15

D-branes and geometry II

In a number of the previous chapters, we probed various systems while remaining largely in the limit where D-branes are pointlike in their transverse directions. However, we learned in chapter 10 that D-branes have an intrinsic geometry of their own, which can be seen when we place a lot of them together to produce a large back-reaction on the spacetime geometry, or if we were to turn up the string coupling (for fixed string tension) such that Newton's constant is strong. Both sorts of situation can and will be forced upon us later, so it is worthwhile trying to understand what we can learn by probing the supergravity geometry with different types of branes (we have already probed extremal p-branes with Dp-branes in section 10.3). If we choose things such that there is some supersymmetry preserved, we can use it to help us learn many useful things.

15.1 Probing p with D$(p-4)$

Let us probe the geometry of the extremal p-branes with a D$(p-4)$-brane. From our analysis of chapter 11, we know that this system is supersymmetric. Therefore, we expect that there should still be a trivial potential for the result of the probe computation, but there is not enough supersymmetry to force the metric to be flat. There are actually two sectors within which the probe brane can move transversely. Let us choose static gauge again, with the probe aligned so that its $p-4$ spatial directions $\xi^1-\xi^{p-4}$ are aligned with the directions x^1-x^{p-4}. Then there are four transverse directions *within* the p-brane background, labelled $x^{p-3}-x^p$, and which we can call $x^i_\parallel$ for short. There are $9-p$ remaining transverse directions which are transverse to the p-brane as well, labelled $x^{p+1}-x^9$, which we'll abbreviate to $x^m_\perp$. The 6–2 case is tabulated as a visual guide below.

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	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
D2-brane	—	—	—	•	•	•	•	•	•	•
6-brane	—	—	—	—	—	—	—	•	•	•

The extremal p-brane supergravity solution is given in equation (10.38). As in section 10.3, we can probe this solution with D-branes, using the world-volume actions described in chapters 5 and 9. Following the same lines of reasoning as used in section 10.3, the determinant which shall go into our Dirac–Born–Infeld Lagrangian is:

$$\det[-G_{ab}] = Z_p^{-\frac{(p-3)}{2}}\left(1 - v_\parallel^2 - Z_p v_\perp^2\right), \tag{15.1}$$

where the velocities come from the time (ξ^0) derivatives of $x_\parallel$ and $x_\perp$. This is nice, since in forming the action by multiplying by the exponentiated dilaton factor and expanding in small velocities, we get the Lagrangian

$$\mathcal{L} = \frac{1}{2}m_{p-4}\left(v_\parallel^2 + Z_p v_\perp^2 - 2\right), \tag{15.2}$$

which again has a constant potential which we can discard, leaving pure kinetic terms. We see that there is a purely flat metric on the moduli space for the motion inside the four dimensions of the p-brane geometry, while there is a metric

$$ds^2 = Z_p(r)\delta_{mn}dx^m dx^n, \tag{15.3}$$

for the transverse motion. This is the Coulomb branch, in gauge theory terms, and the flat metric was on the Higgs branch. (In fact, the Higgs result does not display all of the richness of this system that we have seen. In addition to the flat metric geometry inside the brane that we see here, there is additional geometry describing the Dp–D$(p-4)$ fields corresponding to the full instanton geometry. This comes from the fact that the D$(p-4)$-brane behaves as an instanton of the non-Abelian gauge theory on the world-volume of the coincident Dp-branes. See section 13.4.)

Notice that for the fields we have studied, we obtained a trivial potential for free without having to appeal to a cancellation due to the coupling of the charge μ_{p-4} of the probe. This is good, since there is no electric source of this in the background for it to couple to. Instead, the form of the solution for the background makes it force-free automatically.

15.2 Probing six-branes: Kaluza–Klein monopoles and M-theory

Actually, when $p \geq 5$, something interesting happens. The electric source of $C_{(p+1)}$ potential in the background produces a *magnetic* source of

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$C_{(7-p)}$. The rank of this is low enough for there to be a chance for the D$(p-4)$-probe brane to couple to it even in the Abelian theory. For example, for $p = 5$ there is a magnetic source of C_2 to which the D1-brane probe can couple. Meanwhile for $p = 6$, there is a magnetic source of C_1. The D2-brane probes see this in an interesting way. Let us linger here to study this case a bit more closely. Since there is always a trivial $U(1)$ gauge field on the world volume of a D2-brane probe, corresponding to the centre of mass of the brane, we should include the coupling of the world-volume gauge potential A_a (with field strength F_{ab}) to any of the fields coming from the background geometry.

In fact, as we saw before in section 9.2, there is a coupling

$$2\pi\alpha'\mu_2\int_M C_1\wedge F, \tag{15.4}$$

where $C_1 = C_\phi d\phi$ is the magnetic potential produced by the six-brane background geometry, which is easily computed to be: $C_\phi = -(r_6/g_{\rm s})\cos\theta$, where $r_6 = g_{\rm s}N\alpha'^{1/2}/2$.

The gauge field on the world volume is equivalent to one scalar, since we may exchange A_a for a scalar s by Hodge duality in the (2+1)-dimensional world-volume. (This is of course a feature specific to the p=2 case.) To get the coupling for this extra scalar correct, we should augment the probe computation. As we have seen, the Dirac–Born–Infeld action is modified by an extra term in the determinant:

$$-\det g_{ab}\to-\det(g_{ab}+2\pi\alpha' F_{ab}). \tag{15.5}$$

We can[143, 171] introduce an auxiliary vector field v_a, replacing $2\pi\alpha' F_{ab}$ by the combination $e^{2\phi}\mu_2^{-2}v_av_b$ in the Dirac action, and adding the term

$$2\pi\alpha'\int_M F\wedge v$$

overall. Treating v_a as a Lagrange multiplier, the path integral over v_a will give the action involving F as before. Alternatively, we may treat F_{ab} as a Lagrange multiplier, and integrating it out enforces

$$\epsilon^{abc}\partial_b(-\mu_2\hat{C}_c+v_c)=0. \tag{15.6}$$

Here, $\hat{C}_c$ are the components of the pull-back of C_1 to the probe's world-volume. The solution to the constraint above is

$$-\mu_2\hat{C}_a+v_a=\partial_a s, \tag{15.7}$$

where s is our dual scalar. We may now replace v_a by $\partial_a s+\mu_2\hat{C}_a$ in the action.

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The static gauge computation picks out only $\dot{s}+\mu_2 C_\phi\dot{\phi}$, and recomputing the determinant gives

$$\det = Z_6^{-\frac{3}{2}}\left(1-v_\parallel^2 - Z_6 v_\perp^2 - \frac{Z_6^{\frac{1}{2}}e^{2\Phi}}{\mu_2^2}\left[\dot{s}+\mu_2 C_\phi\dot{\phi}\right]^2\right). \tag{15.8}$$

Again, in the full Dirac–Born–Infeld action, the dilaton factor cancels the prefactor exactly, and including the factor of $-\mu_2$ and the trivial integral over the worldvolume directions to give a factor V_2, the resulting Lagrangian is

$$\mathcal{L} = \frac{1}{2}m_2(v_\parallel^2-2) + \frac{1}{2}V_2\left(\frac{\mu_2 Z_6}{g}v_\perp^2 + \frac{g_s}{\mu_2 Z_6}\left(\dot{s}+\mu_2 C_\phi\dot{\phi}\right)^2\right), \tag{15.9}$$

which is (after ignoring the constant potential) again a purely kinetic Lagrangian for motion in *eight* directions. There is a non-trivial metric in the part transverse to both branes:

$$ds^2 = V(r)\left(dr^2+r^2d\Omega^2\right) + V(r)^{-1}\left(ds+A_\phi d\phi\right)^2,$$
$$\text{with}\quad V(r)=\frac{\mu_2 Z_6}{g_\mathrm{s}}\quad\text{and}\quad A=\frac{\mu_2 r_6}{g_\mathrm{s}}\cos\theta d\phi, \tag{15.10}$$

where $d\Omega^2 = d\theta^2+\sin^2\theta\, d\phi^2$. There is a number of fascinating interpretations of this result. In pure geometry, the most striking feature is that there are now *eleven* dimensions for our spacetime geometry. The D2-brane probe computation has uncovered, in a very natural way, an extra transverse dimension. This extra dimension is compact, since s is periodic, which is inherited from the gauge invariance of the dual worldvolume gauge field. The radius of the extra dimension is proportional to the string coupling, which is also interesting. This eleventh dimension is of course the M-direction we saw in section 12.4. The D2-brane has revealed that the six-brane is a Kaluza–Klein monopole[168] of eleven dimensional supergravity on a circle[152], which is constructed out of a Taub–NUT geometry* in equation (15.10). This fits very well with the fact that the D6 is the Hodge dual of the D0-brane, which we already saw is a Kaluza–Klein electric particle.

15.3 The moduli space of 3D supersymmetric gauge theory

As before, the result also has a field theory interpretation. The $(2+1)$-dimensional $U(1)$ gauge theory (with eight supercharges) on the

* It is a very useful exercise for the reader to take the Taub–NUT metric, times seven flat directions, and use the reduction formula given in insert 12.1 (p. 274) to reproduce the six-brane metric of equation (10.38) directly.

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world-volume of the D2-brane has $N_{\rm f} = N$ extra hypermultiplets coming from light strings connecting it to the $N_{\rm f} = N$ D6-branes. The $SU(N_{\rm f})$ symmetry on the worldvolume of the D6-branes is a global 'flavour' symmetry of the $U(1)$ gauge theory on the D2-brane. A hypermultiplet Ψ has four components Ψ_i corresponding to the four scalar degrees of freedom given by the four positions $\Psi^i \equiv (2\pi\alpha')^{-1}x^i_{\parallel}$. The vector multiplet contains the vector A_a and three scalars $\Phi^m \equiv (2\pi\alpha')^{-1}x^m_{\perp}$. The Yang–Mills coupling is $g^2_{\rm YM} = g_{\rm s}\alpha'^{-1/2}$.

The branch of vacua of the theory with $\Psi \neq 0$ is called the 'Higgs' branch of vacua while that with $\Phi \neq 0$ constitutes the 'Coulomb' branch, since there is generically a $U(1)$ left unbroken. There is a non-trivial four dimensional metric on the Coulomb branch. This is made of the three Φ^m, and the dual scalar of the $U(1)$s photon. Let us focus on the quantities which survive in the low energy limit or 'decoupling limit' $\alpha' \to 0$, holding fixed any sensible gauge theory quantities which appear in our expressions. The surviving parts of the metric (15.10) are:

$$\begin{aligned} &ds^2 = V(U)(dU^2 + U^2 d\Omega_2^2) + V(U)^{-1}(d\sigma + A_\phi d\phi)^2 \\ &\text{where} \quad V(U) = \frac{1}{4\pi^2 g^2_{\rm YM}}\left(1 + \frac{g^2_{\rm YM}N_{\rm f}}{2U}\right); \qquad A_\phi = \frac{N_{\rm f}}{8\pi^2}\cos\theta, \end{aligned} \tag{15.11}$$

where $U = r/\alpha'$ has the dimensions of an energy scale in the gauge theory. Also, $\sigma = \alpha' s$, and we will fix its period shortly.

In fact, the naive tree level metric on the moduli space is that on $\mathbb{R}^3 \times S^1$, of form $ds^2 = g^{-2}_{\rm YM} dx^2_{\perp} + g^2_{\rm YM} d\sigma^2$. Here, we have the tree level and one loop result: $V(U)$ has the interpretation as the sum of the tree level and one-loop correction to the gauge coupling of the 2+1 dimensional gauge theory[237]. Note the factor $N_{\rm f}$ in the one loop correction. This multiplicity comes from the number of hypermultiplets which can run around the loop. Similarly, the cross term from the second part of the metric has the interpretation as a one-loop correction to the naive four dimensional topology, changing it to the (Hopf) fibred structure above.

Actually, the moduli space's dimension had to be a multiple of four, as it generally has to be hyper-Kähler for $D{=}2+1$ supersymmetry with eight supercharges[185]. Our metric is indeed hyper-Kähler since it is the Taub–NUT metric: the hyper-Kähler condition on the metric in the form it is written is the by-now familiar equation: $\nabla \times \mathbf{A} = \nabla V$, which is satisfied.

In fact, we are not quite done yet. With some more care we can establish some important facts quite neatly. We have not been careful about the period of σ, the dual to the gauge field, which is not surprising given all of the factors of 2, π and α'. To get it right is an important task, which will

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yield interesting physics. We can work it out in a number of ways, but the following is quite instructive. If we perform the rescaling $U = \rho/4g_{\rm YM}^2$ and $\psi = 8\pi^2\sigma/N_{\rm f}$, our metric is:

$$
\begin{aligned}
ds^2 &= \frac{g_{\rm YM}^2}{64\pi^2}\, ds_{\rm TN}^2, \qquad \text{where} \\
ds_{\rm TN}^2 &= \left(1+\frac{2N_{\rm f}}{\rho}\right)(d\rho^2+\rho^2 d\Omega_2^2) \\
&\quad + 4N_{\rm f}^2\left(1+\frac{2N_{\rm f}}{\rho}\right)^{-1}(d\psi+\cos\theta d\phi)^2,
\end{aligned}
\tag{15.12}
$$

which is a standard form for the Taub–NUT metric, with mass $N_{\rm f}$, equal to the 'nut parameter' for this self-dual case[186].

This metric is apparently singular at $\rho = 0$, and in fact, for the correct choice of periodicity for ψ, this pointlike structure, called a 'nut', is removable, just like the case of the bolt singularity encountered for the Eguchi–Hanson space. (See insert 7.6, p. 188.) Just for fun, insert 15.1 carries out the analysis and finds that ψ should have period 4π, and so in fact the full $SU(2)$ isometry of the metric is preserved.

What does this all have to do with gauge theory? Let us consider the case of $N_{\rm f} = 1$, which means one six-brane. This is 2+1 dimensional $U(1)$ gauge theory with one hypermultiplet, a rather simple theory. We

Insert 15.1. The 'nut' of Taub–NUT

The metric (15.12) will be singular at at the point $\rho = 0$, for arbitrary periodicity of ψ. This will be a pointlike singularity which is called a 'nut'[83, 82], in contrast to the 'bolt' we encountered for the Eguchi–Hanson space in insert 7.6 (p. 188), which was an S^2. In this case, near $\rho = 0$, if we make the space look like the *origin* of $\mathbb{R}^4$, we can make this pointlike structure into nothing but a coordinate singularity. Near $\rho = 0$, we have, for $R = 2\rho^2$ (see also insert 7.4, p. 180):

$$ds_{\rm TN}^2 = 2N_{\rm f}(dR^2 + R^2 d\Omega_3^2),$$

which is just the right metric for $\mathbb{R}^4$ if $\Delta\psi = 4\pi$, the standard choice for the Euler coordinate. (This may have seemed somewhat heavy-handed for a result one would perhaps have guessed anyway, but it is worthwhile seeing it, in preparation for more complicated examples.)

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see that after restoring the physical scales to our parameters, our original field σ has period $1/2\pi$, and so we see that the dual to the photon is more sensibly defined as $\widetilde{\sigma} = 4\pi^2\sigma$, which would have period 2π, which is a more reasonable choice for a scalar dual to a photon. We shall use this choice later. With this choice, the metric on the Coulomb branch of moduli space is completely non-singular, as should be expected for such a simple theory.

Let us now return to arbitrary N_f. This means that we have N_f hypermultiplets, but still a $U(1)$ 2+1 dimensional gauge theory with a global 'flavour' symmetry of $SU(N_\mathrm{f})$ coming from the six-branes. There is no reason for the addition of hypermultiplets to change the periodicity of our dual scalar and so we keep it fixed and accept the consequences when we return to physical coordinates $(U, \widetilde{\sigma})$: *the metric on the Coulomb branch is singular at* $U = 0$*!* This is so because insert 15.1 told us to give $\widetilde{\sigma}$ a periodicity of $2\pi N_\mathrm{f}$ for freedom from singularities, but we are keeping it as 2π. So our metric in physical units has $\widetilde{\sigma}$ with period 2π appearing in the combination $(2d\widetilde{\sigma} + N_\mathrm{f}\cos\theta d\phi)^2$. This means that the metric is no longer has an $SU(2)$ acting, since the round S^3 has been deformed into a 'squashed' S^3, where the squashing is controlled by N_f. In fact, there is a deficit angle at the origin corresponding to an $A_{N_\mathrm{f}-1}$ singularity.

How are we to make sense of this singularity? Well, happily, this all fits rather nicely with the fact that for $N_\mathrm{f} > 1$ there is an $SU(N_\mathrm{f})$ gauge theory on the six-branes, and so there is a Higgs branch, corresponding to the D2-brane becoming an $SU(N_\mathrm{f})$ instanton! The singularity of the Coulomb branch is indeed a signal that we are at the origin of the Higgs branch, and it also fits that there is no singularity for $N_\mathrm{f} = 1$.

It is worthwhile carrying out this computation for the case of N_f D6-branes in the presence of a negative orientifold six-plane oriented in the same way. In that case we deduce from facts we learned before that the presence of the O6-plane gives global flavour group $SO(2N_\mathrm{f})$ for N_f D6-branes. The D2-brane, however, carries an $SU(2)$ gauge group. This is T-dual to the earlier statement made in section 13.4 about D9-branes in type I string theory carrying $SO(N_\mathrm{f})$ groups while D5s carry $USp(2M)$ groups as we learned in section 8.7: the orientifold forces a pair of D2-branes to travel as one, with a $USp(2) = SU(2)$ group.

So the story now involves 2+1 dimensional $SU(2)$ gauge theory with N_f hypermultiplets. The Coulomb branch for $N_\mathrm{f} = 0$ must be completely non-singular, since again there is no Higgs branch to join to. This fits with the fact that there are no D6-branes; just the O6-plane. The result for the metric on moduli space can be deduced from a study of the gauge theory (with the intuition gained from this stringy situation), and has been proven to be the Atiyah–Hitchin manifold[231]. Some of this will be

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discussed in more detail in subsection 15.6. For the case of $N_\mathrm{f} = 1$, the result is also non-singular (there is again no Higgs branch for one D6-brane) and the result is a certain cover of the Atiyah–Hitchin manifold[232, 248]. The case of general N_f gives certain generalisations of the Atiyah–Hitchin manifold[248, 250]. The manifolds have D_{N_f} singularities, consistent with the fact that there is a Higgs branch to connect to. Note also that a stringy interpretation of this result is that the strong coupling limit of these O6-planes is in fact M-theory on the Atiyah–Hitchin manifold, just like it is Taub–NUT for the D6-brane.

N.B. It is amusing to note – and the reader may have already spotted it – that the story above seems to be describing local pieces of K3, which has ADE singularities of just the right type, with the associated $SU(N)$ and $SO(2N)$ enhanced gauge symmetries appearing also (global flavour groups for the 2+1 dimensional theory here). (The existence of three new exceptional theories, for E_6, E_7, E_8, is then immediate[237].) What we are actually recovering is the fact[153] that there is a strong/weak coupling duality between type I (or $SO(32)$ heterotic) string theory on T^3 and M-theory on K3. We'll recover this fact again via another route in section 16.2.2.

15.4 Wrapped branes and the enhançon mechanism

Despite the successes we have achieved in the previous section with interpretation of supergravity solutions in terms of constituent D-branes, we should be careful, even in the case when we have supersymmetry to steer us away from potential pathologies. It is not always the case that if someone presents us with a solution of supergravity with R–R charges that we should believe that it has an interpretation as being 'made of D-branes'.

Consider again the case of eight supercharges. We studied brane configurations with this amount of supersymmetry by probing the geometry of N (large) Dp-branes with a single D$(p-4)$-brane. As described in previous sections, another simple way to achieve a geometry with eight supercharges from D-branes is to simply wrap branes on a manifold which already breaks half of the supersymmetry[117]. The example mentioned was the four dimensional case of K3. In this case, we learned that if we wrap a D$(p+4)$-brane (say) on K3, we induce precisely one unit of negative Dp-brane charge[115] supported on the unwrapped part of the world-volume (see equation (9.36)). At large N therefore, we might expect[239] that there is a simple supergravity geometry which might be obtained by

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taking the known solution for the D$(p+4)$-Dp system, and modifying the asymptotic charges to suit this situation. The resulting geometry naively should have the interpretation as that due to a large number N of wrapped D$(p+4)$ branes ($p = 1, 2, 3$):

$$\begin{aligned} ds^2 &= Z_p^{-1/2} Z_{p+4}^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + Z_p^{1/2} Z_{p+4}^{1/2} dx^i dx^i \\ &\qquad + V^{1/2} Z_p^{1/2} Z_{p+4}^{-1/2} ds^2_{\mathrm{K3}}, \\ e^{2\Phi} &= g_\mathrm{s}^2 Z_p{}^{(3-p)/2} Z_{p+4}{}^{-(p+1)/2}, \\ C_{(p+1)} &= (Z_p{}^{-1} - 1) g_\mathrm{s}^{-1} dx^0 \wedge dx^1 \wedge \cdots \wedge dx^{p+1} \\ C_{(p+5)} &= (Z_{p+4}^{-1} - 1) g_\mathrm{s}^{-1} dx^0 \wedge dx^1 \wedge \cdots \wedge dx^{p+5}. \end{aligned} \tag{15.13}$$

Here, μ, ν run over the $x^0 - x^{p+1}$ directions, which are tangent to all the branes. Also i runs over the directions transverse to all branes, $x^{p+2} - x^5$, and in the remaining directions, transverse to the induced brane but inside the large brane, ds^2_{K3} is the metric of a K3 surface of unit volume. V is the volume of the K3 as measured at infinity, but the supergravity solution adjusts itself such that $V(r){=}V Z_p/Z_{p+4}$ is the measured volume of the K3 at radius r.

In the above,

$$Z_{p+4} = 1 + \frac{r_{p+1}^{3-p}}{r^{3-p}}, \quad \text{while} \quad Z_p = 1 - \frac{(2\pi\sqrt{\alpha'})^4 r_{p+1}^{3-p}}{V r^{3-p}}, \tag{15.14}$$

where the normalisations are related to those in section 10.2. We have precisely N units of D$(p+4)$-brane charge and $-N$ units of Dp-brane charge. Note that the smaller brane is delocalised in the K3 directions, as it should be, since the same is true of K3's curvature.

15.4.1 Wrapping D6-branes

Let us focus on the case $p = 2$, where we wrap D6-branes, getting induced D2-branes.† The orientations are given as follows.

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
D2	—	—	—	•	•	•	•	•	•	•
D6	—	—	—	•	•	•	—	—	—	—
K3	—	—	—	—	—	—	•	•	•	•

† This will also teach us a lot about the pure $SU(N)$ gauge theory on the remaining 2+1 dimensional world-volume. Wrapping D7-branes ($p = 3$) teaches us[239] about pure $SU(N)$ gauge theory in 3+1 dimensions, where we should make a connection to Seiberg–Witten theory[240, 241] at large N.

The harmonic functions are

$$
\begin{aligned}
Z_2 &= 1+\frac{\hat{r}_2}{r}, \qquad \hat{r}_2 = -\frac{(2\pi)^4 g_{\rm s} N \alpha'^{5/2}}{2V}, \\
Z_6 &= 1+\frac{r_6}{r}, \qquad r_6 = \frac{g_{\rm s} N \alpha'^{1/2}}{2},
\end{aligned}
\tag{15.15}
$$

normalised such that the D2- and D6-brane charges are $Q_2{=}{-}Q_6=-N$.

We worked out the spectrum of type IIA supergravity theory compactified to six dimensions on K3 in section 7.6. Let us remind ourselves of some of the salient features. The six dimensional supergravity theory has an additional 24 $U(1)$s in the R–R sector. Of these, 22 come from wrapping the ten dimensional three-form on the 19+3 two-cycles of K3. The remaining two are special $U(1)$s for our purposes: one of them arises from wrapping IIAs five-form entirely on K3, while the final one is simply the plain one-form already present in the uncompactified theory.

15.4.2 The repulson geometry

It is easy to see that there is something wrong with the geometry which we have just written down, representing the wrapping of the D6-branes on the K3. There is a naked singularity at $r=|\hat{r}_2|$, known as the 'repulson', since‡ it represents a repulsive gravitational potential for small enough r. The curvature diverges there, which is related to the fact that the volume of the K3 goes to zero, and the geometry stops making sense (see figure 15.1).

To characterise the repulsive nature of the geometry we can consider it as a background for particle motion and study geodesics. There is the usual obvious pair of Killing vectors, $\boldsymbol{\xi}=\partial_t$ and $\boldsymbol{\eta}=\partial_\phi$, and so a probe with ten-velocity $\mathbf{u}$ has conserved quantities

$$
e = -\xi\cdot u = -G_{tt}u^t
$$

and

$$
\ell = \eta\cdot u = G_{\phi\phi}u^\phi,
$$

where e and ℓ are the total energy and angular momentum per unit mass, respectively. Since the particle is massive, we have $-1=u\cdot u$. In other words, picking

$$
\mathbf{u} = \left(\frac{dt}{d\tau},\frac{dr}{d\tau},\frac{d\theta}{d\tau},\frac{d\phi}{d\tau},\vec{0}\right),
$$

‡ This is because it is dual[239] to solutions which had earlier become known by that name[257].

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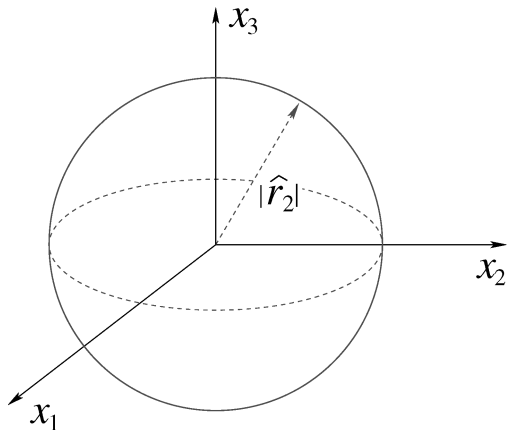

Fig. 15.1. The repulson locus of points; an unphysical naked singularity.

we have, working in the equatorial plane $\theta = \pi/2$,

$$-1 = -G_{tt}\left(\frac{dt}{d\tau}\right)^2 + G_{rr}\left(\frac{dr}{d\tau}\right)^2 + G_{\phi\phi}\left(\frac{d\phi}{d\tau}\right)^2,$$

and so

$$-1 = -\frac{e^2}{G_{tt}} + G_{rr}\left(\frac{dr}{d\tau}\right)^2 + \frac{\ell^2}{G_{\phi\phi}},$$

which we can rewrite as

$$E = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\rm eff},$$

where

$$\begin{aligned} \frac{dr}{d\tau} &= \pm\sqrt{E - V_{\rm eff}(r)}, \qquad E = \frac{e^2-1}{2}, \\ V_{\rm eff}(r) &= \frac{1}{2}\left[\frac{1}{G_{rr}}\left(1 + \frac{\ell^2}{G_{\phi\phi}}\right) - 1\right], \end{aligned} \tag{15.16}$$

and the metric components in the above are in string frame, and we have used that $-G_{tt} = 1/G_{rr}$. For what we wish to analyse, we can consider only purely radial motion, and hence set to zero the angular momentum ℓ which would correspond to a non-zero impact parameter. We sketch the resulting effective potential in figure 15.2.

For large enough r, the effective potential is attractive, and so we need only seek a vanishing first derivative of $V_{\rm eff}(r)$ to see where it becomes

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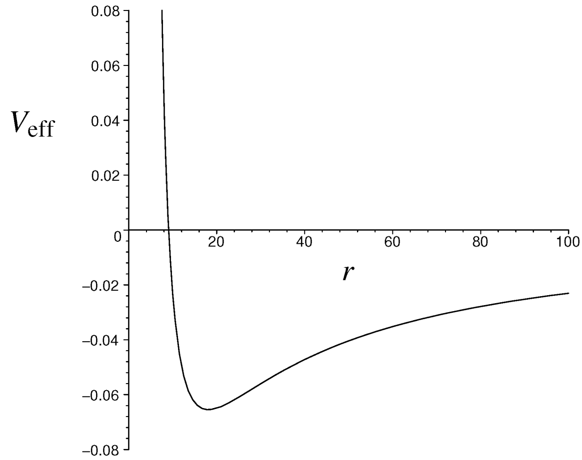

Fig. 15.2. The effective potential for massive particle motion in the geometry. The minimum is at $r = r_{\rm e}$.

negative. This gives the condition:

$$G_{rr}^{-1} = -G'_{tt} = 0, \tag{15.17}$$

which we can write in a number of interesting ways as

$$(Z_2Z_6)' = Z_6Z'_6\left(\frac{Z_2}{Z_6} + \frac{Z'_2}{Z'_6}\right) = Z_2Z_6\left(\frac{Z'_2}{Z_2} + \frac{Z'_6}{Z_6}\right) = 0, \tag{15.18}$$

and the particle begins to be repelled at radii smaller than this. Particles with non-zero angular momentum will of course experience additional centrifugal repulsion, but $r = r_{\rm e}$ is the boundary of the region where there is an intrinsic repulsion in the geometry.

15.4.3 Probing with a wrapped D6-brane

Let us look carefully to see if this is really the geometry produced by the wrapped branes. The object we have made should be a BPS membrane made of N identical objects. These objects feel no force due to each other's presence, and therefore the BPS formula for the total mass is simply (see equation (9.37))

$$\tau_N = \frac{N}{g_{\rm s}}(\mu_6 V - \mu_2) \tag{15.19}$$

with $\mu_6 = (2\pi)^{-6}\alpha'^{-7/2}$ and $\mu_2 = (2\pi)^{-2}\alpha'^{-3/2}$. In fact, the BPS membrane is actually a monopole of one of the six dimensional $U(1)$s. It is

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obvious which $U(1)$ this is; the diagonal combination of the two special ones we mentioned above. The D6-brane component is already a monopole of the IIA R–R one-form, and the D2 is a monopole of the five-form, which gets wrapped.

N.B. As we shall see, the final combination is a non-singular BPS monopole, having been appropriately dressed by the Higgs field associated to the volume of K3. Also, it maps[165] (under the strong/weak coupling duality of the type IIA string on K3 to the heterotic string on T^4) to a bound state of a Kaluza–Klein monopole[168] and an H-monopole[242], made by wrapping the heterotic NS5-brane.[239, 243, 244]

If we are to interpret our geometry as having been made by bringing together N copies of our membrane, we ought to be able to carry out the procedure we described in the previous sections. We should see that the geometry as seen by the probe is potential-free and well-behaved, allowing us the interpretation of being able to bring the BPS probe in slowly from infinity.

The effective action for a D6-brane probe (wrapped on the K3) is:

$$S = -\int_M d^3\xi\, e^{-\Phi(r)}(\mu_6 V(r) - \mu_2)(-\det g_{ab})^{1/2} + \mu_6\int_{M\times \mathrm{K3}} C_7 - \mu_2\int_M C_3. \tag{15.20}$$

Here M is the part of the world-volume in the three non-compact dimensions. As discussed previously (see equation (9.39) and surrounding discussion), the first term is the Dirac–Born–Infeld action with the position dependence of the tension (15.19) taken into account; in particular, $V(r) = VZ_2(r)/Z_6(r)$. The second and third terms are the couplings of the probe charges $(\mu_6, -\mu_2)$ to the background R–R potentials, following from equation (9.36), and surrounding discussion.

Having derived the action, the calculation proceeds very much as we outlined in previous sections, with the result:

$$\begin{aligned}\mathcal{L} = &-\frac{\mu_6 VZ_2 - \mu_2 Z_6}{Z_6 Z_2 g_\mathrm{s}} + \frac{\mu_6 V}{g_\mathrm{s}}(Z_6^{-1} - 1) - \frac{\mu_2}{g_\mathrm{s}}(Z_2^{-1} - 1)\\ &+ \frac{1}{2g_\mathrm{s}}(\mu_6 VZ_2 - \mu_2 Z_6)v^2 + O(v^4).\end{aligned} \tag{15.21}$$

The position-dependent potential terms cancel as expected for a supersymmetric system, leaving the constant potential $(\mu_6 V - \mu_2)/g$ and a

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nontrivial metric on moduli space (the $O(v^2)$ part) as expected with eight supersymmetries. The metric is proportional to

$$ds^2 = \frac{1}{g_\mathrm{s}}\left(\mu_6 V Z_2 - \mu_2 Z_6\right) dx_\perp^2 = \frac{\mu_6 Z_6}{g_s}\left(\frac{Z_2}{Z_6} - \frac{V_*}{V}\right)\left(dr^2 + r^2 d\Omega_2^2\right), \tag{15.22}$$

where we have used $\mu_2/\mu_6 = V_*$. We assume that $V > V_* \equiv (2\pi)^4\alpha'^2$, so that the metric at infinity (and the membrane tension) is positive. However, as r decreases the metric eventually becomes negative, and this occurs at a radius

$$r = \frac{2V}{V - V_*}|\hat{r}_2| \equiv r_\mathrm{e}, \tag{15.23}$$

which is greater than the radius $r_\mathrm{r} = |\hat{r}_2|$ of the repulson singularity. Furthermore, it is precisely the radius at which the geometry becomes repulsive, since $Z_2'/Z_6' = -V_*/V$, and that radius is determined by equation (15.18).

In fact, our BPS monopole is becoming massless as we approach the special radius. This should mean that the $U(1)$ under which it is charged is becoming enhanced to a non-Abelian group. This is the case. There is a purely stringy phenomenon which lies outside supergravity which we have not included thus far. The W-bosons are wrapped D4-branes, which enhance the $U(1)$ to an $SU(2)$. The masses of wrapped D4-branes is computed just like that of the membrane, and so becomes zero when the K3's volume reaches the value $V_* \equiv (2\pi\sqrt{\alpha'})^4$.

The point is that the repulson geometry represents supergravity's best attempt to construct a solution with the correct asymptotic charges. In the solution, the volume of the K3 decreases from its asymptotic value V as one approaches the core of the configuration. At the centre, the K3 radius is zero, and this is the singularity. This ignores rather interesting physics, however. At a finite distance from the putative singularity (where $V_{\mathrm{K3}} = 0$), the volume of the K3 gets to $V = V_*$, so the stringy phenomena – including new massless fields – giving the enhanced $SU(2)$, should have played a role[§]. So the aspects of the supergravity solution near and inside the special radius, called the 'enhançon radius', should not be taken seriously at all, since it ignored this stringy physics.

The supergravity solution should only be taken as physical down to the neighbourhood of the enhançon radius r_e. That locus of points, a two-sphere S^2, is itself called[239] an 'enhançon' (see figure 15.3).

[§] Actually, this enhancement of $SU(2)$ is even less mysterious in the heterotic-on-T^4 dual picture mentioned two pages ago. It is just the $SU(2)$ of a self-dual circle in this picture, which we studied extensively in section 4.3.

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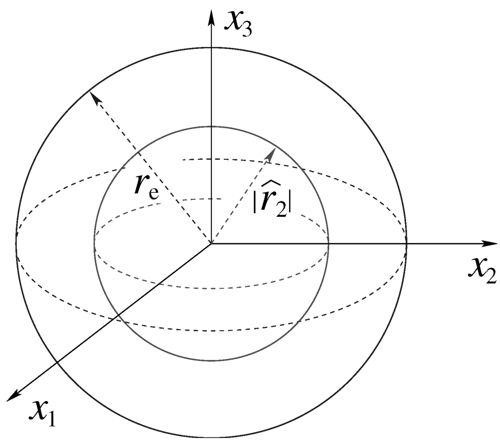

Fig. 15.3. The enhançon locus at which new physics beyond supergravity appears. This happens before the singular repulson locus, signalling that the original geometry inside the enhançon radius was unphysical.

Recall also (see section 13.5.1) that the *size* of the monopole is inverse to the mass of the W-bosons (or the Higgs vev), and so in fact by time our probe gets to the enhançon radius, it has smeared out considerably, and in fact merges into the geometry, forming a 'shell' with the other monopoles at that radius. Since by this argument we cannot place sharp sources inside the enhançon radius, and so the geometry on the inside must be very different from that of the repulson. In fact, to a first approximation, it must simply be flat, forming a junction with the outside geometry at $r = r_e$.

In general, the same sort of reasoning applies for all p. The enhançon locus results from wrapping a D$(p + 4)$-brane on K3 is $S^{4-p} \times \mathbb{R}^{p+1}$, whose interior is $(5 + 1)$-dimensional. This must work since the ratio $\mu_p/\mu_{p+4} = V_*$ and so there will always be wrapped branes becoming massless at the same loci in the geometry, giving physics which goes beyond supergravity. For even p the theory in the interior has an $SU(2)$ gauge symmetry, while for odd p there is the A_1 two-form gauge theory, carried by tensionless strings. The details of the smoothing will be very case dependent, and it should be interesting to work out those details.

One can also study $SO(2N)$, $SO(2N+1)$ and $USp(2N)$ gauge theories with eight supercharges in various dimensions using similar techniques, placing an orientifold O6-plane into the system parallel to the D6-branes. The enhançon then becomes[245] an $\mathbb{RP}^2$.

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15.5 The consistency of excision in supergravity

We can actually use classic General Relativity techniques[259, 260] to carry out the procedure of removing the interior geometry and replacing it by flat space. We should be able to see if this procedure is consistent and makes some physical sense. The standard procedure for this is as follows. If we join two solutions of Einstein's equations across some surface, there will be a discontinuity in the extrinsic curvature at the surface. A rewriting of the equations of motion can be done to show that this discontinuity can be interpreted as a δ-function source of stress-energy located at the surface.

Let's carry this out here[264], performing an incision at arbitrary radius $r = r_{\rm i}$, and then gluing in flat space. The computation must be performed in Einstein frame to enable an interpretation of the discontinuity in the extrinsic curvature as a stress-energy. So we work with the ten dimensional Einstein metric $ds^2_{\rm E} = e^{-\Phi/2}ds^2$ denoting the generic metric components as G_{AB}:

$$\begin{aligned} g_{\rm s}^{1/2}\, ds^2 &= Z_2^{-5/8} Z_6^{-1/8} \eta_{\mu\nu} dx^\mu dx^\nu + Z_2^{3/8} Z_6^{7/8} dx^i dx^i \\ &\quad + V^{1/2} Z_2^{3/8} Z_6^{-1/8} ds^2_{K3} \\ &= G_{\mu\nu} dx^\mu dx^\nu + G_{ij} dx^i dx^j + G_{ab} dx^a dx^b, \end{aligned} \tag{15.24}$$

where Z_2 and Z_6 are given by (15.15).

Since we make a radial slice, we can define unit normal vectors (see insert 10.2):

$$n_\pm^A = \mp \frac{1}{\sqrt{G_{rr}}} \left(\frac{\partial}{\partial r}\right)^A, \tag{15.25}$$

where n_+ (n_-) is the outward pointing normal for the spacetime region $r > r_{\rm i}$ ($r < r_{\rm i}$). Referring to insert 10.2, we see that the extrinsic curvature of the junction surface for each region is

$$K^\pm_{AB} = \frac{1}{2} n^C_\pm \partial_C G_{AB} = \mp \frac{1}{2\sqrt{G_{rr}}} \frac{\partial G_{AB}}{\partial r}. \tag{15.26}$$

We next define the discontinuity in the extrinsic curvature across the junction as $\gamma_{AB} = K^+_{AB} + K^-_{AB}$. The stress-energy tensor supported at the junction is defined in terms of these as:

$$S_{AB} = \frac{1}{\kappa^2}\Big(\gamma_{AB} - G_{AB}\, \gamma^C{}_C\Big), \tag{15.27}$$

where κ is the gravitational coupling defined in (7.44).

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In choosing the metric for flat space, we should ensure that all fields are continuous through the incision by writing the interior solution in appropriate coordinates and gauge:

$$
\begin{aligned}
g_{\rm s}^{1/2}ds^2 &= Z_2(r_{\rm i})^{-5/8}Z_6(r_{\rm i})^{-1/8}\eta_{\mu\nu}dx^\mu dx^\nu + Z_2(r_{\rm i})^{3/8}Z_6(r_{\rm i})^{7/8}dx^i dx^i\\
&\quad + V^{1/2}Z_2(r_{\rm i})^{3/8}Z_6(r_{\rm i})^{-1/8}ds^2_{\rm K3},\\
e^{2\Phi} &= g_{\rm s}^2 Z_2^{1/2}(r_{\rm i})Z_6^{-3/2}(r_{\rm i}),\\
C_{(3)} &= (Z_2(r_{\rm i})g_{\rm s})^{-1}dx^0\wedge dx^1\wedge dx^2,\\
C_{(7)} &= (Z_6(r_{\rm i})g_{\rm s})^{-1}dx^0\wedge dx^1\wedge dx^2\wedge V\,\varepsilon_{\rm K3}.
\end{aligned}
\tag{15.28}
$$

It is straightforward to derive the following results for the discontinuity tensor, and the reader should check the result:

$$
\begin{aligned}
\gamma_{\mu\nu} &= \frac{1}{16}\frac{1}{\sqrt{G_{rr}}}\left(5\frac{Z_2'}{Z_2}+\frac{Z_6'}{Z_6}\right)G_{\mu\nu},\\
\gamma_{ij} &= -\frac{1}{16}\frac{1}{\sqrt{G_{rr}}}\left(3\frac{Z_2'}{Z_2}+7\frac{Z_6'}{Z_6}\right)G_{ij},\\
\gamma_{ab} &= -\frac{1}{16}\frac{1}{\sqrt{G_{rr}}}\left(3\frac{Z_2'}{Z_2}-\frac{Z_6'}{Z_6}\right)G_{ab},
\end{aligned}
\tag{15.29}
$$

where a prime denotes ∂_r and all quantities are evaluated at the incision surface $r = r_{\rm i}$. The trace is:

$$
\gamma^C{}_C = -\frac{1}{16}\frac{1}{\sqrt{G_{rr}}}\left(3\frac{Z_2'}{Z_2}+7\frac{Z_6'}{Z_6}\right),
\tag{15.30}
$$

and the $\mu,\nu = 0,1,2$ index directions along the brane, i,j index the two angular directions (θ,ϕ) transverse to the brane, and a,b index the four K3 directions.

So finally we have the stress-energy tensor of the discontinuity:

$$
\begin{aligned}
S_{\mu\nu} &= \frac{1}{2\kappa^2\sqrt{G_{rr}}}\left(\frac{Z_2'}{Z_2}+\frac{Z_6'}{Z_6}\right)G_{\mu\nu},\\
S_{ij} &= 0,\\
S_{ab} &= \frac{1}{2\kappa^2\sqrt{G_{rr}}}\left(\frac{Z_6'}{Z_6}\right)G_{ab}.
\end{aligned}
\tag{15.31}
$$

Let us consider the physical properties of this object[264]. The last line gives the components of the stress-energy along the K3 direction. It involves only the harmonic function for the pure D6-brane part which is consistent with the fact that there are only D6-branes wrapped there. The middle

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line shows that there is no stress in the directions transverse to the branes, which dovetails nicely with the fact that the constituent branes are BPS with no interaction forces needed to support the shell in the transverse space.

As a first check of this interpretation, we can expand the results in equation (15.31) for large $r_{\rm i}$. Up to an overall sign, the coefficient of the metric components gives an effective tension in the various directions. The leading contributions are simply:

$$\begin{aligned}\tau(r_{\rm i}) &= \frac{1}{2\kappa^2}\frac{r_6}{r_{\rm i}^2}\left(1-\frac{V_*}{V}\right) \qquad (15.32)\\ &= \frac{N}{(2\pi)^6(\alpha')^{7/2}g_s}(V-V_*)\frac{1}{4\pi r_{\rm i}^2 V} = N(\tau_6 V-\tau_2)\left(\frac{1}{4\pi r_{\rm i}^2 V}\right),\\ \tau_{\rm K3}(r_{\rm i}) &= \frac{1}{2\kappa^2}\frac{r_6}{r_{\rm i}^2} = \frac{N}{(2\pi)^6(\alpha')^{7/2}g_s}\left(\frac{1}{4\pi r_{\rm i}^2}\right) = N\tau_6\left(\frac{1}{4\pi r_{\rm i}^2}\right), \qquad (15.33)\end{aligned}$$

which is in precise accord with expectations. In the K3 directions, the effective tension matches precisely that of N fundamental D6-branes, with an additional averaging factor $(1/4\pi r_{\rm i}^2)$ coming from smearing the branes over the transverse space. In the x^0, x^1, x^2 directions, we have an effective membrane tension which, up to the appropriate smearing factor, again matches that for N D6-branes including the subtraction of N units of D2-brane tension as a result of wrapping on the K3 manifold[128] .

Notice that the result for the stress-energy in the unwrapped part of the brane is proportional to $(Z_2 Z_6)'$. As we have already observed in equations (15.18) and (15.23), this vanishes at precisely $r = r_{\rm e}$, where the probe starts to become unphysical, and where the supergravity starts to become repulsive. So, for incision at the enhançon radius, there is a shell of branes of zero tension, as the probe computation showed.

For $r < r_{\rm e}$ we would get a negative tension from the stress-energy tensor, which is problematic even in supergravity. Notice, however, that nothing in our computation shows that we cannot make an incision at any radius of our choosing for $r \geq r_{\rm e}$, and place a shell of branes of the appropriate tension (as in the calculation of the effective tensions at large $r_{\rm i}$ above). This corresponds physically to the fact that constituent branes experience no potential, so they can consistently be placed at any arbitrary position outside the enhançon.

15.6 The moduli space of pure glue in 3D

Note that the Lagrangian (15.21) depends only on three moduli space coordinates, (x^3, x^4, x^5), or (r, θ, ϕ) in polar coordinates. As mentioned

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before, a (2+1) dimensional theory with eight supercharges, should have a moduli space metric which is hyper-Kähler[185]. So we need at least one extra modulus, s. A similar procedure to that used in section 15.2 can be used to introduce the gauge field's correct couplings and dualise to introduce the scalar s. A crucial difference is that one must replace $2\pi\alpha' F_{ab}$ by $e^{2\phi}(\mu_6 V(r) - \mu_2)^{-2} v_a v_b$ in the Dirac–Born–Infeld action, the extra complication being due to the r dependent nature of the tension. The static gauge computation gives for the kinetic term:

$$\mathcal{L} = F(r)\left(\dot{r}^2 + r^2\dot{\Omega}^2\right) + F(r)^{-1}\left(\dot{s}/2 - \mu_2 C_\phi \dot{\phi}/2\right)^2, \qquad (15.34)$$

where

$$F(r) = \frac{Z_6}{2g_\mathrm{s}}\left(\mu_6 V(r) - \mu_2\right) = \frac{\alpha'^{-3/2}}{(2\pi)^2 g_\mathrm{s}}\left(\frac{V}{V_*} - 1 - \frac{g_\mathrm{s} N \alpha'^{1/2}}{r}\right), \qquad (15.35)$$

and $\dot{\Omega}^2 = \dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2$.

Again, there is gauge theory information to be extracted here. We have pure gauge $SU(N)$ theory with no hypermultiplets, and eight supercharges. We should be able to cleanly separate the gauge theory data from everything else by taking the decoupling limit $\alpha' \to 0$ while holding the gauge theory coupling $g^2_\mathrm{YM} = g^2_{\mathrm{YM},p} V^{-1} = (2\pi)^4 g_\mathrm{s} \alpha'^{3/2} V^{-1}$ and the energy scale $U = r/\alpha'$ (proportional to M_W) fixed. In doing this, we get the metric:

$$ds^2 = f(U)\left(\dot{U}^2 + U^2 d\Omega^2\right) + f(U)^{-1}\left(d\sigma - \frac{N}{4\pi^2} A_\phi d\phi\right)^2,$$

$$\text{where}\quad f(U) = \frac{1}{4\pi^2 g^2_\mathrm{YM}}\left(1 - \frac{g^2_\mathrm{YM} N}{U}\right), \qquad (15.36)$$

the $U(1)$ monopole potential is $A_\phi = \pm 1 - \cos\theta$, and $\sigma = s\alpha'$, and the metric is meaningful only for $U > U_\mathrm{e} = \lambda = g^2_\mathrm{YM} N$, the ''t Hooft coupling', a natural gauge theory quantity to hold fixed in the limit of large N, where we make contact with supergravity. This metric, which should be contrasted with equation (15.11), is the hyper-Kähler Taub–NUT metric, but this time with a negative mass. It is singular. For $N = 2$, the full metric, obtained by instanton corrections to this one-loop result, is smooth, as we will discuss. For large N, the instantons are suppressed. We shall discuss this some more in the next section.

15.6.1 Multi-monopole moduli space

Recall that the membrane resulting from wrapping the six-brane is s BPS monopole. Therefore the moduli space of the entire wrapped system

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should be related to the moduli space of N BPS monopoles. In fact, since the low energy dynamics of the branes is $SU(N)$ gauge theory, we learn that BPS monopole moduli space is to be identified with the Coulomb branch of the gauge theory as well[231]. The part of the moduli space corresponding to the motion of a single sub-brane (the probe discussed above) is evidently a submanifold of the full $4N-4$ dimensional metric on the *relative* moduli space[218] of N BPS monopoles which is smooth[219].

This should remind the reader of our study in section 15.3. Recalling that this is also a study of $SU(N)$ gauge theory with no hypermultiplets, we know the result for $N=2$: the metric on the moduli space must be smooth, as there is no Higgs branch to connect to via the singularity. This is true for all $SU(N)$, and matches the monopole result. For $N=2$, we stated that the metric on the moduli space[247] is actually the Atiyah–Hitchin manifold[232]. The metric may be written in the following manifestly $SO(3)$ invariant manner[232, 251]:

$$\begin{gathered} ds^2_{\rm AH} = f^2 d\rho^2 + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2; \\ \frac{2bc}{f}\frac{da}{d\rho} = (b-c)^2 - a^2, \text{ and cyclic perms.}; \quad \rho = 2K\left(\sin\frac{\beta}{2}\right), \end{gathered} \tag{15.37}$$

where the choice $f=-b/\rho$ can be made, the σ_i are defined in (7.4), and $K(k)$ is the elliptic integral of the first kind:

$$K(k) = \int_0^{\frac{\pi}{2}} (1-k^2\sin^2\tau)^{\frac{1}{2}} d\tau. \tag{15.38}$$

Also, $k=\sin(\beta/2)$, the ‘modulus’, runs from 0 to 1, so $\pi \le \rho \le \infty$. In fact, the solution for a, b, c can be written out in terms of elliptic functions, but we shall not do that here. All of the functions entering the metric can be expanded in large ρ, and the result is:

$$ds^2_{\rm TN-} = \left(1-\frac{2}{\rho}\right)\left(d\rho^2+\rho^2 d\Omega^2\right) + 4\left(1-\frac{2}{\rho}\right)^{-1}(d\psi+\cos\theta d\phi)^2. \tag{15.39}$$

Comparing to equation (15.12), we see that this is the Taub–NUT metric, but with a negative mass parameter, i.e. $N_{\rm f}=-1$. Now, as already stated, Taub–NUT has an $SU(2)$ isometry, and the full Atiyah–Hitchin metric has an $SO(3)$. Furthermore, the metric we have here is singular at $\rho=2$, whereas the full metric is smooth everywhere. Therefore there is a lot missing from this approximate metric. In fact, these key differences are invisible at any order in the large ρ expansion, being exponentially

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small in ρ, of the form $e^{-\rho}$. These exponential corrections for smaller ρ remove the singularity: $\rho = 2$ is just an artifact of the large ρ metric in the above form (15.39). As for the isometry, the fact that it is really $SO(3)$ follows from the fact that ψ started out with periodicity 2π and not 4π in the full metric, as required by the requirement that there is no bolt spherical singularity at finite ρ. Expanding in large ρ will not change that periodicity of course, but if one was just presented with the expanded result one would not know of the non-perturbative no-bolt condition. So in this case of two monopoles, there is an $SO(3) = SU(2)/\mathbb{Z}_2$ isometry in the problem, and not the naive $SU(2)$ of the Taub–NUT space, since ψ has period 2π and not 4π. The $SO(3)$ isometry, smoothness, and the condition of hyper-Kählerity actually picks out *uniquely* the Atiyah–Hitchin manifold as the completion of the negative mass Taub–NUT.

Actually, we have described a trivial cover of the true Atiyah–Hitchin space. The two monopole problem has an obvious $\mathbb{Z}_2$ symmetry coming from the fact that the monopoles are identical. Some field configurations described by the manifold as described up to now are overcounted, and so we must divide by this $\mathbb{Z}_2$, resulting in an $\mathbb{RP}^2$ for the bolt instead of an S^2.

What is the relation to our probe result? To see it[258], change variables in our probe metric (15.36) by absorbing a factor of $\lambda/2 = g^2_{\rm YM}N/2$ into the radial variable U, defining $\rho = 2U/\lambda$. Further absorb $\psi = \sigma 8\pi^2/N$ and gauge transform to $A_\phi = -\cos\theta$. Then we get:

$$ds^2 = \frac{g^2_{\rm YM}N^2}{32\pi^2} ds^2_{\rm TN-}, \tag{15.40}$$

showing that we have precisely the form of the Taub–NUT metric that one gets by expanding the Atiyah–Hitchin metric in large ρ and neglecting exponential corrections.

Now for the same reasons as in section 15.3, the periodicity of σ is $1/2\pi$, and we will use $\widetilde{\sigma} = 4\pi^2\sigma$ as our 2π periodic scalar dual to the photon on the probe's world-volume. Looking at the choices we made above, this implies that for the $SU(2)$ case, the coordinate ψ has period 2π, which fits what we stated about the Atiyah–Hitchin manifold above.

The exponential corrections have the expected interpretation in the gauge theory as the instanton corrections which maintain positivity of the metric and the gauge coupling[249]. Translating back to physical variables, we see that these corrections go as $\exp(-U/g^2_{\rm YM})$, which has the correct form of action for a gauge theory instanton. (We have just described a *cover* of the Atiyah–Hitchin manifold needed for the $SU(2)$ case. There is an additional identification to be discussed below.) This completes the story for the $SU(2)$ gauge theory moduli space problem[248].

Can we learn anything from this for our case of general N, especially for large N, to teach us about the enhançon geometry? Notice[258] that the instanton corrections are suppressed at large N if we hold the 't Hooft coupling λ (which sets the Taub–NUT mass) fixed, since there is a bare N in the exponential: $\exp(-NU/\lambda)$. So the smoothing is suppressed at large N, and we recover the macroscopic sharp (relatively) enhançon locus at large N in the supergravity geometry. Notice that if we've fixed our period of $\widetilde{\sigma}$ to be 2π as before, for general N the resulting period of ψ in the scaled variables is $\Delta\psi = 4\pi/N$. Therefore our isometry is not $SO(3)$ but is only $SU(2)/\mathbb{Z}_N$, which is not an isometry at all.

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