi} = \sum_{\{i|(H_i,\psi_i)=(H,\psi)\}} D_i$. This is the notation used in [Par91].

Remark 1.3. If the group $\mathrm{Pic}(Y)$ has no m-torsion, where $m = |G|$, then the branch data determine the building data by [Par91, Proposition 2.1]. In general, the branch data are enough to determine the local geometry of the cover (cf. Proposition 1.6, (ii)).

Remark 1.4. When $G = \mathbb{Z}_2^r$, it is enough to associate with every divisor D_i a non-zero element $g_i \in G$, the generator of H_i. Also, the definition of $\varepsilon^i_{\chi,\chi'}$ is simpler: $\varepsilon^i_{\chi,\chi'}$ is equal to 1 if $\chi(g_i) = \chi'(g_i) = -1$ and it is equal to 0 otherwise.

We now explain how to construct a G-cover from a set of building data. Choose $\chi_1, \ldots, \chi_s \in G^*$ such that G^* is the direct sum of the cyclic subgroups generated by the χ_j. Denote by d_j the order of χ_j and write $L_j := L_{\chi_j}$ and $a^i_j := a^i_{\chi_j}$. By [Par91, Proposition 2.1] for $j = 1, \ldots, s$ there exist isomorphisms:

$$\varphi_j : L_j^{\otimes d_j} \xrightarrow{\sim} \mathcal{O}_Y\left(\sum_i \frac{d_j a^i_j}{m_i} D_i\right).$$

Notice that the coefficients $(d_j a^i_j)/m_i$ in the above formula are integers. Using formulae (2.15) of [Par91] and the isomorphisms φ_j above, one constructs for each pair χ, χ' of non-trivial characters an isomorphism

$$\varphi_{\chi,\chi'} : L_\chi^{-1} \otimes L_{\chi'}^{-1} \xrightarrow{\sim} L_{\chi\chi'}^{-1}\left(-\sum \varepsilon^i_{\chi,\chi'} D_i\right)$$

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such that for every $\chi, \chi', \chi'' \in G^*$ the following diagram commutes (we set $L_1 = \mathcal{O}_Y$):

$$\begin{tikzcd} L_\chi^{-1} \otimes L_{\chi'}^{-1} \otimes L_{\chi''}^{-1} \arrow[r] \arrow[d] & L_{\chi\chi'}^{-1}(-\sum_i \varepsilon^i_{\chi,\chi'} D_i) \otimes L_{\chi''}^{-1} \arrow[d] \\ L_\chi^{-1} \otimes L_{\chi'\chi''}^{-1}(-\sum_i \varepsilon^i_{\chi',\chi''} D_i) \arrow[r] & L_{\chi\chi'\chi''}^{-1}(-\sum_i \delta^i_{\chi,\chi',\chi''} D_i) \end{tikzcd} \tag{3}$$

where $\delta^i_{\chi,\chi',\chi''} = \varepsilon^i_{\chi\chi',\chi''} + \varepsilon^i_{\chi,\chi'} = \varepsilon^i_{\chi,\chi'\chi''} + \varepsilon^i_{\chi',\chi''}$ and the maps are induced by the $\varphi_{\chi,\chi'}$ in the obvious way. We denote by $\mu_{\chi,\chi'} : L_\chi^{-1} \otimes L_{\chi'}^{-1} \to L_{\chi\chi'}^{-1}$ the maps induced by composing $\varphi_{\chi,\chi'}$ with the inclusion $L^{-1}_{\chi\chi'}(-\sum \varepsilon^i_{\chi,\chi'} D_i) \hookrightarrow L^{-1}_{\chi\chi'}$. By the commutativity of (3), the collection of maps $\mu_{\chi,\chi'}$ defines on $\mathcal{E} := \mathcal{O}_Y \oplus \bigoplus_{\chi\neq 1} L_\chi^{-1}$ a commutative and associative algebra structure compatible with the G-action defined by letting G act trivially on $L_1 = \mathcal{O}_Y$ and via the character χ on L_χ^{-1} for $\chi \neq 1$. We define $X := \operatorname{Spec} \mathcal{E}$ with the natural map $\pi : X \to Y$ to be a *standard G-cover* associated with the given set of building data. Notice that, since the L_χ^{-1} are locally free, π is flat and X is S_2 if Y is.

X can be described locally above a point $y \in Y$ as follows. Up to shrinking Y, we may assume that all the L_χ are trivial and that the D_i are defined by equations σ_i. If we denote by z_χ a coordinate on L_χ^{-1}, $\chi \in G^* \backslash \{1\}$, then X is given inside the vector bundle $V(\bigoplus_{\chi\neq 1} L_\chi^{-1}) \cong Y \times \mathbb{K}^{m-1}$ by the following set of equations:

$$z_\chi z_{\chi'} = c_{\chi,\chi'} \Pi_1^k \sigma_i^{\varepsilon^i_{\chi,\chi'}} z_{\chi\chi'}, \quad \chi, \chi' \in G^* \backslash \{1\}, \tag{4}$$

where the $c_{\chi,\chi'}$ are nowhere vanishing regular functions and for $\chi = 1$ we set $z_\chi = 1$. For $1 \neq \chi \in G^*$, denote by d the order of χ and write $\chi|_{H_i} = \psi_i^{a_i}$, with $0 \leqslant a_i < m_i := |H_i|$. Eliminating between the equations in (4), one gets

$$z_\chi^d = b_\chi \Pi_1^k \sigma_i^{(da_i/m_i)}, \tag{5}$$

where b_χ is a nowhere-vanishing function. It follows immediately that X is a variety: indeed, using the decomposition of $\pi_*\mathcal{O}_X$ into G-eigenspaces, we may assume that a nilpotent element is locally of the form $f z_\chi$ for some character χ and some regular function f. Then by (5), $(f z_\chi)^k = 0$ for some k only if $f = 0$. Using the local equations in (4), one can also show the following lemma.

Lemma 1.5. *Use the notation as above. Let $\pi : X \to Y$ be a standard G-cover and $y \in Y$ be a point. Then the inertia subgroup H_y of y is equal to $\sum_{\{i | y \in D_i\}} H_i$.*

Proof. Since the question is local on Y, we may assume that X is given by the equations in (4). Let $x \in X$ be a point lying above y. Then by (5) the coordinate $z_\chi(x)$ does not vanish if and only if $\chi|_{H_i} = 1$ for every i such that $y \in D_i$. Since an element $g \in G$ fixes x if and only if for every $\chi \in G^*$ such that $\chi(g) \neq 1$ the coordinate $z_\chi(x)$ vanishes, this remark proves the claim. □

Given a set of building data, the construction of the standard G-cover $\pi : X \to Y$ depends of course on the choice of the characters $\chi_1, \dots \chi_s$ and of the isomorphisms φ_j. Assume that $\chi'_1, \dots \chi'_t$ are another set of characters of G such that G^* is the direct sum of the cyclic subgroups generated by the χ'_l. Let d'_l be the order of χ'_l, $i = 1, \dots, t$; then by (5) the multiplication maps *induce* for $l = 1, \dots, t$ isomorphisms $\varphi'_l : L_{\chi'_l}^{\otimes d'_l} \xrightarrow{\sim} \mathcal{O}_Y(\sum_i ((k_l b^i_l)/m_i) D_i)$, where $0 \leqslant b^i_l < m_i$ and $\chi'_l|_{H_i} = \psi_i^{b^i_l}$. By the associativity and commutativity of the multiplication, the algebra structure defined on $\mathcal{O}_Y \oplus \bigoplus_{\chi\neq 1} L_\chi^{-1}$ by the φ'_l is the same as that induced by the φ_j. Hence it is enough to analyze to what extent the isomorphism class of π depends on the φ_j.

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Proposition 1.6. (i) (Global case.) *If $H^0(\mathcal{O}_Y^*) = \mathbb{K}^*$, then the building data determine $\pi : X \to Y$ up to isomorphism of G-covers.*

(ii) *In general, given two standard covers $\pi_i : X_i \to Y$, $i = 1, 2$, with the same building data, there exists an étale cover $Y' \to Y$ such that, after base change with $Y' \to Y$, π_1 and π_2 give isomorphic G-covers.*

Proof. (ii) We use the notation introduced above. Let $\mathcal{E}$, $\mathcal{E}'$ be two $\mathcal{O}_Y$-algebra structures on $\mathcal{O}_Y \oplus \bigoplus_{\chi\neq 1} L_\chi^{-1}$ given by isomorphisms φ_j, respectively φ'_j. The isomorphisms φ_j, φ'_j differ by an automorphism of $L_j^{\otimes d_j}$, namely by multiplication by an element $k_j \in H^0(\mathcal{O}_Y^*)$. This automorphism is induced by an automorphism of L_j if and only if k_j has a d_jth root $h_j \in H^0(\mathcal{O}_Y^*)$. So, up to taking an étale cover, one can assume that the roots h_j exist. By [Par91, (2.15)], the h_j can be used to define, for all $\chi \in G^*\setminus\{1\}$, automorphisms ψ_χ of L_χ^{-1} that commute with the isomorphisms $\varphi_{\chi,\chi'}$ and $\varphi'_{\chi,\chi'}$.

To prove statement (i), just observe that if $H^0(\mathcal{O}_Y^*) = \mathbb{K}^*$ no base change is necessary to construct the isomorphism above. □

Remark 1.7. Let $\pi : X \to Y$ be a G-cover with branch data D_i, (G_i, ψ_i), let $y \in Y$, and let σ_i be local equations for D_i near y. Combining Proposition 1.6 with the local equations in (4), we see that, up to passing to an étale cover of (Y, y), X is defined locally near y by the equations

$$z_\chi z_{\chi'} = \prod_{i=1}^{k} \sigma_i^{\varepsilon^i_{\chi,\chi'}} z_{\chi\chi'}, \quad \chi, \chi' \in G^*\setminus\{1\}. \tag{6}$$

1.3 Covers of smooth varieties

Here we find conditions for a G-cover of a smooth variety to be standard. We keep the notation of the previous section.

Definition 1.8. Let Y be a smooth variety and let $\pi : X \to Y$ be a standard G-cover with building data L_χ, D_i, (H_i, ψ_i). By Lemma 1.5, the branch locus D_π of π is the support of the divisor $\sum_i D_i$.

We define the *Hurwitz divisor* of π as the $\mathbb{Q}$-divisor $D := \sum_i((m_i - 1)/m_i)D_i$. Notice that the support of D is equal to D_π.

We say that a variety is *dc* (has *double crossings*) if every point is either smooth or analytically isomorphic to $xy = 0$. We say that a variety is *gdc* (has *generically double crossing*s) if it is dc outside a closed subset of codimension greater than or equal to two.

The following result generalizes the main result of [Par91].

Theorem 1.9. *Let $\pi : X \to Y$ be a G-cover such that Y is smooth and X is S_2. Then the following hold.*

(i) *The variety X is normal if and only if π is standard and every component of the Hurwitz divisor D has multiplicity less than one.*

(ii) *Assume that π is standard. Then X is gdc if and only if every component of D has multiplicity less than or equal to one.*

(iii) *Assume that X is gdc. Then π is standard if and only if for every irreducible divisor F of Y such that X is singular above F one has $H_F = \mathbb{Z}_2^s$ for some s.*

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In the case $G=\mathbb{Z}_2^r$, which is of special interest to us because of the applications in [AP09], Theorem 1.9 reads as follows.

Corollary 1.10. *Let $\pi : X \to Y$ be a $\mathbb{Z}_2^r$-cover such that Y is smooth and X is S_2. Then the following hold.*

(i) *The variety X is normal if and only if π is standard and every component of D has multiplicity less than one.*

(ii) *The variety X is gdc if and only if π is standard and every component of D has multiplicity less than or equal to one.*

Remark 1.11. Let $\pi : X \to Y$ be a standard G-cover with Y smooth and X gdc and let F be a component of the branch divisor D_π. By Lemma 1.5, we have $H_F = \sum_{\{i|D_i=F\}} H_i$. The pairs (subgroup, character) corresponding to F can be determined as follows.

– Assume that F has multiplicity less than one in the Hurwitz divisor D. Then there is precisely one index i with $D_i = F$. In this case, $H_i = H_F$ and the character ψ_i is given by the action of H_i on the tangent space to X at the generic point of an irreducible component of $\pi^{-1}(F)$ (cf. [Par91], §§ 1 and 2).

– Assume that F has multiplicity equal to one in D. Then there are precisely two indices i_1 and i_2 such that $D_{i_1} = D_{i_2} = F$ and H_{i_1} and H_{i_2} have order two. Hence, either $H_F = H_{i_1} = H_{i_2}$ or $H_F = H_{i_1} \oplus H_{i_2}$. In the latter case the proof of Theorem 1.9 shows that H_{i_1} and H_{i_2} are generated by the elements of H_F that interchange the two branches of X at a general point of $\pi^{-1}(F)$.

Proof of Theorem 1.9 Statement (i) is [Par91, Theorem 2.1 and Corollary 3.1].

Therefore, consider the non-normal case. The cover π is flat since Y is smooth and X is S_2, and hence we write, as usual, $\pi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \bigoplus_{\chi\neq 1} L_\chi^{-1}$. The cover is standard if and only if there exist branch data D_i, (H_i, ψ_i) such that for every $\chi, \chi' \in G^*\backslash\{1\}$ the zero divisor of the multiplication map $\mu_{\chi,\chi'} : L_\chi^{-1} \otimes L_{\chi'}^{-1} \to L_{\chi\chi'}^{-1}$ is equal to $\sum_i \varepsilon^i_{\chi,\chi'} D_i$, where the $\varepsilon^i_{\chi,\chi'}$ are defined in § 1.2.

Notice that X, being S_2, is non-normal if and only if it is singular in codimension 1. Fix a component F of D such that X is singular above F. Write $H := H_F$. The cover π factors as $X \to X/H \to Y$, and F is not contained in the branch locus of the map $X/H \to Y$; hence X/H is generically smooth over F. It follows that there is an element of H that exchanges the two branches of X at a general point of $\pi^{-1}(F)$.

Let $\widetilde{X} \to X$ be the normalization, let $\pi^\nu : \widetilde{X} \to Y$ be the induced G-cover, let (H', ψ') be the pair (subgroup, character) corresponding to F for the cover π^ν, and let m' be the order of H' (if π^ν is not branched on F, we take H' and ψ' to be trivial). Since the normalization map $\widetilde{X} \to X$ is G-equivariant, we have a short exact sequence:

$$0 \to H' \to H \to \mathbb{Z}_2 \to 0. \tag{7}$$

We consider the H-covers $p : X \to Z := X/H$ and $p^\nu : \widetilde{X} \to \widetilde{X}/H = Z$, and we study the algebras $\mathcal{A} := p_*\mathcal{O}_{X,F'}$ and $\mathcal{A}^\nu := p^\nu_*\mathcal{O}_{\widetilde{X},F'}$, where F' is an irreducible component of the inverse image of F in Z. We denote by $t \in \mathcal{O}_{Z,F'}$ a local parameter.

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We distinguish three cases.

Case (a): $|H| = 2$.

In this case $H' = \{0\}$, and X is given locally by $z^2 = at^2$, where $a \in \mathcal{O}^*_{Z,F'}$.

Case (b): *H is cyclic of order $2m' \geqslant 4$.*

Let $\psi \in H^*$ be a generator that restricts to ψ' on H'. The algebra $\mathcal{A}^\nu$ is generated by elements z, w such that

$$z^{m'} = atw, \quad w^2 = b \tag{8}$$

where $a, b \in \mathcal{O}^*_{Z,F'}$ and H acts on z via the character ψ and on w via the character $\psi^{m'}$. The eigenspace corresponding to ψ^j is generated by $z_j := z^j$ for $0 \leqslant j < m'$, and by $z_j := wz^{j-m'}$ for $m' \leqslant j < 2m'$. Since the inclusion $\mathcal{A} \subset \mathcal{A}^\nu$ is G-equivariant, $\mathcal{A}$ is generated by elements of the form $t^{a_j} z_j$ for suitable $a_j \geqslant 0$.

Since H fixes $p^{-1}(F')$ pointwise, by the argument in the proof of Lemma 1.5 $\mathcal{A}$ is contained in the subalgebra $\mathcal{B}$ of $\mathcal{A}^\nu$ generated by

$$1, z^{m'} = tw, z_j, \quad 1 \leqslant j \leqslant 2m' - 1, \quad j \neq m'.$$

The algebra $\mathcal{B}$ is also generated by $z_1 = z$, $z_{m'+1} = wz$, with the only relation $bz_1^2 = z_{m'+1}^2$; hence Spec $\mathcal{B}$ is gdc and the map Spec $\mathcal{B} \to$ Spec $\mathcal{A}$ is an isomorphism. So $\mathcal{A} = \mathcal{B}$.

Case (c): *H is not cyclic.*

In this case m' is even and $H \cong H' \times \mathbb{Z}_2$. We denote by $\psi \in H^*$ a character that restricts to ψ' on H' and by ϕ the character such that $H' = \ker \phi$. $\mathcal{A}^\nu$ is generated by z, w such that

$$z^{m'} = at, \quad w^2 = b, \tag{9}$$

where $a, b \in \mathcal{O}^*_{Z,F'}$ and H acts on z via the character ψ and on w via the character ϕ. Arguing as in the previous case, one checks that $\mathcal{A}$ is generated by

$$1, z_1 := z, \ldots, z^{m'-1}, tw, z_{m'+1} := zw, \ldots, z^{m'-1}w.$$

The algebra $\mathcal{A}$ can also be generated by $z_1, z_{m'+1}$ with the only relation $bz_1^2 = z_{m'+1}^2$.

For $\chi_1, \chi_2 \in G^* \backslash \{1\}$, denote by $\varepsilon_{\chi_1,\chi_2}$ the order of vanishing on F of the multiplication map $\mu_{\chi_1,\chi_2} : L_{\chi_1}^{-1} \otimes L_{\chi_2}^{-1} \to L_{\chi_1\chi_2}^{-1}$. Using the above analysis and arguing as in the proof of [Par91, Theorem 2.1], one obtains the following rules, up to exchanging χ_1 and χ_2.

Case (a). In this case we have:
$\varepsilon_{\chi_1,\chi_2} = 2$ if $\chi_1, \chi_2 \notin H^\perp$; and
$\varepsilon_{\chi_1,\chi_2} = 0$ otherwise.

Case (b). For $i = 1, 2$, write $\chi_i|_H = \psi^{\alpha_i m' + \beta_i}$, where $\alpha_i = 0$ or 1 and $0 \leqslant \beta_i < m'$. Then we have:
$\varepsilon_{\chi_1,\chi_2} = 2$ if $\alpha_1 = \alpha_2 = 1, \beta_1 = \beta_2 = 0$;
$\varepsilon_{\chi_1,\chi_2} = 1$ if $\alpha_1 = 1, \beta_1 = 0, \beta_2 > 0$; and
$\varepsilon_{\chi_1,\chi_2} = [(\beta_1 + \beta_2 - 1)/m']$ in the remaining cases.

Case (c). For $i = 1, 2$, write $\chi_i|_H = \phi^{\alpha_i}\psi^{\beta_i}$,where $\alpha_i = 0$ or 1 and $0 \leqslant \beta_i < m'$. Then we have:
$\varepsilon_{\chi_1,\chi_2} = 2$ if $\alpha_1 = \alpha_2 = 1, \beta_1 = \beta_2 = 0$;
$\varepsilon_{\chi_1,\chi_2} = 1$ if $\alpha_1 = 1, \beta_1 = 0, \beta_2 > 0$; and
$\varepsilon_{\chi_1,\chi_2} = [(\beta_1 + \beta_2)/m']$ in the remaining cases.

In the above analysis the group $\mathbb{Z}_2^s$ appears in case (a) and case (c) for $m' = 2$. In case (a), the cover π is standard: F appears twice among the branch data, both times with label H.

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In case (c), π is standard for $m' = 2$: F appears twice among the branch data, with labels H_1 and H_2 corresponding to the subgroups of order two of H distinct from H'. Moreover, it is not difficult to check that in case (b) and in case (c) for $m' \neq 2$ the cover is not standard. So we have proven (iii) and also that every component of the Hurwitz divisor D of a standard gdc cover has multiplicity less than or equal to one.

Vice versa, assume that π is standard and F appears in D with multiplicity less than or equal to one. If the multiplicity is less than one then the cover is normal over F. If the multiplicity is equal to 1, then F appears twice among the branch data, and the corresponding subgroups H_1 and H_2 have order two. If $H_1 = H_2$, then the cover is given over the generic point of F by the equation $z^2 = ut^2$, with u a unit, so it is gdc. If $H_1 \neq H_2$, then the cover is given by the equations $z_1^2 = at$, $z_2^2 = bt$, with a and b units. These equations are equivalent to $az_2^2 = bz_1^2$, so the cover is gdc. This completes the proof of (ii). $\square$

1.4 Covers of normal varieties

Let $\pi : X \to Y$ be a G-cover such that Y is normal and X is S_2. Let Y_0 be the non-singular locus of Y. Then the restriction $\pi_0 : X_0 \to Y_0$ is a G-cover, and by Lemma 1.2 π is the unique S_2-extension of π_0 to Y. Thus the theory in the normal case is the immediate extension of the non-singular case. We record the following changes.

(i) The sheaves $\mathcal{F}_\chi$ are no longer invertible but they are S_2, i.e. in this case reflexive, divisorial sheaves. The multiplication maps are

$$\mathcal{F}_\chi \times \mathcal{F}_{\chi'} \to \mathcal{F}_\chi \otimes \mathcal{F}_{\chi'} \to (\mathcal{F}_\chi \otimes \mathcal{F}_{\chi'})^{**} \to \mathcal{F}_{\chi\chi'}.$$

(ii) The branch divisors D_g are Weil divisors.

Otherwise, the same fundamental relations between $\mathcal{F}_\chi$ and D_g must hold.

One has to be careful that the morphism π may be not flat; indeed, it is flat if and only if all $\mathcal{F}_\chi$ are invertible. Also, for a singular Y the branch locus may have non-divisorial components.

Example 1.12. Let $X = \mathbb{A}^2 = \operatorname{Spec} k[x, y]$, $G = \mathbb{Z}_2$ acting by $x \mapsto -x$, $y \mapsto -y$, and let Y be the quotient $\operatorname{Spec} k[x^2, xy, y^2]$, a quadratic cone. Then π is ramified only over the vertex P of the cone. The divisors D_g are zero. The eigensheaves are $\mathcal{F}_1 = \mathcal{O}_Y$ and $\mathcal{F}_{-1}$, and the divisorial sheaf corresponding to a line ℓ through the vertex. $\mathcal{F}_{-1}$ is also isomorphic to the $\mathcal{O}_Y$-submodule of $\mathcal{O}_X$ generated by x and y.

The fundamental relation in this case is $2\mathcal{F}_{-1} = 0$.

1.5 Covers of non-normal varieties

Now we assume that Y is a non-normal gdc and S_2 variety. Let C be the divisorial part of the singular locus of Y, let $\nu : \widetilde{Y} \to Y$ be the normalization, let C' be the inverse image of C in $\widetilde{Y}$, and let $\widetilde{C'} \to C'$ be the normalization. Since Y is gdc, there is a biregular involution ι on $\widetilde{C'}$ induced by the degree two map $\widetilde{C'} \to C' \to C$. (If the components of Y are smooth, then $\widetilde{C'}$ is a union of several pairs of varieties, exchanged by the involution ι. In general, some components of $\widetilde{C}$ map to themselves.) Consider a commutative diagram:

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ \widetilde{Y} & \longrightarrow & Y \end{array}$$

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where X and X' are gdc and S_2 varieties, the vertical arrows are G-covers, $X' \to \widetilde{Y}$ is a cover as in the previous section, and $X' \to X$ is a birational morphism.

We denote by B, B' the preimages of C, C' in X, X', and by $\widetilde{B'}$ the normalization of B'.

$$\begin{tikzcd}
\widetilde{B'} \arrow[loop left, "j"] \arrow[r] \arrow[dd] & B' \arrow[rr] \arrow[dr, hook] \arrow[dd] & & B \arrow[dr, hook] \arrow[dd] & \\
& & X' \arrow[rr] \arrow[dd] & & X \arrow[dd] \\
\widetilde{C'} \arrow[loop left, "\iota"] \arrow[r] & C' \arrow[rr] \arrow[dr, hook] & & C \arrow[dr, hook] & \\
& & \widetilde{Y} \arrow[rr, "\nu"] & & Y
\end{tikzcd} \tag{10}$$

We first give two constructions for the cover $X \to Y$ starting with $X' \to \widetilde{Y}$ and the appropriate data for the double locus. One construction proceeds by S_2-fication of the 'nice' part. The second one is by a gluing procedure, and the result is very convenient for computing the invariants of X. Finally, we show that indeed every $X \to Y$ comes from these constructions.

Theorem 1.13. *Suppose we are given:*

(i) Y, $\widetilde{Y}$, C', $(\widetilde{C'}, \iota)$;

(ii) *a G-cover $X' \to \widetilde{Y}$, with X' an S_2 and gdc variety.*

Let $B' \to C'$ be the induced cover and let $\widetilde{B'} \to B'$ be its normalization.

Then X' can be glued to a cover $X \to Y$ with X gdc and S_2 if and only if it is generically smooth along B', and there exists an involution $j : \widetilde{B'} \to \widetilde{B'}$ that covers the involution $\iota : \widetilde{C'} \to \widetilde{C'}$ and commutes with the action of G on $\widetilde{B'}$.

Proof by S_2-fication. Assume that X exists. Then the map $\widetilde{B'} \to X$ induces an involution j as required. In addition, if X' were not generically smooth along a component F of B', then X would have generically at least three branches along the image of F. Thus these two conditions on X' are necessary for the existence of X.

Next we show that they are also sufficient. We start by identifying the 'bad locus'. It includes the singular locus of $\widetilde{Y}$, the intersection of branch divisors between themselves and with C'. The image of this bad locus in Y has codimension greater than or equal to two. Let Y_0 be its complement, and restrict all varieties and covers to Y_0.

The condition that the involution j commutes with the G-action implies that for any irreducible component F of B' the subgroup H of elements of G that fix F pointwise is the same as the subgroup of elements that fix jF pointwise. Since X' is generically smooth along B', one has (cf. [Par91, § 1]) $H = \mathbb{Z}_n$ for some n and, working étale locally, H acts locally by $(x, x_2, \dots, x_n) \mapsto (\xi x, x_2, \dots, x_n)$ near F and by $(y, y_2, \dots, y_n) \mapsto (\xi^a y, y_2, \dots, y_n)$ near jF for some primitive root $\xi^n = 1$ and $(a, n) = 1$. Here $y_i = j^* x_i$, $i = 2, \dots, n$.

We glue X'_0 along $B_0 := \widetilde{B'}_0 / j = B'_0 / \iota$ to obtain a variety X_0 with a finite morphism to Y_0. The G-action extends to X_0, because j commutes with the G-action, and is of the type (smooth) $\times$ (compatible action of curves), where 'compatible' means that, working étale locally, $\mathbb{Z}_n$ acts on $xy = 0$ by $x \mapsto \xi x$, $y \mapsto \xi^a y$.

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Over the double locus we have $\mathbb{K}[x,y]/(xy)$ and the ring of $\mathbb{Z}_n$-invariants is $\mathbb{K}[u,v]/(uv)$, where $u=x^n$ and $v=y^n$. Thus, X_0 has only normal crossings and $X_0\to Y_0$ is a G-cover.

Finally, we apply Lemma 1.2 to obtain an S_2 and gdc cover $X\to Y$ by taking S_2-fication. $\square$

Proof by explicit gluing. We obtain X by gluing X' along the involution $j:\widetilde{B'}\to\widetilde{B'}$, i.e. as the pushout of the following commutative diagram.

$$\begin{tikzcd} \widetilde{B'} \arrow[r] \arrow[d] & \widetilde{B'}/j \\ X' & \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathcal{O}_{\widetilde{B'}} & \mathcal{O}_{\widetilde{B'}/j} \arrow[l] \\ \mathcal{O}_{X'} \arrow[u] & \end{tikzcd}$$

Since all varieties are affine over Y, $\mathcal{O}_X$ is the fiber product of the corresponding diagram of $\mathcal{O}_Y$-algebras, in which we identify sheaves with their pushforwards on Y. We can rewrite this fiber product diagram by saying that $\mathcal{O}_X$ is the kernel in the exact sequence

$$0\to\mathcal{O}_X\to\mathcal{O}_{X'}\oplus\mathcal{O}_{\widetilde{B'}/j}\xrightarrow{\beta}\mathcal{O}_{\widetilde{B'}}.$$

Further, we have

$$0\to\mathcal{O}_{\widetilde{B'}/j}\to\mathcal{O}_{\widetilde{B'}}\to\mathcal{A}\to 0,$$

where $\mathcal{A}$ is the alternating part (if char $\mathbb{K}\neq 2$ then $\mathcal{O}_{\widetilde{B'}}=\mathcal{O}_{\widetilde{B'}/j}\oplus\mathcal{A}$), and the image of β contains $\mathcal{O}_{\widetilde{B'}/j}$. Hence, we have induced exact sequences

$$0\to\mathcal{O}_X\to\mathcal{O}_{X'}\xrightarrow{\alpha}\mathcal{A},\quad 0\to\mathcal{O}_X\to\mathcal{O}_{X'}\xrightarrow{\alpha}\operatorname{im}\alpha\to 0. \tag{11}$$

The variety X thus defined is S_2 by the next Lemma 1.16, since $\operatorname{im}\alpha$ is a subsheaf of $\mathcal{A}$ and so obviously does not have embedded primes. It is gdc again by looking in codimension 1 as in the previous proof. The G-action on X' descends to a G-action on X since j commutes with the G-action on $\widetilde{B'}$ and by construction the subalgebra of G-invariants is the algebra of $\widetilde{Y}$ glued along $\widetilde{C'}/\iota$, i.e. $\mathcal{O}_Y$. $\square$

The varieties X obtained in the two proofs coincide, since they both have finite morphisms to Y, they are both S_2, and they coincide over an open subset $Y_0\subset Y$ with $\operatorname{codim}(Y\backslash Y_0)\geqslant 2$.

Warning 1.14. It may happen that there is no covering involution of B' but only of its normalization $\widetilde{B}'$. Then the double locus of X is obtained from $\widetilde{B}'/j$ by some additional gluing in codimension 1 (codimension 2 for X). As a consequence, branches of X may not be S_2. However, the variety X is S_2. Multiple examples of this phenomenon are contained in [AP09, § 5.4].

On the other hand, the involution j need not be unique. For instance, if $g\in G$ has order two, then jg is another involution satisfying the assumptions for gluing. The next example shows that gluing via different involutions can give rise to non-isomorphic covers.

Example 1.15. Let $Y=\{u^2-wv^2=0\}\subset\mathbb{A}_{u,v,w}$. The normalization of Y is the map $\widetilde{Y}=\mathbb{A}^2_{s,t}\to Y$ defined by $u=st$, $v=t$, $w=s^2$. Here $C=\{u=v=0\}$, $\widetilde{C'}=C'=\{t=0\}$ and the involution ι of $\widetilde{C'}$ is given by $s\mapsto -s$.

Let $X'=\{\epsilon^2=1\}\subset\mathbb{A}^3_{s,t,\epsilon}$ and let $p:X'\to\widetilde{Y}$ be the trivial $\mathbb{Z}_2$ cover, given by the projection on the coordinates s,t. The $\mathbb{Z}_2$-action is $\epsilon\mapsto-\epsilon$ and $B'=\widetilde{B'}=\{t=0,\epsilon^2=1\}$. There are two involutions of $\widetilde{B'}$ that lift ι, namely $j_1:=(s,\epsilon)\mapsto(-s,\epsilon)$ and $j_2:=(s,\epsilon)\mapsto(-s,-\epsilon)$. The cover $X_1\to Y$ obtained by gluing via j_1 is obviously the trivial $\mathbb{Z}_2$-cover.

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We describe the cover $X_2 \to Y$ obtained by gluing via j_2 following the second proof of Theorem 1.13. The map $\widetilde{B'} \to \widetilde{B'}/j_2$ corresponds to the inclusion $\mathbb{K}[s\epsilon] \to \mathbb{K}[s,\epsilon]/(\epsilon^2-1)$ and the map $\widetilde{B'} \to X'$ corresponds to the surjection $\mathbb{K}[s,t,\epsilon] \to \mathbb{K}[s,\epsilon]/(\epsilon^2-1)$. The fiber product of these two ring maps can be identified with $R := \mathbb{K}[s,t,\epsilon t]/(\epsilon^2-1) \subset \mathbb{K}[s,t,\epsilon]/(\epsilon^2-1)$. The map $R \to \mathbb{K}[x,y,z]/(x^2-y^2)$ defined by $s \mapsto z$, $t \mapsto x$, $\epsilon t \mapsto y$ is an isomorphism, and hence X_2 is the union of two copies of $\mathbb{A}^2$ glued along a line. The cover $X_2 \to Y$ is given by $(x,y,z) \mapsto (x, yz, z^2)$, and the $\mathbb{Z}_2$-action on X is given by $(x,y,z) \mapsto (x,-y,-z)$. Thus $(0,0,0) \in Y$ is the only branch point. Hence the ramification locus of a standard G-cover has always pure codimension 1, but this not true for the G-covers obtained from a standard cover by gluing, and the analogue of Lemma 1.5 does not hold.

Lemma 1.16. *Using the notations as given in the second proof by gluing, assume that X' is S_n for some $n \geqslant 2$. Then X is S_n if and only if* $\operatorname{im}\alpha$ *is S_{n-1}.*

Proof. We use the cohomological interpretation of depth using local cohomology [Har67, 3.8] (alternatively and equivalently one can use $\operatorname{Ext}^i(\mathcal{O}_{X,Z}/m_{X,Z}, \bullet)$). A sheaf $\mathcal{E}$ satisfies S_n if and only if for every irreducible subvariety $Z \subset Y$ one has $H^i_Z(\mathcal{E}) = 0$ for all $i < \min(n, \operatorname{codim} Z)$. Looking at the long exact sequence of cohomologies corresponding to the short exact sequence (11), we get $H^i_Z(\mathcal{O}_X) = H^{i-1}_Z(\operatorname{im}\alpha)$ for all $i < \min(n, \operatorname{codim} Z)$. The statement now follows. □

We spell out Theorem 1.13 in a special case, which is of interest to us because of the applications in [AP09].

Example 1.17. Take $G = \mathbb{Z}_2^r$. For simplicity of exposition, we assume that $Y = Y_1 \cup Y_2$ is the gdc union of two smooth projective surfaces that intersect along a smooth rational curve C, but all our considerations generalize straightforwardly to the case of a gdc surface with smooth components whose double locus is a union of smooth rational curves.

We have $\widetilde{Y} = Y_1 \sqcup Y_2$, and hence an S_2 and gdc G-cover $X' \to \widetilde{Y}$ is the disjoint union of S_2 and gdc covers $\pi_i : X'_i \to Y_i$, $i = 1, 2$. By Corollary 1.10, the covers π_i are standard. We denote by $D^{(i)}_1, \dots, D^{(i)}_{r_i}$, $g^{(i)}_1, \dots, g^{(i)}_{r_i}$ the branch data of π_i, $i = 1, 2$. We write $\widetilde{C'} = C' = C'_1 \sqcup C'_2$, $B' = B'_1 \sqcup B'_2$ and $\widetilde{B'} = \widetilde{B'_1} \sqcup \widetilde{B'_2}$. We denote by γ_i the generator of subgroup $H_{C'_i}$. An involution j of $\widetilde{B'}$ as in Theorem 1.13 exists if and only if there is an isomorphism $\widetilde{B'_1} \to \widetilde{B'_2}$ compatible with the G-action. This is equivalent to the following conditions.

(i) One has $\gamma_1 = \gamma_2 =: \gamma$.

(ii) For $y \in C$, denote by $m^{(1)}_{y,h}$ the intersection multiplicity at y of $D^{(1)}_h$ with $C = C_1$, $h = 1, \dots, r_1$ and by $m^{(2)}_{y,s}$ the intersection multiplicity at y of $D^{(2)}_s$ with $C = C_2$, $s = 1, \dots, r_2$. Then
$$\sum_h m^{(1)}_{y,t} g^1_h = \sum_s m^{(2)}_{y,s} g^2_s \mod \gamma, \ \forall y \in C.$$

Indeed, condition (i) follows immediately by the fact that j commutes with the action of G. In addition, by the normalization algorithm of [Par91, § 3] condition (ii) is equivalent to requiring that the branch data of the normalizations $\widetilde{B'_1} \to C$ and $\widetilde{B'_2} \to C$ of the $G/\!\!/\langle\gamma\rangle$-coverings of $C = C_1 = C_2$ induced by π_1 and π_2 are the same. Since C is smooth rational, the branch data are enough to determine the building data (cf. Remark 1.3). Since C is projective, the building data determine the cover up to isomorphism by Proposition 1.6.

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Assume that the gluing conditions are satisfied. Giving an involution of $\widetilde{B'}$ that commutes with the G action is the same as giving an isomorphism of G-covers $\alpha: \widetilde{B'_1} \to \widetilde{B'_2}$. Then any other such map α' is equal to αg for some $g \in G$ and the automorphism of $X' = X'_1 \sqcup X'_2$ defined by $x \mapsto x$ if $x \in X'_1$ and $x \mapsto gx$ if $x \in X'_2$ induces an isomorphism of the cover of Y obtained by gluing via α with the one obtained by gluing via α'. Hence in this case all the possible involutions give isomorphic covers.

THEOREM 1.18 (The reverse). *Vice versa, every G-cover $X \to Y$ with gdc S_2 varieties X, Y is obtained via the gluing construction of Theorem 1.13.*

Proof. Given $X \to Y$ and the normalization $\widetilde{Y} \to Y$, let X'' be the fiber product $X'' = X \times_Y \widetilde{Y}$. We define X' as $X' := S_2(X''_{\mathrm{red}}) \to X''_{\mathrm{red}} \to X''$. Thus, X' is S_2 by definition, and it maps to $\widetilde{Y}$. By the universality of taking the reduced part and S_2-fication, there is an induced G-action on X'. By the universal property of G-quotients, we also have a morphism $X'/G \to Y$. We claim that it is an isomorphism.

It is enough to check this in codimension 1 over the double locus. We claim that generically over the double locus of Y, the cover is (smooth) $\times$ (admissible action of curves), where 'admissible' means that, working étale locally, X is given by $xy = 0$, and the action is $x \mapsto \xi x$, $y \mapsto \xi^a y$ for some primitive root $\xi^n = 1$ and $(a, n) = 1$. Indeed, let H_F be the subgroup of elements that restrict to the identity on an irreducible component F of the double locus of X. Then on the normalization on both branches we have the same subgroup for the preimages F' and jF'. Since generically F', jF' are smooth, $H_F = \mathbb{Z}_n$ for some $n \geqslant 1$ (note that one possibly has $n = 1$).

Thus, étale locally the morphism $X \to Y$ can be written as

$$(\text{smooth}) \times \mathbb{K}[u, v]/(uv) \to \mathbb{K}[x, y]/(xy), \quad u \mapsto x^n, v \mapsto y^n,$$

where G acts as $x \mapsto \xi x$, $y \mapsto \xi^a y$, $\xi^n = 1$, $(a, n) = 1$. By computation, we get that X'' corresponds to (smooth) $\times\, \mathbb{K}[x, y]/(xy, y^n) \oplus \mathbb{K}[x, y]/(xy, x^n)$, and X' to $\mathbb{K}[x] \oplus \mathbb{K}[y]$. The quotient X'/G is then $\mathbb{K}[u] \oplus \mathbb{K}[v]$, i.e. $\widetilde{Y}$.

This proves that $\phi: X'/G \to \widetilde{Y}$ is an isomorphism outside a closed subset of codimension greater than or equal to two. Since both are finite over Y and S_2, ϕ is an isomorphism. □

2. Singularities of covers

2.1 The canonical divisor and slc singularities

Let Z be a variety, let B_j, $j = 1, \ldots, n$, be effective Weil divisors on X, possibly reducible, and let b_j be rational numbers with $0 \leqslant b_j \leqslant 1$. Set $B = \sum_j b_j B_j$.

DEFINITION 2.1. Assume that Z is a *normal* variety. Then Z has a canonical Weil divisor K_Z defined up to linear equivalence. The pair (Z, B) is called *log canonical* if the following apply.

(i) The divisor $K_Z + B$ is $\mathbb{Q}$-Cartier, i.e. some positive multiple is a Cartier divisor.

(ii) Every prime divisor of Z has multiplicity less than or equal to one in B and for every proper birational morphism $h: Z' \to Z$ with normal Z', in the natural formula

$$K_{Z'} + h_*^{-1} B = h^*(K_Z + B) + \sum a_i E_i$$

one has $a_i \geqslant -1$. Here, E_i are the irreducible exceptional divisors of h, the pull back h^* is defined by extending $\mathbb{Q}$-linearly the pullback on Cartier divisors, and $h_*^{-1} B = \sum b_j h_*^{-1} B_j$ is the strict

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preimage of B. The coefficients a_i are called *discrepancies*. For the non-exceptional divisors, already appearing on Z, one defines $a(B_j)=-b_j$.

If $\operatorname{char}\mathbb{K}=0$, then Z has a resolution of singularities $h: Z' \to Z$ such that $\operatorname{Supp}(h_*^{-1}B)\cup E_i$ is a normal crossing divisor; then it is sufficient to check the condition $a_i \geqslant -1$ for this morphism h only.

Definition 2.2. A pair (Z, B) is called *semi log canonical* if the following apply.

(i) The variety Z satisfies Serre's condition S_2.

(ii) The variety Z is gdc, and no divisor B_j contains any component of the double locus of Z.

(iii) Some multiple of the Weil $\mathbb{Q}$-divisor K_Z+B, well defined thanks to the previous condition, is Cartier.

(iv) Denoting by $\nu: \widetilde{Z}\to Z$ the normalization, the pair $(\widetilde{Z}, (\text{double locus}) + \nu_*^{-1}B)$ is log canonical.

Lemma 2.3. *Let $f: X\to Y$ be a finite morphism of degree d between equidimensional S_2 varieties. Assume that either* $\operatorname{char}\mathbb{K}=0$ *or f is Galois and* $\operatorname{char}\mathbb{K}$ *does not divide d.*

Let Y_0 be an open subset and denote by $f_0: X_0\to Y_0$ the induced cover. Assume that the following are true.

– *One has* $\operatorname{codim}(Y\backslash Y_0)\geqslant 2$ *and both X_0 and Y_0 are dc.*

– *There exist effective $\mathbb{Q}$-divisors B^X of X and B^Y of Y, not containing any component of the double locus, such that $(f_0)^*(K_{Y_0}+B^{Y_0}) = (K_{X_0}+B^{X_0})$, where B^{Y_0} is the restriction of B^Y to Y_0 and B^{X_0} is the restriction of B^X to X_0.*

Then the following hold.

(i) *The divisor K_Y+B^Y is $\mathbb{Q}$-Cartier if and only if K_X+B^X is also $\mathbb{Q}$-Cartier.*

(ii) *The pair (Y, B^Y) is* slc *if and only if the pair (X, B^X) is also* slc.

Proof. (i) Let $i: X_0\to X$ be the inclusion map. If the sheaf $L=\mathcal{O}_Y(N(K_Y+B^Y))$ is invertible then we have a homomorphism

$$\mathcal{O}_X(N(K_X+B^X)) = i_*(\mathcal{O}_{X_0}(N(K_{X_0}+B^{X_0}))) \to f^*L$$

which is an isomorphism outside codimension 2. Hence it must be an isomorphism by the S_2 condition. Similarly, if the sheaf $L'=\mathcal{O}_X(N(K_X+B^X))$ is invertible, then the sheaf $L=\mathcal{O}_Y(Nd(K_Y+B^Y))$ is isomorphic to the norm of L', so is invertible.

(ii) Assume first that X and Y are normal. In this case the statement, due to Shokurov, is very well known. We recall the proof because usually it is only stated and proved in characteristic zero. Let $h_Y: Y'\to Y$ be some partial resolution with normal Y', X' be the normalization of $X\times_Y Y'$, and let $h_X: X'\to X$, $f': X'\to Y'$ be the induced maps.

Pick an irreducible divisor E on Y', and let F be an irreducible divisor on X' over it. By our condition on $\operatorname{char}\mathbb{K}$, the field extension $\mathbb{K}(F)/\mathbb{K}(E)$ is separable, and if π_X, π_Y are uniformizing parameters in the discrete valuation rings $\mathcal{O}_{X',F}$ and $\mathcal{O}_{Y',E}$, then one has $\pi_Y = u\cdot\pi_X^e$ for a unit u and some integer e dividing d and hence coprime to $\operatorname{char}\mathbb{K}$.

Then the Riemann–Hurwitz formula applies and says that generically along E and F one has $(f')^*(K_{Y'}+E) = K_{X'}+F$. Comparing this to the identity $(f')^*h_Y^*(K_Y+B^Y) = h_X^*(K_X+B^X)$ and the definition of the log discrepancy, one obtains that $1+a_F = e(1+a_E)$. Thus, $a_F\geqslant -1 \iff a_E\geqslant -1$. This proves that (X, B^X) is lc if and only if (Y, B^Y) is lc.

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Now consider the general gdc case. Let $\nu_X : \widetilde{X} \to X$ be the normalization. We have

$$K_{\widetilde{X}} + B^{\widetilde{X}} := \nu_X^*(K_X + B^X) = K_{\widetilde{X}} + \nu_{X*}^{-1} B^X + (\text{double locus}),$$

and similarly for Y. Thus, the double loci appear in the divisors $B^{\widetilde{X}}$, $B^{\widetilde{Y}}$ with coefficient 1. By the Riemann–Hurwitz formula again, for the normalizations we still have $\tilde{f}^*(K_{\widetilde{Y}} + B^{\widetilde{Y}}) = K_{\widetilde{X}} + B^{\widetilde{X}}$. We finish by applying the normal case. □

We now extend Definition 1.8 of the Hurwitz divisor to the case of a gdc base Y.

Definition 2.4. Let $\pi : X \to Y$ be a G-cover of S_2 and gdc varieties. For a prime Weil divisor $F \subset Y$, we define $\rho_F \in \mathbb{Q}$ as follows.

- If F is contained in the double locus of Y, then $\rho_F = 0$.
- If F is not contained in the double locus of Y, but $\pi^{-1}(F)$ is contained in the double locus of X, then $\rho_F = 1$.
- If F is not contained in the double locus of Y, $\pi^{-1}(F)$ is not contained in the double locus of X and m is the ramification order of π at F, then $\rho_F = (m-1)/m$.

We define the Hurwitz divisor D of π to be the $\mathbb{Q}$-divisor $\sum_F \rho_F F$.

Notice that if $X \to Y$ is a standard G-cover with X gdc this definition coincides with Definition 1.8 by Theorem 1.9.

Note that D does not contain any components of the double locus of Y.

Proposition 2.5. *Let $\pi : X \to Y$ be a G-cover as in Definition 2.4 and let D be the Hurwitz divisor of π, let $X' \to \widetilde{Y}$ be the corresponding S_2 and gdc G-cover (cf. § 1.5). Then the following hold.*

(i) *The divisor K_X is $\mathbb{Q}$-Cartier if and only if $K_Y + D$ is also $\mathbb{Q}$-Cartier, and then $K_X = \pi^*(K_Y + D)$.*

(ii) *The variety X is* slc *if and only if the pair (Y, D) is also* slc.

Proof. Recall that $|G|$ and $\operatorname{char} \mathbb{K}$ are coprime by assumption. So Lemma 2.3 applies and we may assume that Y is dc. We need to show that $K_X = \pi^*(K_Y + D)$. This is equivalent to the following equality for the cover $\tilde{\pi} : \widetilde{X} \to \widetilde{Y}$, where $\widetilde{X}$ is the normalization of X' (and of X):

$$K_{\widetilde{X}} + (\text{double locus}) = \tilde{\pi}^*(\mathrm{K}_{\widetilde{\mathrm{Y}}} + (\text{double locus}) + \nu^* \mathrm{D}).$$

In view of Definition 2.4 the formula follows easily by the usual Hurwitz formula. □

2.2 Cohen–Macaulay covers

By Lemma 1.1, a G-cover over a smooth base is CM. Here, we give a partial generalization of this case to the case of a non-normal base. We use the notations of Theorem 1.13 and the exact sequence (11).

Proposition 2.6. *Assume that X' is CM (for example, $\widetilde{Y}$ is smooth). Then X is CM if and only if the sheaf* $\operatorname{im} \alpha$ *is CM.*

Proof. The proof is immediate by Lemma 1.16. □

Using Proposition 2.6 it is not hard to give examples of abelian covers $X \to Y$ such that Y is CM and gdc, and X is gdc and S_2 but not CM.

Example 2.7. We take $G=\mathbb{Z}_2$ and assume char $\mathbb{K}\neq 2$; for any prime p one can construct similar examples with $G=\mathbb{Z}_p$ and char $\mathbb{K}\neq p$.

Let $Y=Y_1\cup Y_2$ be the union of two copies of $\mathbb{P}^3$ glued transversally along a plane C. Let L_1 and L_2 be distinct lines on C, and for $i=1,2$ let $D_i\subset Y_i$ be a quadric that restricts to $2L_i$ on C. For a generic choice, D_i is a quadric cone with vertex $y_i\in L_i$, and the points y_1, y_2 and $y_3:=L_1\cap L_2$ are distinct. Let $X'_i\to Y_i$ be the double cover of Y_i branched on D_i, and let $X'=X'_1\sqcup X'_2$. Then X' is Gorenstein, and it has an ordinary double point over y_1 and y_2 and no other singularity. Write $C'=C'_1\sqcup C'_2$ and $B'=B'_1\sqcup B'_2$; then B'_i is the union of two copies of C'_i glued transversally along L_i and $\widetilde{B'}\to C'$ is the trivial $\mathbb{Z}_2$-cover. Hence there exists an involution j of $\widetilde{B'}$ that commutes with the $\mathbb{Z}_2$-action, and by Theorem 1.13 X' can be glued to an S_2 and gdc cover $X\to Y$. The dc locus of X is the complement of the preimage of $L_1\cup L_2$.

In the exact sequence (11) each term splits under the G-action and the maps are compatible with the splitting, so we get two exact sequences, one for each character of G. Since $\mathcal{A}=\mathcal{O}_C\oplus\mathcal{O}_C$ and $\mathbb{Z}_2$ acts on $\mathcal{A}$ by switching the two summands, the sequence for the non-trivial character is

$$0\to\mathcal{F}_-\to\mathcal{O}_{Y_1}(-1)\oplus\mathcal{O}_{Y_2}(-1)\xrightarrow{\alpha^-}\mathcal{O}_C,$$

where $\mathcal{F}_-$ (respectively $\mathcal{A}^-$) is the antiinvariant summand of $\mathcal{O}_X$ (respectively of $\mathcal{A}$). By definition, the map $\mathcal{O}_{Y_i}(-1)\to\mathcal{O}_C$ factorizes as $\mathcal{O}_{Y_i}(-1)\to\mathcal{O}_C(-L_i)\to\mathcal{O}_C$. Hence, $\operatorname{im}\alpha^-$ coincides with $\mathcal{I}_{y_3}\mathcal{O}_C$, the maximal ideal of y_3 in C, and therefore it is not S_2. It follows by Proposition 2.6 that X is not CM over y_3.

Let $\bar{y}\in L_1$ be a point distinct from y_3; in a neighborhood of $\bar{y}$ we have $(D_1+D_2)\cap Y_2=L_1$, and thus D_1+D_2 is not $\mathbb{Q}$-Cartier. Since Y is Gorenstein, it follows that $2K_Y+D_1+D_2$ is not $\mathbb{Q}$-Cartier either, and hence K_X is not $\mathbb{Q}$-Cartier by Proposition 2.5.

2.3 Cartier index of K_X

All the statements in this section are étale local, so we often pass to a smaller neighborhood of a point without explicit mention of the fact.

For convenience, we write 'K_X' to denote the divisorial sheaf ω_X (recall that X is Gorenstein in codimension 1 and S_2). We also use the additive notation D_1+D_2 for the sheaf $(\mathcal{O}_X(D_1)\otimes\mathcal{O}_X(D_1))^{**}$.

2.3.1 *Standard covers with Y normal.* We consider the following situation.

- Suppose that Y is a normal variety and C is a reduced effective divisor on Y such that K_Y+C is Cartier.
- Suppose that $\pi:X\to Y$ is a standard gdc G-cover (so X is automatically S_2 by Lemma 1.1). We assume that X is generically smooth over C, and we denote by B the preimage of C in X. Therefore, B is also a reduced effective divisor.

Let D be the Hurwitz divisor of π; then we have

$$K_X+B=\pi^*(K_Y+D+C).$$

Thus, if d is the exponent of G, then the divisor $d(K_Y+D+C)$ is Cartier (recall that the divisors D_i are Cartier by the definition of a standard cover in § 1.2), and thus $d(K_X+B)$ is also Cartier.

Fix a point $y\in Y$; the purpose of this section is to compute the Cartier index of K_X+B at a point $x\in X$ such that $\pi(x)=y$. Here we are interested mainly in the case $B=0$, but the case of a pair is needed in the next section to treat the case Y non-normal.

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In order to state our result we need some notation. We label the branch data $D_i, (H_i, \psi_i)$, $i = 1, \ldots, k$, in such a way that $D_i \subseteq C$ if and only if $i \leqslant p$. Since the question is local on Y we may assume that $y \in D_i$ for every i. Consider the map $\overline{G} := \oplus H_i \to G$. By Lemma 1.5, the image of this map is the inertia subgroup H_y; we denote by N the kernel. We let $\overline{\chi} \in \overline{G}^*$ be the character $\psi_{p+1} \cdots \psi_k$.

Reminder. Since the group G is finite abelian, the map $G^* \to H_y^*$ is surjective. Hence the character $\overline{\chi}$ is the pullback of a character of H_y if and only if it is the pullback of a character of G.

Proposition 2.8. *Notation and assumptions are as given previously.*

The Cartier index of $K_X + B$ at x is equal to the order of $N/(N \cap \ker \overline{\chi})$.

In particular, $K_X + B$ is Cartier if and only if $\overline{\chi}$ is the pullback of a character $\chi \in G^*$.

Proof. Since the question is local, we may assume that the line bundles L_χ, $\mathcal{O}_Y(D_i)$ and $\mathcal{O}_Y(K_Y + C)$ are trivial. The map $X \to X/H_y$ is étale. Hence, up to replacing Y by X/H_y, we may assume that $H_y = G$, or, equivalently, that $\pi^{-1}(y) = \{x\}$. We denote by $u_1, \ldots, u_k$ local equations of $D_1, \ldots, D_k$ near y. By Remark 1.7, up to passing to an étale cover of Y we may assume that X is given by

$$z_\chi z_{\chi'} = \Pi_1^k u_i^{\varepsilon^i_{\chi,\chi'}} z_{\chi\chi'}, \quad \chi, \chi' \in G^* \backslash \{1\}. \tag{12}$$

The equations:

$$z_1^{m_1} = u_1, \quad \ldots \quad z_k^{m_k} = u_k \tag{13}$$

define inside $Y \times \mathbb{K}^k$ a $\overline{G}$-cover $\overline{X} \to Y$ ($\overline{G}$ acts on z_i via the character ψ_i), the *maximal totally ramified cover* of Y with branch data $D_i, (H_i, \psi_i)$ (here we regard H_i as a subgroup of $\overline{G}$). Since Y is gdc by assumption and $X \to Y$ and $\overline{X} \to Y$ have the same Hurwitz divisor, $\overline{X}$ is also gdc by Theorem 1.9.

For every $\chi \in G^*$, write $\chi = \psi_1^{a^1_\chi} \cdots \psi_k^{a^k_\chi}$, with $0 \leqslant a_i^\chi < m_i$ for $i = 1, \ldots, k$; then setting $z_\chi = z_1^{a^1_\chi} \cdots z_k^{a^k_\chi}$ defines a map $p : \overline{X} \to X$ which is the quotient map for the action of the kernel N of $\overline{G} \to G$. The map p is unramified in codimension 1 and $p^{-1}(x)$ consists of just one point $\overline{x}$.

Denote by $\overline{B}$ the preimage of C (and of B) in $\overline{X}$; observe that $K_Y + D + C$ pulls back to $K_X + B$ on X and to $K_{\overline{X}} + \overline{B}$ on $\overline{X}$. If τ is a generator of $\mathcal{O}_Y(K_Y + C)$ then $\mathcal{O}_{\overline{X}}(K_{\overline{X}} + \overline{B})$ is generated by the residue σ on $\overline{X}$ of the rational differential form:

$$\frac{(z_1^{m_1-1} \cdots z_p^{m_p-1}) dz_1 \wedge \cdots \wedge dz_k \wedge \tau}{(z_1^{m_1} - u_1) \cdots (z_k^{m_k} - u_k)}.$$

Thus $\mathcal{O}_{\overline{X}}(K_{\overline{X}} + \overline{B})$ is invertible and G acts on the local generator σ via the character $\overline{\chi}$. Set $Z := \overline{X}/(N \cap \ker \overline{\chi})$. The map $\overline{X} \to Z$ is unramified in codimension 1 and σ descends on Z to a generator of $\mathcal{O}_Z(K_Z + B_Z)$, where B_Z is the image of $\overline{B}$. The map $Z \to X$ is a cyclic cover with Galois group $N/(N \cap \ker \overline{\chi})$ with the following properties.

- It is unramified in codimension 1 and the preimage of x consists only of one point.
- The pullback of $\mathcal{O}_X(K_X + B)$ is a line bundle on which the Galois group acts via a primitive character.

It follows that $Z \to X$ is a canonical cover and that the Cartier index of $K_X + B$ at x is equal to $[N : N \cap \ker \overline{\chi}]$. $\square$

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Corollary 2.9. *Let $\pi : X \to Y$ be a standard abelian with X and Y gdc and Y Gorenstein, let $y \in Y$ and let $x \in X$ be a point such that $\pi(x) = y$. Then X is Gorenstein at x if and only if the character $\overline{\chi}$ descends to a character χ of H_y.*

Proof. The variety X is Cohen–Macaulay by Lemma 1.1 and K_X is Cartier by Proposition 2.8. □

Remark 2.10. Corollary 2.9 is proven in [Iac06] under the assumption that X is normal and Y is smooth.

2.3.2 *The case Y non-normal.* Here we consider the problem of determining the Cartier index of K_X at a point $x \in X$ of a G-cover $X \to Y$ with Y non-normal of Cartier index 1. The situation is much more complicated than in the case Y normal and we are able to give only a partial answer that is, however, sufficient for the applications in [AP09]. The main difficulty is that one does not know how to write down an analogue of the maximal totally ramified cover used in the proof of Proposition 2.8.

We consider the following setup:

- we assume that $Y = Y_1 \cup \cdots \cup Y_t$, where Y_i is irreducible for $i = 1, \ldots, t$, is a gdc and S_2 variety; $\widetilde{Y} = \widetilde{Y}_1 \sqcup \cdots \sqcup \widetilde{Y}_t \to Y$ is the normalization;
- we assume that $\pi : X \to Y$ is an S_2 and gdc G-cover obtained by gluing a cover $X' = X'_1 \sqcup \cdots \sqcup X'_t \to \widetilde{Y}$ such that $X'_i \to \widetilde{Y}_i$ is standard for every i;
- we assume that $y \in Y$ and $x \in X$ are points such that $\pi(x) = y$; we assume that y lies on every component of the branch locus of π.

We denote by $D_i, (H_i, \psi_i)$, $i = 1, \ldots, k$ the branch data of the standard cover $X' \to \widetilde{Y}$, and we assume that D_i is contained in the preimage C' of the double locus of Y if and only if $i \leqslant p$. Consider the map $\overline{G} := \oplus H_i \to G$. As in the case Y normal, we denote by $\overline{\chi} \in \overline{G}^*$ the character $\psi_{p+1} \cdots \psi_k$. Then we have the following proposition.

Proposition 2.11. *In the above setup, if K_X is Cartier, then the following are true.*

(i) *The divisor $K_Y + D$ is $\mathbb{Q}$-Cartier.*

(ii) *The character $\overline{\chi}$ is the pullback of a character $\chi \in G^*$.*

Proof. (i) Part (i) follows immediately by Proposition 2.5.

(ii) For every $i = 1, \ldots, t$, denote by $C'_i \subset \widetilde{Y}_i$ (respectively $B'_i \subset X'_i$) the preimage of the double locus of Y in $\widetilde{Y}_i$ (respectively in X'_i). Let $\chi \in G^*$ be the character via which G acts on $\mathcal{O}_X(K_X) \otimes \mathbb{K}(x)$ at x. Let $x'_i \in X'_i$ be a point that maps to x and let y_i be the image of x'_i in $\widetilde{Y}_i$. Since K_X pulls back to $K_{X'_i} + B'_i$ on X'_i, the inertia subgroup H_{y_i} acts on $\mathcal{O}_{X'_i}(K_{X'_i} + B'_i) \otimes \mathbb{K}(x'_i)$ via the restriction of χ. Set $\overline{G}_{y_i} := \bigoplus_{\{j | y_i \in D_j\}} H_j$ and let $\overline{\chi}_{y_i}$ be the restriction of $\overline{\chi}$ to $\overline{G}_{y_i}$; the map $\overline{G}_{y_i} \to H_{y_i}$ is a surjection by Lemma 1.5. By the proof of Proposition 2.8, χ pulls back on $\overline{G}_{y_i}$ to $\overline{\chi}_{y_i}$. Since $\overline{G} = \sum_{\{y' \in \widetilde{Y} | y' \mapsto y\}} \overline{G}_{y'}$, it follows that χ pulls back to $\overline{\chi}$ on $\overline{G}$. □

We now prove a partial converse of Proposition 2.11. Assume that for every component $\widetilde{Y}_i$ of $\widetilde{Y}$ the map $\widetilde{Y} \to Y$ induces a homeomorphism $\widetilde{Y}_i \to Y_i$ onto its image (this is always true up to an étale cover). Then we associate to (Y, y) an incidence graph $\Gamma_{Y,y}$ as follows.

- The vertices of $\Gamma_{Y,y}$ are indexed by the branches of (Y, y).
- The edges are indexed by the components of the double locus C of Y.

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– The edge corresponding to a component F of C connects the vertices corresponding to the two branches of Y through F.

Proposition 2.12. *In the above setup, assume the following:*

(i) *the graph $\Gamma_{Y,y}$ is a tree;*

(ii) *the divisor K_Y is Cartier and there exists m such that $m(K_Y+D)$ is Cartier and $(m, \operatorname{char} \mathbb{K})=1$;*

(iii) *the character $\overline{\chi}$ is the pullback of a character $\chi \in G^*$.*

Then K_X is Cartier.

Proof. Let $C_i' \subset \widetilde{Y}_i$ the restriction of the double locus C' of $\widetilde{Y}$ and let $B_i' \subset X_i'$ be the preimage of C_i'. Let $y_i \in \widetilde{Y}_i$ be the only point that maps to $y \in Y$; let $\overline{G}_{y_i}$ and $\overline{\chi}_i$ be defined as in the proof of Proposition 2.11.

By assumption (iii), the divisor $K_{X_i'}+B_i'$ is Cartier by Proposition 2.8. By the following Lemma 2.13, up to replacing (Y,y) by an étale neighborhood we may assume that for $i=1,\dots,t$ the sheaf $\mathcal{O}_{X_i'}(K_{X_i'}+B_i')$ is trivial and has a generator σ_i on which G acts via χ. By Proposition 2.5, there exists a local generator τ of $\mathcal{O}_X(mK_X)$ near x. For every i, by Lemma 2.13, τ pulls back on X_i' to $h_i\sigma_i^m$ where h_i is a nowhere vanishing regular function on $\widetilde{Y}_i$. Up to passing to an étale cover of Y we may assume that h_i has an mth root f_i for every i. Hence we may replace σ_i by $f_i\sigma_i$ and assume that τ pulls back to σ_i^m for every i.

Now let $U \subset X$ be an open set such that U is dc and the complement of U has codimension greater than one. Let F be an irreducible component of the double locus C of Y and let Y_a, Y_b be the components of Y that contain F. Choose an irreducible component E of the inverse image of F in U. It makes sense to compare σ_a and σ_b along E, since they both restrict to local generators of $\mathcal{O}_E(K_E)$. Since $\sigma_a^m=\sigma_b^m$, there exists $\zeta \in \mu_m$ such that $\sigma_a=\zeta\sigma_b$ along E. Since G acts on σ_a and σ_b via the same character χ and G acts transitively on the components of the preimage of F, $\zeta_F:=\zeta$ depends only on F. Hence $\{\zeta_F\}$ represents a class in $H^1(\Gamma_{Y,y},\mu_m)$. Since $\Gamma_{Y,y}$ is a tree, we can find $\lambda_i \in \mu_m$ such that the local generators $\lambda_i\sigma_i$ glue to give a local generator σ of $\mathcal{O}_X(K_X)$ on which G acts via χ. □

We complete the proof of Proposition 2.12 by proving the following lemma.

Lemma 2.13. *Let $Z \to W$ be a standard G-cover with building data $L_\chi, D_i, (H_i,\psi_i)$.*

Let $w \in W$ be a point and let H be the inertia subgroup of w. Let L be a G-linearized line bundle of Z, let $z \in Z$ be a point that maps to w, and let $\phi \in H^*$ be the character via which H acts on $L \otimes \mathbb{K}(z)$. Then we have the following.

(i) *Let $\chi \in G^*$ be such that $\chi|_H=\phi$; then, up to replacing W by an étale neighborhood of w, there exists a generator σ of L such that G acts on σ via the character χ.*

(ii) *The generator σ is uniquely determined by χ up to multiplication by a nowhere vanishing regular function of W.*

Proof. (ii) Assume that σ,σ' are generators of L on which G acts via the character χ. Then $f:=\sigma/\sigma'$ is a regular H-invariant function on Z, so it is a function on W.

(i) We break the proof into three steps.

Step 1: *the case $H=G$.* Let s be a generator of L near z. The group H acts on the vector space V of local sections of L spanned by the elements h_*s, $h \in H$; V is finite-dimensional, and decomposes under the G-action as a direct sum of eigenspaces. Since $s(z) \neq 0$ and $s \in V$, there exists an eigenvector $\sigma \in V$ such that $\sigma(z) \neq 0$. Since G acts on $L \otimes \mathbb{K}(z)$ via χ, σ belongs to the eigenspace corresponding to χ.

Step 2: *the case in which $G = H \oplus N$ for some N.* Consider the factorization $Z \to Z' := Z/N \to W$. The map $Z' \to W$ is an H-cover such that the preimage of w consists of one point $z' \in Z'$. The subgroup N acts freely on Z, and hence L descends to an H-linearized line bundle L' on Z'. Then by Step 1 there exists a local generator σ' of L' near z' such that H acts on σ' via ϕ. Pulling back to Z we get a generator τ of L on which H acts via ϕ and N acts trivially.

Denote by ϕ' the restriction of χ to N, so that $\chi = (\phi, \phi')$. Consider the factorization $Z \to Z'' := Z/H \to W$. The map $Z'' \to W$ is a étale N-cover. Hence there exists a nowhere-vanishing function f on Z'' such that N acts on f via the character ϕ. Thus G acts on $\sigma := f\tau$ via the character χ.

Step 3: *the general case.* Choose a finite abelian group N with a surjective map $G_0 := H \oplus N \to G$ that extends the inclusion $H \to G$, and let T be the kernel of $G_0 \to G$. By Proposition 1.6, up to replacing W by an étale neighborhood of w, we may also assume (cf. (4)) that $Z \to W$ is given inside $W \times \mathbb{K}^k$ by the equations

$$y_\chi y_{\chi'} = \Pi_1^k u_i^{\varepsilon^i_{\chi,\chi'}} y_{\chi\chi'}, \quad \chi, \chi' \in G^* \backslash \{1\}, \tag{14}$$

where u_i is a local equation for D_i, $i = 1, \dots, k$. The branch data for Z can be interpreted in an obvious way as branch data for a G_0-cover. Letting $Z_0 \to W$ be the G_0-cover given by the equations analogous to (14), we have $Z = Z_0/T$ by construction. Let L_0 be the pullback of L to Z_0; L_0 has a natural G_0-linearization and H is a direct summand of G_0, and hence by Step 2 there exists a generator σ_0 of L_0 on which G_0 acts via the character χ_0 of G_0 induced by χ. Since T acts freely on Z_0 and $T \subset \ker \chi_0$ by construction, σ_0 descends to a generator σ of L on Z on which G acts via χ. $\square$

3. Semi log canonical $\mathbb{Z}_2^r$-covers of surfaces

3.1 Setup

In this section we make a detailed study of $\mathbb{Z}_2^r$-covers of surfaces. We use freely the notation introduced in § 1.4. In particular, we refer the reader to the commutative diagram (10) and Theorem 1.13.

The situation that we consider is the following.

- The surface Y is a gdc surface with smooth irreducible components $Y_1, \dots, Y_t$. The irreducible components $F_1, \dots, F_s$ of the double curve C of Y are smooth, Y is dc at the smooth points of C, and it is analytically isomorphic to the cone over a cycle of rational curves at the singular points of C. In particular, Y is Gorenstein.
- The group $G = \mathbb{Z}_2^r$ and $\pi : X \to Y$ is a G-cover with X gdc and S_2, obtained as in Theorem 1.13 by gluing a cover $X' \to \widetilde{Y} = Y_1 \sqcup \cdots \sqcup Y_t$ such that for every $i = 1, \dots, t$ the restricted cover $\pi_i : X'_i \to Y_i$ is standard with building data $L_{i,\chi}$, D_{i,j_i}.
- The D_{i,j_i} and the components of the double curve C' are 'lines' of Y, namely they are smooth and meet pairwise transversally.
- The intersection points of the support of the Hurwitz divisor D of π with the double curve C of Y are smooth points of C.

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- The divisor $K_Y + D$ (or, equivalently, D, since Y is Gorenstein) is 2-Cartier and the pair (Y, D) is slc, so that by Proposition 2.5 X is slc and K_X is 2-Cartier. Recall that, since we assume that the components of $\nu^* D$ and of C' are lines, the pair (Y, D) is slc if and only if on $\widetilde{Y}$ the divisor $\nu^* D + C$ has components of multiplicity less than or equal to one and has multiplicity less than or equal to two at every point.
- For every $y \in Y$ that is singular for C, label the components $Y_1, \dots, Y_q$ of Y containing y in such a way that, for every $i = 1, \dots, q$, the surfaces Y_i and Y_{i+1} meet along an irreducible curve F_i containing y (the indices are taken modulo q) and let $g_i \in G$ be the generator of the inertia subgroup of F_i. By Theorem 1.13, for every i we have $g_{i-1} = g_{i+1} \mod g_i$. We assume that the natural map $\langle g_i \rangle \oplus \langle g_{i+1} \rangle \longrightarrow H_y$ is an isomorphism for every $i = 1, \dots, q$. These conditions imply that the fibre of $X \to Y$ over y consists of $2^r/|H_y|$ points. At each of these points X is analytically isomorphic to the cone over a cycle of q smooth rational curves.

All the above assumptions are satisfied in the cases considered in [AP09].

3.2 Numerical invariants

Here we assume that the surface Y is projective.

By Proposition 2.5, K_X^2 can be computed as follows:

$$K_X^2 = 2^r (K_{\widetilde{Y}} + \nu^* D + (\text{double locus}))^2 = \sum_i 2^r (K_{Y_i} + D|_{Y_i} + (\text{double locus})|_{Y_i})^2. \tag{15}$$

To compute the cohomology of $\mathcal{O}_X$, we are going to write down explicitly in the above situation the sequences (11) in the second proof of Theorem 1.13 (as usual we push forward to Y all the sheaves). Since all the maps are G-equivariant, the sequences (11) split as sums of exact sequences:

$$0 \to \mathcal{F}_\chi \to \oplus_{i=1}^t L_{i,\chi}^{-1} \xrightarrow{\alpha} \mathcal{A}_\chi, \quad 0 \to \mathcal{F}_\chi \to \oplus_{i=1}^t L_{i,\chi}^{-1} \xrightarrow{\alpha} (\operatorname{im} \alpha)_\chi \to 0, \tag{16}$$

where χ varies in G^* and G acts in $\mathcal{F}_\chi$, $\mathcal{A}_\chi$ and $(\operatorname{im} \alpha)_\chi$ via χ.

To describe the sheaves $\mathcal{A}_\chi$ and $(\operatorname{im} \alpha)_\chi$, we need to introduce some more notation. Given a component F_l of C we denote by $g_l \in G$ the generator of the inertia subgroup of F_l and by Y_{a_l} and Y_{b_l} the two components of Y that contain F_l. We denote by E_l (respectively E_{l,a_l}, E_{l,b_l}) the preimages of F_l in X (respectively X'_{a_l}, X'_{b_l}) and by $\widetilde{E_l}$ the common normalization of E_l, E_{l,a_l}, E_{l,b_l} (cf. Example 1.17). In the commutative diagram

$$\begin{tikzcd}
& \widetilde{E_l} \arrow[dl] \arrow[d] \arrow[dr] & \\
E_{l,a_l} \arrow[r] \arrow[dr] & E_l \arrow[d] & E_{l,b_l} \arrow[l] \arrow[dl] \\
& F_l &
\end{tikzcd}$$

the maps to F_l are $G/\langle g_l \rangle$-covers and the remaining maps are finite and birational. The cover $E_{l,a_l} \to F_l$ is standard and its building data can be recovered from those of $X'_{a_l} \to Y_{a_l}$ as follows.

- We identify $(G/\langle g_l \rangle)^*$ with $\langle g_l \rangle^\perp \subseteq G^*$, and for every $\chi \in \langle g_l \rangle^\perp$ we restrict $L_\chi^{a_l}$ to F_l.
- For every $D_j^{a_l}$ with $g_j \neq g_l$, we label each point of $D_j^{a_l}|_{F_l}$ with the image of g_j in $G/\langle g_l \rangle$.

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The same can be done of course for $E_{l,b_l} \to F_l$. Let $y \in F_l$ be a point such that ν^*D has multiplicity one at the points of $\widetilde{Y}$ that map to y (since we assume that $2D$ is Cartier, the multiplicity of ν^*D is the same at all points lying over y). Recall that by assumption Y is dc at y; denote by $\alpha_{y,1}$ $\alpha_{y,2}$ the elements of G associated to the two branch lines of $X'_{a_l} \to Y_{a_l}$ containing y and by $\beta_{y,1}$, $\beta_{y,2}$ the elements of G associated to the two branch lines of $X'_{b_l} \to Y_{b_l}$ containing y. We have $\alpha_{y,1} + \alpha_{y,2} = \beta_{y,1} + \beta_{y,2}$ modulo g_l (cf. Example 1.17). Then E_{l,a_l} is singular over y if and only if $\alpha_{y,1}$ and $\alpha_{y,2}$ are both different from g_l, namely if and only if there exists a character χ with $\chi(g_l) = 1$ and $\chi(\alpha_{1,y}) = \chi(\alpha_{2,y}) = -1$. For each $\chi \in G^*$ and l such that $\chi(g_l) = 1$ we denote by $A_{l,\chi}$ the set of points $y \in F_l$ such that $\chi(\alpha_{1,y}) = \chi(\alpha_{2,y}) = -1$, and we take $A_{l,\chi}$ to be the empty set if $\chi(g_l) \neq 1$. We define $B_{l,\chi}$ in a similar way by considering the cover $E_{l,b_l} \to F_l$. We have the following lemma.

Lemma 3.1. *For $\chi \in \langle g_l \rangle^\perp$ denote by $M^{-1}_{l,\chi}$ the eigensheaf of $\mathcal{O}_{\widetilde{E_l}}$ corresponding to χ. Then the maps $\widetilde{E_l} \to E_{l,a_l}$ and $\widetilde{E_l} \to E_{l,b_l}$ induce isomorphisms:*

$$L^{-1}_{a_l,\chi} \otimes \mathcal{O}_{F_l} \cong M^{-1}_{l,\chi}(-A_{l,\chi}), \quad L^{-1}_{b_l,\chi} \otimes \mathcal{O}_{F_l} \cong M^{-1}_{l,\chi}(-B_{l,\chi}).$$

Proof. The lemma follows by the normalization algorithm of [Par91, § 3]. □

Let $N_{l,\chi} := A_{l,\chi} \cap B_{l,\chi}$ and let T_χ be the set of points y such that C is singular at y and $\chi|_{H_y}$ is trivial. We are now ready to describe $(\operatorname{im}\alpha)_\chi$.

Proposition 3.2. *For every $\chi \in G^* \setminus \{1\}$, there is an exact sequence:*

$$0 \to (\operatorname{im}\alpha)_\chi \longrightarrow \oplus_{\{l|\chi(g_l)=1\}} M^{-1}_{l,\chi}(-N_{l,\chi}) \longrightarrow \mathcal{O}_{T_\chi} \to 0.$$

Proof. In our setup, the map $\widetilde{B'} \to \widetilde{C'}$ is the disjoint union of two copies of $\widetilde{B} = \bigsqcup_{l=1}^s \widetilde{E_l} \to \bigsqcup_{l=1}^s F_l$ that are switched by the involution j. So by Lemma 3.1 the first sequence in (16) can be rewritten as:

$$0 \to \mathcal{F}_\chi \to \oplus_{i=1}^t L^{-1}_{i,\chi} \to \oplus_{\{l|\chi(g_l)=1\}} M^{-1}_{l,\chi}. \tag{17}$$

In addition, if F_l is a component of C contained in Y_{a_l} and Y_{b_l}, then again by Lemma 3.1 the image of the map $L^{-1}_{a_l,\chi} \oplus L^{-1}_{b_l,\chi} \to M^{-1}_{l,\chi}$ is equal to $M^{-1}_{l,\chi}(-N^l_\chi)$, so we have an exact sequence:

$$0 \to (\operatorname{im}\alpha)_\chi \to \oplus_{\{l|\chi(g_l)=1\}} M^{-1}_{l,\chi}(-N^l_\chi) \to \mathcal{C}_\chi \to 0, \tag{18}$$

where the cokernel $\mathcal{C}_\chi$ is concentrated on the set T_χ. Using the description of the singularities of X at these points given in § 3.1, one checks that $\mathcal{C}_\chi$ has length 1 at points y such that $\chi|_{H_y}$ is trivial and it is 0 elsewhere, so $\mathcal{C}_\chi = \mathcal{O}_{T_\chi}$. □

Remark 3.3. Let $y \in C$ be a smooth point, let F be the irreducible component of C that contains y, and let Y_1, Y_2 be the two components of Y that contain F. Let H the subgroup of G generated by the inertia subgroups of F and of the components of D that contain y. Of course, one has $H \subseteq H_y$, but in the present setup equality actually holds. Indeed, if $\chi \in H^\perp$ is a non-trivial character, then by Proposition 3.2 the second sequence in (16) can be written near y as $0 \to \mathcal{F}_\chi \to \mathcal{O}_{Y_1} \oplus \mathcal{O}_{Y_2} \xrightarrow{\alpha_\chi} \mathcal{O}_F \to 0$, where α_χ is given by $(f_1, f_2) \mapsto (f_1 - f_2)|_F$. By Lemma 1.5, there exist $z_i \in \mathcal{O}_{Y_i}$, $i = 1, 2$, that correspond to functions on X'_i that do not vanish at any point of $\pi^{-1}(y)$. Up to multiplying, say, z_1 by a nowhere-vanishing regular function on Y_1 we can arrange that $z_\chi := z_1 - z_2 \in \mathcal{F}_\chi$. Hence z_χ corresponds to a function on X that is non-zero near $\pi^{-1}(y)$ and on which G acts via the character χ. It follows that G/H acts freely on $\pi^{-1}(y)$, i.e. that $H = H_y$.

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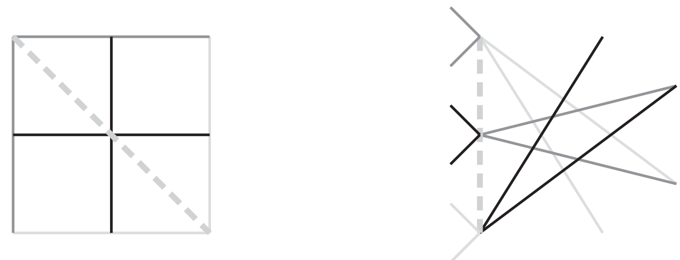

Figure 1. The $\mathbb{Z}_2^2$-cover of Example 3.5.

We say that a point $y \in C$ is *relevant* if and only if either it is singular for C or there exists l, χ with $\chi(g_l) = 1$ such that $y \in N^l_\chi$. Observe that, in view of the assumptions of 3.1, by Proposition 2.12 and by the description of singularities of § 3.4 the set of relevant points can be described intrinsically as the set of points of C over which X is Gorenstein but not dc.

Corollary 3.4. *Let* Rel *be the set of relevant points and let* $\widetilde{B} = \bigsqcup_{l=1}^s \widetilde{E_l}$ *be the normalization of the double locus* B *of* X*. Then*

$$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{X'}) - \chi(\mathcal{O}_{\widetilde{B}}) + \sum_{y \in \mathrm{Rel}} [G : H_y].$$

Proof. The claim follows immediately by Proposition 3.2, by (16) and by the fact that for $\chi = 1$ one has the exact sequence

$$0 \to (\operatorname{im} \alpha)_1 \to \oplus_{l=1}^s \mathcal{O}_{F_l} \to \mathcal{O}_T \to 0,$$

where T is the set of singular points of C. □

We close this section by computing the numerical invariants of two of the degenerations of Burniat surfaces described in [AP09].

Example 3.5. Let $G = \mathbb{Z}_2^2$, let g_1, g_2, g_3 be the non-zero elements of G, and for $i = 1, 2, 3$ let $\chi_i \in G^*$ be the non-zero character such that $\chi_i(g_i) = 1$. Let $Y_1 = \mathbb{P}^1 \times \mathbb{P}^1$, $Y_2 = \mathbb{P}^2$, and let Y be the surface obtained by gluing Y_1 and Y_2 along a smooth rational curve C which is of type $(1, 1)$ on Y_1 and is a line on Y_2. Fix three distinct points $y_1, y_2, y_3 \in C$. For $i = 1, 2, 3$, let $D_{1,j} \subset Y_1$ be the union of a fibre and a section through y_{j-1} and let $D_{2,j} \subset Y_2$ be a pair of lines through y_{j+1} (the index j varies in $\mathbb{Z}_3$). In Figure 1, Y_1 is represented on the left and Y_2 on the right, the curve C is shown as a solid dashed line, light gray lines correspond to $D_{i,1}$, black lines correspond to $D_{i,2}$, and medium gray lines correspond to $D_{i,3}$.

For $i = 1, 2$, we let $\pi_i : X_i' \to Y_i$ be the standard G-cover with branch data $D_{i,j}$, g_j, $j = 1, 2, 3$. Solving (2), we get $L_{1,i} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$ and $L_{2,j} = \mathcal{O}_{\mathbb{P}^2}(2)$, $j = 1, 2, 3$, where $L_{i,j}^{-1}$ denotes the subsheaf of $\mathcal{O}_{X_i'}$ corresponding to the character χ_j. Notice that the line bundles $L_{i,j}^{-1}$ have no cohomology, and hence, in particular, $\chi(\mathcal{O}_{X_1'}) = \chi(\mathcal{O}_{X_2'}) = 1$.

By [Par91, § 3], for $i = 1, 2$ the normalization of the cover of C induced by π_i is the trivial G-cover. So, by Theorem 1.13, we can glue $X_1' \sqcup X_2' \to Y_1 \sqcup Y_2$ to a cover $\pi : X \to Y$. By (15) we have

$$K_X^2 = 4(K_{Y_1} + \tfrac{1}{2}(D_{1,1} + D_{1,2} + D_{1,3}) + C)^2 + 4(K_{Y_2} + \tfrac{1}{2}(D_{2,1} + D_{2,2} + D_{2,3}) + C)^2 = 2 + 4 = 6.$$

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Figure 2. The $\mathbb{Z}_2^2$-cover of Example 3.6.

The curve C is smooth and the points y_1, y_2 and y_3 are relevant points with $H_{y_i} = G$, so Corollary 3.4 gives:

$$\begin{aligned}\chi(\mathcal{O}_X) &= \chi(\mathcal{O}_{X'_1}) + \chi(\mathcal{O}_{X'_2}) - \chi(\mathcal{O}_{\widetilde{B}}) \\ &\quad + [G : H_{y_1}] + [G : H_{y_2}] + [G : H_{y_3}] = 1 + 1 - 4 + 1 + 1 + 1 = 1.\end{aligned}$$

For $\chi = 1$, we have an isomorphism $(\operatorname{im}\alpha)_1 \cong \mathcal{O}_C$. Hence $(\operatorname{im}\alpha)_1$ has no cohomology in degree $i > 0$, and the exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_{Y_1} \oplus \mathcal{O}_{Y_2} \to (\operatorname{im}\alpha)_1 = \mathcal{O}_C \to 0$$

implies that $h^i(\mathcal{O}_Y) = 0$ for $i > 0$. Next we compute the cohomology of the sheaves $\mathcal{F}_\chi$. By Proposition 3.2, for $j = 1, 2, 3$ we have $(\operatorname{im}\alpha)_{\chi_j} = \mathcal{O}_C(-y_j)$. Hence (16) gives an exact sequence:

$$0 \to \mathcal{F}_{\chi_j} \to L_{1,j}^{-1} \oplus L_{2,j}^{-1} \to \mathcal{O}_C(-y_j) \to 0.$$

Therefore $h^1(\mathcal{F}_{\chi_j}) = h^2(\mathcal{F}_{\chi_j}) = 0$ for $j = 1, 2, 3$, and thus $h^1(\mathcal{O}_X) = h^2(\mathcal{O}_X) = 0$.

Example 3.6. Let $Y = Y_1 \cup \cdots \cup Y_6$ be the union of six copies of $\mathbb{P}^2$ glued in a cycle along lines as shown in Figure 2.

As in the previous example, let $G = \mathbb{Z}_2^2$, and for $i \in \mathbb{Z}_6$ let $\pi_i : X'_i \to Y_i$ be the G-cover branched on the lines pictured with three shades of gray in Figure 2. For every i, two of the sheaves $L_{i,\chi}$ are $\mathcal{O}_{Y_1}(2)$, and the remaining one is $\mathcal{O}_{Y_1}(1)$. Hence the $L_{i,\chi}^{-1}$ have no cohomology, and $\chi(X'_i) = 1$. It's easy to check using Theorem 1.13 that the cover $X'_1 \sqcup \cdots \sqcup X'_6 \to Y_1 \sqcup \cdots \sqcup Y_6$ can be glued to a G-cover $\pi : X \to Y$. The normalization $\widetilde{B} \to C$ of the induced cover of the double curve C is the disjoint union of six smooth rational curves, each mapping two-to-one onto a component of C. The only relevant point is the singular point y of C. So, applying (15) and Corollary 3.4, we get

$$K_X^2 = 6, \quad \chi(\mathcal{O}_X) = 1.$$

Let $F_1, \ldots, F_6$ be the irreducible components of C. For $\chi = 1$, as in the proof of Corollary 3.4 we have an exact sequence,

$$0 \to (\operatorname{im}\alpha)_1 \to \oplus_{l=1}^{6} \mathcal{O}_{F_l} \to \mathbb{K}(y) \to 0,$$

which gives $h^i((\operatorname{im}\alpha)_1) = 0$ for $i > 0$. By Proposition 3.2, for $\chi \neq 0$ the sheaf $(\operatorname{im}\alpha)_\chi$ is isomorphic to the direct sum of two copies of $\mathcal{O}_{\mathbb{P}^1}$, and hence it has no higher cohomology. So by (16) we have $h^i(\mathcal{F}_\chi) = 0$ for $i > 0$, and therefore $h^1(\mathcal{O}_X) = h^2(\mathcal{O}_X) = 0$.

3.3 Singularities: the case Y smooth.

We wish to describe the singularities of a $\mathbb{Z}_2^r$-cover $\pi : X \to Y$ as in § 3.1. Since the question is local, we fix $y \in Y$ and we study X locally above Y in the étale topology. By the assumptions in § 3.1, the singularities of X over a point $y \in Y$ lying on $q > 2$ components of Y are degenerate

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Table 1. One, two, three, and four reduced lines.

No.	$\|H\|$	Relations	ι	Singularity
0.1	1	none	1	smooth
1.1	2	none	1	smooth
2.1	4	none	1	smooth
2.2	2	12	1	A_1
3.1	8	none	1	A_1
3.2	4	12	1	A_3
3.3	4	123	2	$\frac{1}{4}(1,1)$
3.4	2	12,13	1	D_4
4.1	16	none	1	elliptic, $F^2=-4$
4.2	8	12	1	elliptic, $F^2=-2$
4.3	8	123	2	$T_{2,2,2,2}$, $F^2=-4$
4.4	8	1234	1	elliptic, $F^2=-8$
4.5	4	12 13	1	elliptic, $F^2=-1$
4.6	4	12 34	1	elliptic, $F^2=-4$
4.7	4	12 134	2	$T_{2,2,2,2}$, $F^2=-3$
4.8	2	12 13 14	1	elliptic, $F^2=-2$

cusps such that the exceptional divisor of its minimal semiresolution is a cycle of q rational curves (cf. [KS88, Definition 4.20]). So it is enough to analyze two cases.

- The surface Y is smooth.
- The surface $Y = Y_1 \cup Y_2$ dc and π is obtained by gluing standard covers $\pi_i : X_i' \to Y_i$, $i = 1, 2$.

Remark 3.7. All the singularities listed in Tables 1–9, actually occur on some stable surface of general type. To give examples of the singularities that appear when the base Y of the cover is smooth, one can take $G = \mathbb{Z}_2^r$, $2 \leqslant r \leqslant 4$, a set of generators $g_1, \ldots, g_k$ of G, $k \leqslant 4$, and lines $L_1, \ldots, L_k$ through a point $y \in \mathbb{P}^2$ such that the pair $(\mathbb{P}^2, (L_1 + \cdots + L_k)/2)$ is lc. If $g = g_i$, define $D_{g_i} = L_i$, where D_i' is a general curve of even degree, and for $g \neq 1, g_1, \ldots, g_k$ let D_g be a general curve of odd degree. The divisors D_g so defined are the branch data for a G-cover $X \to \mathbb{P}^2$ (the relations in (2) are easily seen have a solution in this case). By Proposition 2.5, the surface X is slc and it is of general type as soon as the degree of the Hurwitz divisor D is greater than 6. There is only one point $x \in X$ mapping to y; all the singularities (X, x) with $|H| \geqslant 4$ listed in Tables 1–3 can be realized in this way (for the definition of H, see below). The singularities with $|H| = 2$ can be obtained by taking a double cover $X \to \mathbb{P}^2$, branched on the sum of k lines through y and a general curve of degree d such that $d + k$ is even and greater than or equal to 8.

Since all the curves in the construction are general, the singularities of $X\backslash\{x\}$ are at most A_1 points.

Similar constructions, slightly more involved, can be used to realize the singularities of Tables 4–9.

We study the case Y smooth in this section, and the case Y reducible in § 3.4.

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Table 2. Double line + zero, one, or two reduced lines.

No.	$\|H\|$	Relations	ι	Singularity	$\widetilde{X}$	$C_{\widetilde{X}} \to C_X \to C_Y$	X^{sr}
$2'.1$	4	none	1	semismooth	2(1.1)	$2\Delta \to \Delta \to \Delta$	dc
$2'.2$	2	12	1	semismooth	2(0.1)	$2\Delta \to \Delta \to \Delta$	dc
$3'.1$	8	none	1	semismooth	2(2.1)	$2\Delta \to \Delta \xrightarrow{2} \Delta$	dc
$3'.2$	4	12	1	semismooth	2(1.1)	$2\Delta \to \Delta \xrightarrow{2} \Delta$	dc
$3'.3$	4	13	1	semismooth	(2.1)	$\Delta \xrightarrow{2} \Delta \to \Delta$	pinch
$3'.4$	4	123	2	$(3'.1)/\mathbb{Z}_2$	2(2.2)	$2\Delta \to \Delta \to \Delta$	dc
$3'.5$	2	12 13	1	semismooth	(1.1)	$\Delta \xrightarrow{2} \Delta \to \Delta$	pinch
$4'.1$	16	none	1	deg.cusp(2)	2(3.1)	$2\Gamma_2 \to \Gamma_2 \xrightarrow{22} \Delta$	dc
$4'.2$	8	12	1	deg.cusp(2)	2(2.1)	$2\Gamma_2 \to \Gamma_2 \xrightarrow{22} \Delta$	dc
$4'.3$	8	13	1	deg.cusp(1)	(3.1)	$\Gamma_2 \to \Delta \xrightarrow{2} \Delta$	dc
$4'.4$	8	34	1	deg.cusp(6)	2(3.2)	$2\Gamma_2 \to \Gamma_2 \to \Delta$	dc
$4'.5$	8	123	2	$(4'.1)/\mathbb{Z}_2$	2(3.2)	$2\Delta \to \Delta \xrightarrow{2} \Delta$	dc
$4'.6$	8	134	2	$(4'.1)/\mathbb{Z}_2$	(3.1)	$\Gamma_2 \xrightarrow{22} \Gamma_2 \to \Delta$	pinch
$4'.7$	8	1234	1	deg.cusp(2)	2(3.3)	$2\Gamma_2 \to \Gamma_2 \to \Delta$	dc
$4'.8$	4	12 13	1	deg.cusp(1)	(2.1)	$\Gamma_2 \to \Delta \xrightarrow{2} \Delta$	dc
$4'.9$	4	13 14	1	deg.cusp(3)	(3.2)	$\Gamma_2 \to \Delta \to \Delta$	dc
$4'.10$	4	12 34	1	deg.cusp(2)	2(2.2)	$2\Gamma_2 \to \Gamma_2 \to \Delta$	dc
$4'.11$	4	13 24	1	deg.cusp(1)	(3.3)	$\Gamma_2 \to \Delta \to \Delta$	dc
$4'.12$	4	12 134	2	$(4'.2)/\mathbb{Z}_2$	(2.1)	$\Gamma_2 \xrightarrow{22} \Gamma_2 \to \Delta$	pinch
$4'.13$	4	13 124	2	$(4'.3)/\mathbb{Z}_2$	(3.2)	$\Delta \xrightarrow{2} \Delta \to \Delta$	pinch
$4'.14$	4	123 34	2	$(4'.4)/\mathbb{Z}_2$	2(3.4)	$2\Delta \to \Delta \to \Delta$	dc
$4'.15$	2	12 13 14	1	deg.cusp(1)	(2.2)	$\Gamma_2 \to \Delta \to \Delta$	dc

Table 3. Two double lines.

No.	$\|H\|$	Relations	ι	Singularity	$\widetilde{X}$	$C_{\widetilde{X}} \to C_X \to C_Y$	X^{sr}
$4''.1$	16	none	1	deg.cusp(4)	4(2.1)	$4\Gamma_2 \to \Gamma_4 \xrightarrow{2222} \Gamma_2$	dc
$4''.2$	8	12	1	deg.cusp(4)	4(1.1)	$4\Gamma_2 \to \Gamma_4 \xrightarrow{2211} \Gamma_2$	dc
$4''.3$	8	13	1	deg.cusp(2)	2(2.1)	$2\Gamma_2 \to \Gamma_2 \xrightarrow{22} \Gamma_2$	dc
$4''.4$	8	123	2	$(4''.1)/\mathbb{Z}_2$	2(2.1)	$2\Gamma_2 \xrightarrow{1122} \Gamma_3 \xrightarrow{211} \Gamma_2$	pinch
$4''.5$	8	1234	1	deg.cusp(4)	4(2.2)	$4\Gamma_2 \to \Gamma_4 \to \Gamma_2$	dc
$4''.6$	4	12 13	1	deg.cusp(2)	2(1.1)	$2\Gamma_2 \to \Gamma_2 \xrightarrow{21} \Gamma_2$	dc
$4''.7$	4	12 34	1	deg.cusp(4)	4(0.1)	$4\Gamma_2 \to \Gamma_4 \to \Gamma_2$	dc
$4''.8$	4	13 24	1	deg.cusp(2)	2(2.2)	$2\Gamma_2 \to \Gamma_2 \to \Gamma_2$	dc
$4''.9$	4	12 134	2	$(4''.2)/\mathbb{Z}_2$	2(1.1)	$2\Gamma_2 \xrightarrow{2211} \Gamma_3 \to \Gamma_2$	pinch
$4''.10$	4	13 124	2	$(4''.3)/\mathbb{Z}_2$	(2.1)	$\Gamma_2 \xrightarrow{22} \Gamma_2 \to \Gamma_2$	pinch
$4''.11$	2	12 13 14	1	deg.cusp(2)	2(0.1)	$2\Gamma_2 \to \Gamma_2 \to \Gamma_2$	dc

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Table 4. C not in the branch locus, zero, or two, or four reduced lines.

No.	$\|H\|$	Relations	ι	χ	Singularity	$\widetilde{X}$	$C_{\widetilde{X}} \to C_X \to C_Y$	X^{sr}
E0.1	1	none	1	0	dc	$(0.1) \sqcup (0.1)$	$2\Delta \to \Delta \to \Delta$	dc
E2.1	2	12	1	0	dc	$(1.1) \sqcup (1.1)$	$2\Delta \to \Delta \xrightarrow{2} \Delta$	dc
E4.1	8	1234	1	2^{r-3}	deg.cusp(4)	$2(2.1) \sqcup 2(2.1)$	$2\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_4 \xrightarrow{2222} \Delta$	dc
E4.2	4	12 34	1	2^{r-2}	deg.cusp(4)	$2(2.2) \sqcup 2(2.2)$	$2\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_4 \to \Delta$	dc
E4.3	4	13 24	1	2^{r-2}	deg.cusp(2)	$(2.1) \sqcup (2.1)$	$\Gamma_2 \sqcup \Gamma_2 \to \Gamma_2 \xrightarrow{22} \Delta$	dc
E4.4	2	12 13 14	1	2^{r-1}	deg.cusp(2)	$(2.2) \sqcup (2.2)$	$\Gamma_2 \sqcup \Gamma_2 \to \Gamma_2 \to \Delta$	dc

Table 5. C not in the branch locus, a double line + two reduced lines.

No.	$\|H\|$	Relations	ι	χ	Singularity	$\widetilde{X}$	$C_{\widetilde{X}} \to C_X \to C_Y$	X^{sr}
E4′.1	8	1234	1	2^{r-3}	deg.cusp(6)	$4(1.1) \sqcup 2(2.1)$	$4\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_6 \xrightarrow{112\ldots2} \Gamma_2$	dc
E4′.2	4	12 34	1	2^{r-2}	deg.cusp(6)	$4(0.1) \sqcup 2(2.2)$	$4\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_6 \to \Gamma_2$	dc
E4′.3	4	13 24	1	2^{r-2}	deg.cusp(3)	$2(1.1) \sqcup (2.1)$	$2\Gamma_2 \sqcup \Gamma_2 \to \Gamma_3 \xrightarrow{122} \Gamma_2$	dc
E4′.4	2	12 13 14	1	2^{r-1}	deg.cusp(3)	$2(0.1) \sqcup (2.2)$	$2\Gamma_2 \sqcup \Gamma_2 \to \Gamma_3 \to \Gamma_2$	dc

Table 6. C not in the branch locus, two pairs of double lines.

No.	$\|H\|$	Relations	ι	χ	Singularity	$\widetilde{X}$	$C_{\widetilde{X}} \to C_X \to C_Y$	X^{sr}
E4″.1	8	1234	1	2^{r-3}	deg.cusp(8)	$4(1.1) \sqcup 4(1.1)$	$4\Gamma_2 \sqcup 4\Gamma_2 \to \Gamma_8 \xrightarrow{112\ldots211} \Gamma_3$	dc
E4″.2	4	12 34	1	2^{r-2}	deg.cusp(8)	$4(0.1) \sqcup 4(0.1)$	$4\Gamma_2 \sqcup 4\Gamma_2 \to \Gamma_8 \to \Gamma_3$	dc
E4″.3	4	13 24	1	2^{r-2}	deg.cusp(4)	$2(1.1) \sqcup 2(1.1)$	$2\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_4 \xrightarrow{1221} \Gamma_3$	dc
E4″.4	2	12 13 14	1	2^{r-1}	deg.cusp(4)	$2(0.1) \sqcup 2(0.1)$	$2\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_4 \to \Gamma_3$	dc

We let $(D_1, g_1), \ldots, (D_k, g_k)$ be the branch data of π. We may assume that $y \in D_i$ for every i. So, by the condition that D is slc, we have $k \leqslant 4$ and no three of the D_i coincide. Whenever the D_i are not all distinct, we assume $D_1 = D_2$.

All the possible cases are listed in Tables 1–3. The first digit in the label given to each case is equal to the number k of components through y, followed by $'$ if $D_1 = D_2$ and by $''$ if $D_1 = D_2$ and $D_3 = D_4$ (obviously this case occurs only for $k = 4$). So, for instance, a label of the form $3'.m$, where m is any positive integer, means that y belongs to three components of D, two of which coincide.

The entries in the columns have the following meanings.

- The column marked $|H|$ contains the order of the subgroup H the subgroup generated by $g_1, \ldots, g_k$.
- The column marked *Relations* contains the relations between $g_1, \ldots, g_k$. For instance, 123 means $g_1 + g_2 + g_3 = 0$.
- *Singularity.* The notations are mostly standard: $\frac{1}{4}(1, 1)$ denotes a cyclic singularity $\mathbb{A}^2/\mathbb{Z}_4$ with weights 1,1. $T_{2,2,2,2}$ denotes an arrangement consisting of four disjoint -2-curves $G_1, \ldots, G_4$ and of a smooth rational curve F intersecting each of the G_i transversely at one point. The self-intersection F^2 is given in the table. In the non-normal case (Tables 2 and 3) we use the notations of [KS88], where Kollár and Shepherd-Barron classified all slc surface singularities over $\mathbb{C}$. We work in any characteristic not equal to 2, but only

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Table 7. C in the branch locus, zero, or two, or four reduced lines.

No.	$\|H\|$	Relations	ι	χ	Singularity	$\widetilde{X}$	$C_{\widetilde{X}} \to C_X \to C_Y$	X^{sr}
R0.1	2	none	1	0	dc	$(1.1) \sqcup (1.1)$	$\Delta \sqcup \Delta \to \Delta \to \Delta$	dc
R2.1	4	12	1	0	dc	$(2.1) \sqcup (2.1)$	$\Delta \sqcup \Delta \to \Delta \xrightarrow{2} \Delta$	dc
R2.3	2	12 01	2	0	$(\mathrm{R}2.1)/\mathbb{Z}_2$	$(2.2) \sqcup (2.2)$	$\Delta \sqcup \Delta \to \Delta \to \Delta$	dc
R2.2	4	012			same as R2.1			
R4.1	16	1234	1	2^{r-4}	deg.cusp(4)	$2(3.1) \sqcup 2(3.1)$	$2\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_4 \xrightarrow{2\dots2} \Delta$	dc
R4.2	8	1234 01	2	0	$(\mathrm{R}4.1)/\mathbb{Z}_2$	$2(3.2) \sqcup (3.1)$	$2\Delta \sqcup \Gamma_2 \to \Gamma_2 \xrightarrow{22} \Delta$	dc
R4.3	8	1234 012	1	2^{r-3}	deg.cusp(4)	$2(3.3) \sqcup 2(3.3)$	$2\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_4 \to \Delta$	dc
R4.4	8	1234 013	1	2^{r-3}	deg.cusp(2)	$(3.1) \sqcup (3.1)$	$\Gamma_2 \sqcup \Gamma_2 \to \Gamma_2 \xrightarrow{22} \Delta$	dc
R4.5	8	12 34	1	2^{r-3}	deg.cusp(12)	$2(3.2) \sqcup 2(3.2)$	$2\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_4 \to \Delta$	dc
R4.6	4	12 34 01	2	0	$(\mathrm{R}4.5)/\mathbb{Z}_2$	$2(3.4) \sqcup (3.2)$	$2\Delta \sqcup \Gamma_2 \to \Gamma_2 \to \Delta$	dc
R4.7	4	12 34 013	1	2^{r-2}	deg.cusp(6)	$(3.2) \sqcup (3.2)$	$\Gamma_2 \sqcup \Gamma_2 \to \Gamma_2 \to \Delta$	dc
R4.8	8	13 24			same as R4.4			
R4.9	4	13 24 01	2	0	$(\mathrm{R}4.8)/\mathbb{Z}_2$	$(3.2) \sqcup (3.2)$	$\Delta \sqcup \Delta \to \Delta \xrightarrow{2} \Delta$	dc
R4.10	4	13 24 012	1	2^{r-2}	deg.cusp(2)	$(3.3) \sqcup (3.3)$	$\Gamma_2 \sqcup \Gamma_2 \to \Gamma_2 \to \Delta$	dc
R4.11	4	12 13 14			same as R4.7			
R4.12	2	12 13 14 01	2	0	$(\mathrm{R}4.11)/\mathbb{Z}_2$	$(3.4) \sqcup (3.4)$	$\Delta \sqcup \Delta \to \Delta$	dc
R4.13	16	01234			same as R4.1			
R4.14	8	12 034	1	2^{r-3}	deg.cusp(8)	$2(3.2) \sqcup 2(3.3)$	$2\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_4 \to \Delta$	dc
R4.15	8	13 024			same as R4.4			
R4.16	8	123 04			same as R4.2			
R4.17	4	12 13 014	1	2^{r-2}	deg.cusp(4)	$(3.2) \sqcup (3.3)$	$\Gamma_2 \sqcup \Gamma_2 \to \Gamma_2 \to \Delta$	dc
R4.18	4	12 134 01	2	0	$(\mathrm{R}4.14)/\mathbb{Z}_2$	$2(3.4) \sqcup (3.3)$	$2\Delta \sqcup \Gamma_2 \to \Gamma_2 \to \Delta$	dc
R4.19	4	13 124 01			same as R4.9			

the singularities from the list in [KS88] appear. The notation 'deg.cusp(k)' means a degenerate cusp (cf. [KS88, Definition 4.20]) such that the exceptional divisor in the minimal semiresolution has k components.

- The column marked ι contains the index of $x \in X$. It is equal to 1 if all the relations have even length and it is equal to 2 otherwise (cf. Proposition 2.8).
- The column marked $\widetilde{X}$ describes the normalization of X (the entries refer to the cases in Table 1).
- The column marked $C_{\widetilde{X}} \to C_X \to C_Y$ describes the inverse image in $\widetilde{X}$ of the double curve C_X of X and C_Y is the image of C_X in Y. The symbol Δ denotes the germ of a smooth curve, and Γ_k is the seminormal curve obtained by gluing k copies of Δ at one point. The notation $\Gamma_k \xrightarrow{a_1,\dots,a_k} C$ means that the map restricts to a degree a_i map on the ith component of Γ_k (we do not specify the a_i when they are all equal to 1).
- The column marked X^{sr} describes the minimal semiresolution of X. We write 'dc' when X^{sr} has only normal crossings and 'pinch' if it has also pinch points.

Theorem 3.8. *The singularities of* slc *covers* $\pi : X \to Y$ *with smooth* Y *are listed in Tables 1–3.*

Since all these singularities can be studied in a similar way, we just explain the method and work out two cases as an illustration. We start with some general remarks.

(i) We always assume $G = H$. Indeed, the cover π factors as $X \xrightarrow{\pi_2} X/H \xrightarrow{\pi_1} Y$. By Lemma 1.5, the map π_1 is étale near y, while for every $z \in \pi_1^{-1}(y)$ the fiber $\pi_2^{-1}(z)$ consists

Table 8. C in the branch locus, a double line + two reduced lines.

No.	$\|H\|$	Relations	ι	χ	Singularity	$\widetilde{X}$	$C_{\widetilde{X}} \to C_X \to C_Y$	X^{sr}
R4′.1	16	1234	1	2^{r-4}	deg.cusp(6)	$4(2.1) \sqcup 2(3.1)$	$4\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_6 \xrightarrow{2\ldots2} \Gamma_2$	dc
R4′.2	8	1234 01	2	0	$(\mathrm{R4}'.1)/\mathbb{Z}_2$	$2(2.1) \sqcup (3.1)$	$2\Gamma_2 \sqcup \Gamma_2 \xrightarrow{221111} \Gamma_4 \xrightarrow{1122} \Gamma_2$	pinch
R4′.3	8	1234 03	2	0	$(\mathrm{R4}'.1)/\mathbb{Z}_2$	$2(2.1) \sqcup 2(3.2)$	$2\Gamma_2 \sqcup 2\Delta \to \Gamma_3 \xrightarrow{222} \Gamma_2$	dc
R4′.4	8	1234 012	1	2^{r-3}	deg.cusp(6)	$4(2.2) \sqcup 2(3.3)$	$4\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_6 \to \Gamma_2$	dc
R4′.5	8	1234 013	1	2^{r-3}	deg.cusp(3)	$2(2.1) \sqcup (3.1)$	$2\Gamma_2 \sqcup \Gamma_2 \to \Gamma_3 \xrightarrow{222} \Gamma_2$	dc
R4′.6	8	12 34	1	2^{r-3}	deg.cusp(10)	$4(1.1) \sqcup 2(3.2)$	$4\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_6 \xrightarrow{221\ldots1} \Gamma_2$	dc
R4′.7	4	12 34 01	2	0	$(\mathrm{R4}'.6)/\mathbb{Z}_2$	$2(1.1) \sqcup (3.2)$	$2\Gamma_2 \sqcup \Gamma_2 \xrightarrow{221\ldots1} \Gamma_4 \to \Gamma_2$	pinch
R4′.8	4	12 34 03	2	0	$(\mathrm{R4}'.6)/\mathbb{Z}_2$	$2(1.1) \sqcup 2(3.4)$	$2\Gamma_2 \sqcup 2\Delta \to \Gamma_3 \xrightarrow{211} \Gamma_2$	dc
R4′.9	4	12 34 013	1	2^{r-2}	deg.cusp(5)	$2(1.1) \sqcup (3.2)$	$2\Gamma_2 \sqcup \Gamma_2 \to \Gamma_3 \xrightarrow{211} \Gamma_2$	dc
R4′.10	8	13 24			same as R4′.5			
R4′.11	4	13 24 01	2	0	$(\mathrm{R4}'.10)/\mathbb{Z}_2$	$(2.1) \sqcup (3.2)$	$\Gamma_2 \sqcup \Delta \xrightarrow{211} \Gamma_2 \xrightarrow{12} \Gamma_2$	
R4′.12	4	13 24 012	1	2^{r-2}	deg.cusp(3)	$2(2.2) \sqcup (3.3)$	$2\Gamma_2 \sqcup \Gamma_2 \to \Gamma_3 \to \Gamma_2$	dc
R4′.13	4	12 13 14			same as R4′.9			
R4′.14	2	12 13 14 01	2	0	$(\mathrm{R4}'.13)/\mathbb{Z}_2$	$(1.1) \sqcup (3.4)$	$\Gamma_2 \sqcup \Delta \xrightarrow{211} \Gamma_2 \to \Gamma_2$	pinch
R4′.15	8	13 024			same as R4′.5			
R4′.16	8	12 034	1	2^{r-3}	deg.cusp(6)	$4(1.1) \sqcup 2(3.3)$	$4\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_6 \xrightarrow{221\ldots1} \Gamma_2$	dc
R4′.17	8	13 024			same as R4′.5			
R4′.18	8	34 012	1	2^{r-3}	deg.cusp(10)	$4(2.2) \sqcup 2(3.2)$	$4\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_6 \to \Gamma_2$	dc
R4′.19	8	123 04			same as R4′.3			
R4′.20	8	134 02			same as R4′.2			
R4′.21	4	12 13 014	1	2^{r-2}	deg.cusp(3)	$2(1.1) \sqcup (3.3)$	$2\Gamma_2 \sqcup \Gamma_2 \to \Gamma_3 \xrightarrow{211} \Gamma_2$	dc
R4′.22	4	13 14 012	1	2^{r-2}	deg.cusp(5)	$2(2.2) \sqcup (3.2)$	$2\Gamma_2 \sqcup \Gamma_2 \to \Gamma_3 \to \Gamma_2$	dc
R4′.23	4	12 134 01	2	0	$(\mathrm{R4}'.16)/\mathbb{Z}_2$	$2(1.1) \sqcup (3.3)$	$2\Gamma_2 \sqcup \Gamma_2 \to \Gamma_3 \xrightarrow{211} \Gamma_2$	pinch
R4′.24	4	13 124 01			same as R4′.11			
R4′.25	4	34 123 03	2	0	$(\mathrm{R4}'.18)/\mathbb{Z}_2$	$2(2.2) \sqcup 2(3.4)$	$2\Gamma_2 \sqcup 2\Delta \to \Gamma_3 \to \Gamma_2$	dc

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TABLE 9. C in the branch locus, two pairs of double lines.

No.	$\|H\|$	Relations	ι	χ	Singularity	$\widetilde{X}$	$C_{\widetilde{X}} \to C_X \to C_Y$	X^{sr}
R4″.1	16	1234	1	2^{r-4}	deg.cusp(8)	$4(2.1) \sqcup 4(2.1)$	$4\Gamma_2 \sqcup 4\Gamma_2 \to \Gamma_8 \xrightarrow{2\ldots2} \Gamma_3$	dc
R4″.2	8	1234 01	2	0	(R4″.1)/$\mathbb{Z}_2$	$2(2.1) \sqcup 2(2.1)$	$2\Gamma_2 \sqcup 2\Gamma_2 \xrightarrow{221\ldots1} \Gamma_5 \xrightarrow{11222} \Gamma_3$	pinch
R4″.3	8	1234 012	1	2^{r-3}	deg.cusp(8)	$4(2.2) \sqcup 4(2.2)$	$4\Gamma_2 \sqcup 4\Gamma_2 \to \Gamma_8 \to \Gamma_3$	dc
R4″.4	8	1234 013	1	2^{r-3}	deg.cusp(4)	$2(2.1) \sqcup 2(2.1)$	$2\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_4 \xrightarrow{2222} \Gamma_3$	dc
R4″.5	8	12 34	1	2^{r-3}	deg.cusp(8)	$4(1.1) \sqcup 4(1.1)$	$4\Gamma_2 \sqcup 4\Gamma_2 \to \Gamma_8 \xrightarrow{22111122} \Gamma_3$	dc
R4″.6	4	12 34 01	2	0	(R4″.5)/$\mathbb{Z}_2$	$2(1.1) \sqcup 2(1.1)$	$2\Gamma_2 \sqcup 2\Gamma_2 \xrightarrow{221\ldots1} \Gamma_5 \xrightarrow{11112} \Gamma_3$	pinch
R4″.7	4	12 34 013	1	2^{r-2}	deg.cusp(4)	$2(1.1) \sqcup 2(1.1)$	$2\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_4 \xrightarrow{2112} \Gamma_3$	dc
R4″.8	8	13 24			same as R4″.4			
R4″.9	4	13 24 01	2	0	(R4″.8)/$\mathbb{Z}_2$	$(2.1) \sqcup (2.1)$	$\Gamma_2 \sqcup \Gamma_2 \xrightarrow{212} \Gamma_3 \xrightarrow{121} \Gamma_3$	pinch
R4″.10	4	13 24 012	1	2^{r-2}	deg.cusp(4)	$2(2.2) \sqcup 2(2.2)$	$2\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_4 \to \Gamma_3$	dc
R4″.11	4	12 13 14			same as R4″.7			
R4″.12	2	12 13 14 01	2	0	(R4″.11)/$\mathbb{Z}_2$	$(1.1) \sqcup (1.1)$	$\Gamma_2 \sqcup \Gamma_2 \xrightarrow{2112} \Gamma_3 \to \Gamma_3$	pinch
R4″.13	16	01234			same as R4″.1			
R4″.14	8	12 034	1	2^{r-3}	deg.cusp(8)	$4(1.1) \sqcup 4(2.2)$	$4\Gamma_2 \sqcup 4\Gamma_2 \to \Gamma_8 \xrightarrow{221\ldots1} \Gamma_3$	dc
R4″.15	8	13 024			same as R4″.4			
R4″.16	8	123 04			same as R4″.2			
R4″.17	4	12 13 014	1	2^{r-2}	deg.cusp(4)	$2(1.1) \sqcup 2(2.2)$	$2\Gamma_2 \sqcup 2\Gamma_2 \to \Gamma_4 \xrightarrow{2111} \Gamma_3$	dc
R4″.18	4	12 134 01	2	0	(R4″.14)/$\mathbb{Z}_2$	$2(1.1) \sqcup 2(2.2)$	$2\Gamma_2 \sqcup 2\Gamma_2 \xrightarrow{221\ldots1} \Gamma_5 \to \Gamma_3$	pinch
R4″.19	4	13 124 01			same as R4″.9			

only of one point. Since G acts transitively on each fiber of π, it is enough to describe the singularity of X above any point $z\in\pi_1^{-1}(x)$.

(ii) The cover X is normal at x if and only if $[D]=0$. It is non-singular at x if and only if either $k=1$ or $k=2$, $D_1\neq D_2$, $g_1\neq g_2$. Assume that X is not normal, and let F be an irreducible divisor that appears in D with multiplicity one. This means that, say, $F=D_1$ and $F=D_2$. The normalization of X along F is a G-cover of Y with branch data (D_i,g_i), for $i\neq 1,2$, and, if $g_1+g_2\neq 0$, (F,g_1+g_2) (cf. [Par91, § 3]).

(iii) The cover X is said to be *simple* if the *set* $\{g_1,\dots,g_k\}$ is a basis of $|H|$ (for instance, X is simple if the g_i are all equal). In this case, X is a complete intersection, and it is very easy to write down equations for it (see Case $4'.1$ below).

(iv) The double curve C_X maps onto the divisors that appear in D with multiplicity equal to one. Since for a semismooth surface the double curve is locally irreducible, X is never semismooth in the cases $4''$. In addition, if X is semismooth then the pullback $C_{\widetilde{X}}$ of C_X to the normalization is smooth. Using this remark, it is easy to check that X is never semismooth in the cases $4'$, either.

(v) In order to compute the minimal semiresolution X^{sr}, we consider the blow up $\widehat{Y}\to Y$ of Y at y, pull back X and normalize along the exceptional curve E to get a cover $\widehat{X}\to\widehat{Y}$. The branch locus of $\widehat{X}\to\widehat{Y}$ is supported on a dc divisor and, by construction, the singularities of $\widehat{X}$ are only of type 1, 2 or $3'$. Looking at the tables, one sees that either $\widehat{X}$ is semismooth or it has points of type 2.2 or $3'.4$ (cf. Table 1). In the former case, $\widehat{X}$ is the minimal semiresolution. In the latter case, blowing up $\widehat{Y}$ at the non-semismooth points and taking base change and normalization along the exceptional divisor, one gets a semismooth cover $\widehat{\widehat{X}}\to\widehat{\widehat{Y}}$. The semiresolution $\widehat{\widehat{X}}\to X$ is minimal, except in cases $4''.5$, $4''.10$. In these cases the minimal semiresolution X^{sr} is obtained by contracting the inverse image in $\widehat{\widehat{X}}$ of the exceptional curve of the blow up $\widehat{Y}\to Y$.

Next we analyze in detail two cases.

Case $4'.1$. By remark (ii) above, the normalization $\widetilde{X}$ is an H-cover with branch data (D_1,g_1+g_2), (D_3,g_3) and (D_4,g_4). Hence g_1 acts on X without fixed points and X is the disjoint union of two copies of the cover (3.1). We choose local parameters u,v on Y such that $D_1=D_2$ is given by $u=0$, D_3 is defined by $v=0$ and D_4 by $u+v=0$.

The cover X is defined étale locally above y by the following equations:

$$z_1^2=u,\quad z_2^2=u,\quad z_3^2=v,\quad z_4^2=(u+v). \tag{19}$$

In particular, X is a complete intersection (see remark (iii) above). The element g_i acts on z_j as multiplication by $(-1)^{\delta_{ij}}$. The double curve C_X is the inverse image of $u=0$, hence it is defined by $z_1=z_2=0$, $z_3=\pm z_4$ and the map $C_X\to D_1$ is given by $z_3\mapsto z_3^2$, so C_X is isomorphic to Γ_2, with each component mapping 2-to-1 to $D_1\simeq\Delta$. The curve $C_{\widetilde{X}}$ is the inverse image of D_1 in $\widetilde{X}$, so it has two connected components, each isomorphic to Γ_2, that are glued together in the map $\widetilde{X}\to X$.

To compute the minimal semiresolution, consider the blow up $\widehat{Y}\to Y$ of Y at y and the cover $\widehat{X}\to\widehat{Y}$ obtained by pulling back $X\to Y$ and normalizing along the exceptional curve E. The branch data for $\widehat{X}$ are $(E,g_1+g_2+g_3+g_4)$ and, for $i=1,\dots,4$, $(\widehat{D_i},g_i)$, where $\widehat{\ }$ indicates the strict transform. The cover is singular precisely above $\widehat{D_1}=\widehat{D_2}$, and it is easy, using the local equations, to check that it is dc there. Hence $\widehat{X}$ is the minimal semiresolution of X. The exceptional divisor is the inverse image F of E in X. Applying the normalization algorithm to the restricted cover $F\to E$, one sees that the normalization $\widetilde{F}$ of F is the union of two smooth rational curves F_1 and F_2. The map $\widetilde{F} \to F$ identifies the two points of F_1 that lie over the point $E \cap D_1'$ with the corresponding two points of F_2. Hence $\widehat{X}$ is the minimal semiresolution of X and the singularity is a degenerate cusp solved by a cycle of two rational curves.

Case $4'.5$. As in the previous case, $\widetilde{X}$ and $C_{\widetilde{X}}$ can be computed by the normalization algorithm. One obtains that $\widetilde{X}$ is the disjoint union of two copies of (3.2) and $C_{\widetilde{X}}$ is the disjoint union of two copies of Δ. This singularity is the quotient of a cover X_0 of type $(4'.1)$ by the element $g_0 := g_1 + g_2 + g_3$. Since this element has odd length, the index ι of X at x is equal to 2.

Since the only fixed point of g_0 on X is $x := \pi^{-1}(y)$, the double curve C_X is the quotient of the double curve C_{X_0} of X_0. The two components of C_{X_0} are identified by g_0, and thus C_X is irreducible and maps two-to-one onto D_1.

To compute the minimal semiresolution, again we blow up $\widehat{Y} \to Y$ at y and consider the cover $\widehat{X} \to \widehat{Y}$ obtained by pull back and normalization along the exceptional curve E. As usual, we denote by $\widehat{F}$ the strict transform on $\widehat{Y}$ of a curve F of Y. The branch data for $\widehat{X}$ are $(\widehat{D_1}, g_1)$, $(\widehat{D_2}, g_2)$, $(\widehat{D_3}, g_1+g_2)$, $(\widehat{D_4}, g_4)$, and (E, g_4). Hence $\widehat{X}$ has normal crossings over $\widehat{D_1}$, it has four A_1 points over the point $\hat{y} := \widehat{D_4} \cap E$, and it is smooth elsewhere (cf. Tables 1 and 2). We blow up at $\hat{y}$ and take again pull back and normalization along the exceptional curve E_2. We obtain a cover $\widehat{\widehat{X}} \to \widehat{\widehat{Y}}$ which is dc over the strict transform $\widehat{\widehat{D_1}}$ of $\widehat{D_1}$ and has no other singularity, so $\widehat{\widehat{X}} \to X$ is a semismooth resolution. Let E_1 denote the strict transform on $\widehat{\widehat{Y}}$ of the exceptional curve E of the first blow up. Arguing as in Case $4'.1$, one sees that the inverse image of E_1 is the union of two smooth rational curves F_1^1 and F_2^1 that intersect transversely precisely at one point of the double curve, and the inverse image of E_2 consists of four disjoint curves $F_2^1, \dots, F_2^4$. All these curves pull back to -2 curves on the normalization of $\widehat{\widehat{X}}$ and, up to relabeling, F_1^1, F_1^2, F_2^2 and F_1^2, F_2^3, F_2^4 form two disjoint A_3 configurations. Hence $\widehat{\widehat{X}}$ is the minimal semiresolution of X. In the notation of [KS88, Definition 4.26], $\widehat{\widehat{X}}$ is obtained by gluing two copies of (A, Δ) along Δ.

3.4 Singularities: the case Y reducible

Here we repeat the local analysis of the previous section for the case in which $Y = Y_1 \cup Y_2$ is dc, keeping as far as possible the same notations. So we fix $y \in C$, where C is the double curve of Y, and describe X locally over y. We assume that $X \to Y$ is obtained by gluing standard covers $\pi_i : X_i' \to Y_i$, $i = 1, 2$, such that y lies on all the components of the Hurwitz divisor D. We let $(D_1, g_1), \dots, (D_k, g_k)$ be the union of the branch data of π_1 and π_2 such that D_i is distinct from the double curve C of Y (hence $D = (D_1 + \cdots + D_k)/2$). We denote by g_0 the generator of the inertia subgroup of C for π_1 and π_2. By Remark 3.3, the inertia subgroup H_y is equal to $H := \langle g_0, g_1, \dots, g_k \rangle$, so up to an étale cover we may assume that $G = H$ and that $\pi^{-1}(y) = \{x\}$.

Since D is $\mathbb{Q}$-Cartier, there are the same number of D_i on Y_1 and on Y_2. We order them so that all components on Y_1 come first. Recall that $k \leqslant 4$ by the assumption that (Y, D) is slc. The cases in the tables are labeled E ('étale') if $g_0 = 0$ and R ('ramified') if $g_0 \neq 0$. The first digit of the label is the number k of branch lines through y. It is followed by $'$ if $D_1 = D_2$ and by $''$ if $D_1 = D_2$ and $D_3 = D_4$. For instance, in the cases E$4'.m$ the map π is generically étale over C and there are four branch lines $D_1, \dots, D_4$ with $D_1 = D_2$, and $D_3 \neq D_4$.

The singularities that we get here are non-normal, and as in [KS88, Theorems 4.21, 4.23] they turn out to be either semismooth or degenerate cusps in the Gorenstein case and $\mathbb{Z}_2$-quotients of these otherwise.

The tables here contain the same columns as those of § 3.3 plus an extra one, denoted χ: this is the contribution of y in the formula for $\chi(\mathcal{O}_X)$ of Corollary 3.4 (recall $|G| = 2^r$). By Propositions 2.11 and 2.12 the index ι is equal to 1 if all relations have even length when reduced modulo g_0 and it is equal to 2 otherwise.

Theorem 3.9. *The singularities of* slc *covers* $\pi : X \to Y$ *where* Y *is the dc union of two smooth surfaces are given in Tables 4–9.*

The analysis of the singularities in the reducible case is similar to the case Y smooth. One blows up Y at the point y and takes pull back and normalization of X along the exceptional divisor. Repeating this process, if necessary, one obtains a semiresolution $X_0 \to X$. If X_0 is not minimal, then the minimal semiresolution $X^{\mathrm{sr}} \to X$ is obtained by blowing down the -1-curves of X_0.

As the computations are all similar, we work out a only a couple of cases to show the method.

Case R4′.1. The normalization $\widetilde{X}$ is equal to $\widetilde{X'_1} \sqcup \widetilde{X'_2}$, where $\widetilde{X'_i}$ is the normalization of X'_i. The branch data of $\widetilde{X'_1} \to Y_1$ are $(D_1, g_1 + g_2)$, (D_0, g_0), so $\widetilde{X'_1}$ is étale locally the disjoint union of four copies of the cover (2.1). Also, $X'_2 = \widetilde{X'_2}$ is étale locally the disjoint union of two copies (3.1).

The image C_Y of the double curve C_X is equal to $C \cup D_1$. The preimage in $\widetilde{X'_1}$ of C_Y is the disjoint union of four copies of Γ_2. The preimage of C_Y in $\widetilde{X'_2}$ is equal to two copies of Γ_2. Hence $C_{\widetilde{X}} = 4\Gamma_2 \sqcup 2\Gamma_2$. Each component of $C_{\widetilde{X}}$ maps two-to-one onto its image. The map $C_{\widetilde{X}} \to C_X$ identifies in pairs the four components of the preimage of D_1 and the eight components of the preimage of C. Hence C_X is Γ_6, with two components mapping two-to-one onto D_1 and four components mapping two-to-one onto C.

To compute the semiresolution, blow up $y \in Y$ to get $\widehat{Y} \to Y$. Let $E_1 \subset Y_1$ and $E_2 \subset Y_2$ be the irreducible components of the exceptional divisor. Let $\widehat{\pi} : \widehat{X} \to \widehat{Y}$ be the G-cover obtained from $X \to Y$ by taking pull back and normalizing along E_1 and E_2. Denoting by $\widehat{\ }$ the strict transform on $\widehat{Y}$, the branch data of $\widehat{\pi}$ are $(E_1, g_0 + g_1 + g_2)$, $(E_2, g_0 + g_3 + g_4 = g_0 + g_1 + g_2)$, $(\widehat{D_1}, g_1)$, $(\widehat{D_2} = \widehat{D_1}, g_2)$, $(\widehat{D_3}, g_3)$ and $(\widehat{D_4}, g_4)$, $(\widehat{C}, g_0)$. Hence $\widehat{X}$ is dc by the tables of § 3.3, and it is therefore the semiresolution X^{sr} of X. The preimage of E_1 is the union of four smooth rational curves meeting in pairs over the point $E_1 \cap \widehat{D_1}$. The preimage of E_2 is the disjoint union of two rational curves, which together with the components of the preimage of E_1 form a cycle of six rational curves. The singularity $x \in X$ is Gorenstein by Proposition 2.12, and hence it is 'deg.cusp(6)'.

Case R4′.2. This is a $\mathbb{Z}_2$-quotient of R4′.2, and it is not Gorenstein by Proposition 2.11. The normalization $\widetilde{X}$ is equal to $\widetilde{X'_1} \sqcup \widetilde{X'_2}$, where $\widetilde{X'_i}$ is the normalization of X'_i. The branch data of $\widetilde{X'_1} \to Y$ are $(D_1, g_0 + g_2)$, (D_0, g_0), so $\widetilde{X'_1}$ is étale locally the disjoint union of two copies of the cover (2.1). The image C_Y of the double curve C_X is equal to $C \cup D_1$. The preimage in $\widetilde{X_1}$ of C_Y is the disjoint union of two copies of Γ_2. The preimage of C_Y in $\widetilde{X'_2}$ is Γ_2. Hence $C_{\widehat{X}} = 2\Gamma_2 \sqcup \Gamma_2$. Each component of $C_{\widetilde{X}}$ maps two-to-one onto its image in C_Y. The map $C_{\widetilde{X}} \to C_X$ glues to itself each of the two components of the preimage of D_1, and it identifies in pairs the four components of the preimage of C. Hence C_X is Γ_4, with two components mapping one-to-one onto D_1 and two components mapping two-to-one onto C.

To compute the semiresolution, blow up $y \in Y$ to get $\widehat{Y} \to Y$. Let $E_1 \subset Y_1$ and $E_2 \subset Y_2$ be the irreducible components of the exceptional divisor. Let $\widehat{\pi} : \widehat{X} \to \widehat{Y}$ be the G-cover obtained from $X \to Y$ by taking pull back and normalizing along E_1 and E_2. Denoting by $\widehat{\ }$ the strict transform on $\widehat{Y}$, the branch data of $\widehat{\pi}$ are (E_1, g_2), $(E_2, g_0 + g_3 + g_4 = g_2)$, $(\widehat{D_1}, g_1 = g_0)$, $(\widehat{D_2} = \widehat{D_1}, g_2)$,

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$(\widehat{D_3}, g_3)$, $(\widehat{D_4}, g_4)$ and $(\widehat{C}, g_0)$. By the tables of § 3.3, $\widehat{X}$ has two pinch points over the point $\widehat{D_1} \cap E_1$ and is at most dc elsewhere; hence it is equal to the minimal semiresolution X^{sr}. The preimage of E_1 is a pair of smooth rational curves meeting over the point $E_1 \cap \widehat{D_1}$. The preimage of E_2 is a smooth rational curve, meeting each component of the preimage of E_1 at a point lying over $\widehat{C} \cap E_1 = \widehat{C} \cap E_2$.

In the notation of [KS88, Definition 4.26], X^{sr} is a chain consisting of copy of $(A, 2\Delta)$ (namely the second component of X^{sr}) in the middle and two copies of $(A, 2\Delta)$ with Δ pinched at the ends.

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