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LOCAL-GLOBAL PRINCIPLE FOR THE FINITENESS AND ARTINIANNESS OF GENERALISED LOCAL COHOMOLOGY MODULES

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Abstract

Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules, where R is a commutative Noetherian ring. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R and let M and N be finite R-modules. We prove that if N and $H^i_{\mathfrak{a}}(M, N)$ belong to $\mathcal{S}$ for all $i < n$ and if $n \le \text{f-grad}(\mathfrak{a}, \mathfrak{b}, N)$, then $\operatorname{Hom}_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M, N)) \in \mathcal{S}$. We deduce that if either $H^i_{\mathfrak{a}}(M, N)$ is finite or $\operatorname{Supp} H^i_{\mathfrak{a}}(M, N)$ is finite for all $i < n$, then $\operatorname{Ass} H^n_{\mathfrak{a}}(M, N)$ is finite. Next we give an affirmative answer, in certain cases, to the following question. If, for each prime ideal $\mathfrak{p}$ of R, there exists an integer $n_{\mathfrak{p}}$ such that $\mathfrak{b}^{n_{\mathfrak{p}}} H^i_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = 0$ for every i less than a fixed integer t, then does there exist an integer n such that $\mathfrak{b}^n H^i_{\mathfrak{a}}(M, N) = 0$ for all $i < t$? A formulation of this question is referred to as the local-global principle for the annihilation of generalised local cohomology modules. Finally, we prove that there are local-global principles for the finiteness and Artinianness of generalised local cohomology modules.

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1. Introduction

Throughout this paper R denotes a commutative Noetherian ring with identity and $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are ideals of R. We denote by $\mathbb{N}$ and $\mathbb{N}_0$ the set of positive and nonnegative integers, respectively. The notion of generalised local cohomology functors was introduced by Herzog, in [9], over a local ring and then continued by Suzuki in [18]. Later this concept was studied by Bijan-Zadeh, in [1], over any commutative Noetherian ring. For each integer i, the ith generalised local cohomology functor $H^i_{\mathfrak{a}}(\cdot\,,\cdot)$ is defined by

$$H^i_{\mathfrak{a}}(M, N) = \varinjlim_n \operatorname{Ext}^i_R(M/\mathfrak{a}^n M, N)$$

for all R-modules M and N. Clearly, this notion is a generalisation of the usual local cohomology functor [4]. On the other hand, the concept of a filter regular sequence has been studied in [12, 15, 17, 21] and has led to some interesting results. We denote the common length of all maximal $\mathfrak{a}$-filter regular M-sequences contained in $\mathfrak{b}$ by f-grad $(\mathfrak{a}, \mathfrak{b}, M)$ and call it the $\mathfrak{a}$-filter grade of $\mathfrak{b}$ on M. We briefly recall, in Section 2, the concept of a filter regular sequence and basic properties of f-grad $(\mathfrak{a}, \mathfrak{b}, M)$, but refer the reader to [8, 19] for more details. It is clear that an R-filter regular M-sequence is just a weak M-sequence [2] and f-grad $(R, \mathfrak{b}, M) =$ grad $(\mathfrak{b}, M)$. If $(R, \mathfrak{m})$ is a local ring, then f-grad $(\mathfrak{m}, \mathfrak{b}, M)$ is just the well-known notion f-depth $(\mathfrak{b}, M)$; see [11] for some characterisations of f-depth $(\mathfrak{b}, M)$. Filter regular sequences were employed in [19] to establish some finiteness results on usual local cohomology modules. In this paper we use those sequences to obtain some finiteness and Artinianness results on generalised local cohomology modules.

Recall that a class $\mathcal{S}$ of R-modules is a Serre subcategory of the category of R-modules if it is closed under taking submodules, quotients and extensions. In Theorem 2.2, for finite R-modules M and N, we prove that if N and $H^i_{\mathfrak{a}}(M, N)$ belong to $\mathcal{S}$ for all $i < n$ and $n \le$ f-grad $(\mathfrak{a}, \mathfrak{b}, N)$, then $\mathrm{Hom}_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M, N)) \in \mathcal{S}$. We deduce that if either $H^i_{\mathfrak{a}}(M, N)$ is finite or Supp $H^i_{\mathfrak{a}}(M, N)$ is finite for all $i < n$, then Ass $H^n_{\mathfrak{a}}(M, N)$ is finite. In a certain case, when $M = R$, this is the main result of [13]. Therefore Theorem 2.2 provides a generalisation of the main result of [13]. Notice that Ass $H^n_{\mathfrak{a}}(M, N)$ is not finite in general; see, for example, [10, 16].

Let M, N be finite R-modules. As a generalisation of the $\mathfrak{b}$-finiteness dimension $f^{\mathfrak{b}}_{\mathfrak{a}}(N)$ of N with respect to $\mathfrak{a}$, we define

$$f^{\mathfrak{b}}_{\mathfrak{a}}(M, N) = \inf\{i \in \mathbb{N}_0 \mid \mathfrak{b} \not\subseteq \sqrt{(0 :_R H^i_{\mathfrak{a}}(M, N))}\}$$

and denote $f^{\mathfrak{a}}_{\mathfrak{a}}(M, N)$ by $f_{\mathfrak{a}}(M, N)$. In fact, by Proposition 3.1,

$$f_{\mathfrak{a}}(M, N) = \inf\{i \in \mathbb{N}_0 \mid H^i_{\mathfrak{a}}(M, N) \text{ is not finite}\}.$$

In Section 3 we give some properties of $f^{\mathfrak{b}}_{\mathfrak{a}}(M, N)$. In particular, we prove that $f^{\mathfrak{b}}_{\mathfrak{a}}(N) \le f^{\mathfrak{b}}_{\mathfrak{a}}(M, N)$. We present an example to show that the above inequality may be strict (Example 3.6). Thus the result [5, Proposition 2.10] of Chu is not correct. Moreover, Example 3.6 shows that the result [5, Lemma 2.9] is no longer true.

The local-global principle for the finiteness of local cohomology modules, investigated by Faltings in [6, 7], states that, for all nonnegative integers r, $f_{\mathfrak{a}}(N) > r$ if and only if $f_{\mathfrak{a}R_{\mathfrak{p}}}(N_{\mathfrak{p}}) > r$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$. Also we say that Faltings' local-global principle for the annihilation of local cohomology modules holds at level r if

$$f^{\mathfrak{b}}_{\mathfrak{a}}(N) > r \Leftrightarrow f^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(N_{\mathfrak{p}}) > r \quad \text{for all } \mathfrak{p} \in \mathrm{Spec}(R)$$

is true for all finite R-modules N and all ideals $\mathfrak{a}, \mathfrak{b}$. Raghavan proved, in [14], that the local-global principle for the annihilation of local cohomology modules holds at level 1, while Brodmann *et al.* proved it is true at level 2 [3, Theorem 2.6]. As a generalisation of this, we say that Faltings' local-global principle for the annihilation of generalised local cohomology modules holds at level r if

$$f^{\mathfrak{b}}_{\mathfrak{a}}(M, N) > r \Leftrightarrow f^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) > r \quad \text{for all } \mathfrak{p} \in \mathrm{Spec}(R) \qquad (\dagger)$$

is true for all finite R-modules M, N and all ideals $\mathfrak{a}$, $\mathfrak{b}$. We show, in Proposition 4.2, that the local-global principle for the annihilation of generalised local cohomology modules holds at levels 0, 1, 2. Now let $\mathfrak{b} \subseteq \mathfrak{a}$. Then we prove the following statements, in Theorem 4.4.

(i) $f_{\mathfrak{a}}(M, N) \geq \inf\{f_{\mathfrak{a}}^{\mathfrak{b}}(M, N), \text{f-grad}\,(\mathfrak{a}, \mathfrak{b}, N) + 1\}$. In particular, $f_{\mathfrak{a}}(M, N) = f_{\mathfrak{a}}^{\mathfrak{b}}(M, N)$ whenever $f_{\mathfrak{a}}^{\mathfrak{b}}(M, N) \leq \text{f-grad}\,(\mathfrak{a}, \mathfrak{b}, N) + 1$.

(ii) Assume that $r \leq \text{f-grad}\,(\mathfrak{a}, \mathfrak{b}, N) + 1$. Then

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M, N) > r \Leftrightarrow f_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) > r \quad \text{for all } \mathfrak{p} \in \operatorname{Spec}(R).$$

(iii) If $\operatorname{Supp} N/\mathfrak{b}N \subseteq V(\mathfrak{a})$, then the statement

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M, N) > r \Leftrightarrow f_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) > r \quad \text{for all } \mathfrak{p} \in \operatorname{Spec}(R)$$

holds for all r.

(iv) Faltings' local-global principle for the finiteness of generalised local cohomology modules holds. In other words, for any positive integer r, $H^i_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$ is finite for all $i \leq r$ and for all $\mathfrak{p} \in \operatorname{Spec} R$ if and only if $H^i_{\mathfrak{a}}(M, N)$ is finite for all $i \leq r$.

Finally, in Theorem 5.3, for finite R-modules M and N and for a positive integer n, we prove that $H^i_{\mathfrak{a}}(M, N)$ is Artinian for all $i < r$ if and only if $H^i_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$ is Artinian for all $i < r$ and for all $\mathfrak{p} \in \operatorname{Spec} R$. We observe that this result improves the main result of [20].

2. Preliminary results

We first recall some basic properties of filter regular sequences. The reader is referred to [8] for more details. Assume that M and N are finite R-modules. We say that a sequence $x_1, \ldots, x_n$ of elements of R is an $\mathfrak{a}$-filter regular M-sequence if $x_i \notin \mathfrak{p}$ for all

$$\mathfrak{p} \in \operatorname{Ass}(M/(x_1, \ldots, x_{i-1})M) \setminus V(\mathfrak{a})$$

and for all $i = 1, \ldots, n$. If, in addition, $x_1, \ldots, x_n \in \mathfrak{b}$, then we say that $x_1, \ldots, x_n$ is an $\mathfrak{a}$-filter regular M-sequence in $\mathfrak{b}$. There exists an $\mathfrak{a}$-filter regular M-sequence in $\mathfrak{b}$ of infinite length if and only if $\operatorname{Supp} M/\mathfrak{b}M \subseteq V(\mathfrak{a})$. Now assume that $\operatorname{Supp} M/\mathfrak{b}M \nsubseteq V(\mathfrak{a})$. Then we denote the common length of all maximal $\mathfrak{a}$-filter regular M-sequences contained in $\mathfrak{b}$ by $\text{f-grad}\,(\mathfrak{a}, \mathfrak{b}, M)$ and we call it the $\mathfrak{a}$-filter grade of $\mathfrak{b}$ on M. We set $\text{f-grad}\,(\mathfrak{a}, \mathfrak{b}, M) = \infty$ whenever $\operatorname{Supp} M/\mathfrak{b}M \subseteq V(\mathfrak{a})$. Also, notice that

$$\begin{aligned}
\text{f-grad}\,(\mathfrak{a}, \mathfrak{b}, M) &= \inf\{i \in \mathbb{N}_0 \mid \operatorname{Supp} \operatorname{Ext}^i_R(R/\mathfrak{b}, M) \nsubseteq V(\mathfrak{a})\} \\
&= \inf\{i \in \mathbb{N}_0 \mid \operatorname{Supp} H^i_{\mathfrak{b}}(M) \nsubseteq V(\mathfrak{a})\}, \\
\text{f-grad}\,(\mathfrak{a}, \operatorname{Ann} N, M) &= \inf\{i \in \mathbb{N}_0 \mid \operatorname{Supp} \operatorname{Ext}^i_R(N, M) \nsubseteq V(\mathfrak{a})\}, \\
\text{f-grad}\,(\mathfrak{a}, \mathfrak{b} + \operatorname{Ann} N, M) &= \inf\{i \in \mathbb{N}_0 \mid \operatorname{Supp} H^i_{\mathfrak{b}}(N, M) \nsubseteq V(\mathfrak{a})\}.
\end{aligned}$$

Since f-grad $(R, \mathfrak{b}, M) = \operatorname{grad}(\mathfrak{b}, M)$, we have the following well-known properties ([1, Proposition 5.5], [4, Theorem 6.2.7]):

$$\operatorname{grad}(\mathfrak{b}, M) = \inf\{i \in \mathbb{N}_0 \mid \operatorname{Ext}^i_R(R/\mathfrak{b}, M) \neq 0\} = \inf\{i \in \mathbb{N}_0 \mid H^i_{\mathfrak{b}}(M) \neq 0\}$$

and

$$\operatorname{grad}(\mathfrak{b} + \operatorname{Ann} N, M) = \inf\{i \in \mathbb{N}_0 \mid H^i_{\mathfrak{b}}(N, M) \neq 0\}.$$

If $(R, \mathfrak{m})$ is a local ring, then f-grad $(\mathfrak{m}, \mathfrak{b}, M)$ is just the well-known notion f-depth $(\mathfrak{b}, M)$; see [11] for some properties of f-depth $(\mathfrak{b}, M)$. The following lemma is of assistance in the proof of the next theorem.

Lemma 2.1. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules, M be a finite R-module and $N \in \mathcal{S}$. Then $\operatorname{Ext}^i_R(M, N) \in \mathcal{S}$ for all $i \in \mathbb{N}_0$.*

Proof. Since $\operatorname{Ext}^i_R(M, N)$ is a subquotient of N^α for some $\alpha \in \mathbb{N}_0$, the result is clear. □

Theorem 2.2. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules. Let $n \in \mathbb{N}_0$ and let M and N be finite R-modules such that N and $H^i_{\mathfrak{a}}(M, N)$ belong to $\mathcal{S}$ for all $i < n$. If* f-grad $(\mathfrak{a}, \mathfrak{b}, N) \geq n$, *then* $\operatorname{Hom}_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M, N)) \in \mathcal{S}$. *In particular,* $\operatorname{Hom}_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M, N)) \in \mathcal{S}$ *whenever* $\operatorname{Supp} N/\mathfrak{b}N \subseteq V(\mathfrak{a})$.

Proof. We prove the assertion by induction on n. Since $H^0_{\mathfrak{a}}(M, N) \cong \operatorname{Hom}_R(M, \Gamma_{\mathfrak{a}}(N))$, the result is clear for $n = 0$ by Lemma 2.1. Assume that $n > 0$ and that the result has been proved for $n - 1$. Let f-grad $(\mathfrak{a}, \mathfrak{b}, N) \geq n$ and suppose that $x \in \mathfrak{b}$ is an $\mathfrak{a}$-filter regular N-sequence. The exact sequence

$$0 \to \Gamma_{\mathfrak{a}}(N) \to N \to N/\Gamma_{\mathfrak{a}}(N) \to 0$$

induces the long exact sequence

$$\cdots \to H^i_{\mathfrak{a}}(M, \Gamma_{\mathfrak{a}}(N)) \xrightarrow{f^i} H^i_{\mathfrak{a}}(M, N) \to H^i_{\mathfrak{a}}(M, N/\Gamma_{\mathfrak{a}}(N)) \to H^{i+1}_{\mathfrak{a}}(M, \Gamma_{\mathfrak{a}}(N)) \to \cdots .$$

Since, by [23, Lemma 1.1],

$$H^i_{\mathfrak{a}}(M, \Gamma_{\mathfrak{a}}(N)) \cong \operatorname{Ext}^i_R(M, \Gamma_{\mathfrak{a}}(N)) \quad \text{for all } i \in \mathbb{N}_0,$$

we use Lemma 2.1 and the above long exact sequence to see that $H^i_{\mathfrak{a}}(M, N) \in \mathcal{S}$ if and only if $H^i_{\mathfrak{a}}(M, N/\Gamma_{\mathfrak{a}}(N)) \in \mathcal{S}$. Also $N/\Gamma_{\mathfrak{a}}(N) \in \mathcal{S}$ and f-grad $(\mathfrak{a}, \mathfrak{b}, N) =$ f-grad $(\mathfrak{a}, \mathfrak{b}, N/\Gamma_{\mathfrak{a}}(N))$. On the other hand, since $\operatorname{im} f^n \in \mathcal{S}$, the induced exact sequence

$$0 \to \operatorname{Hom}_R(R/\mathfrak{b}, \operatorname{im} f^n) \to \operatorname{Hom}_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M, N)) \to \operatorname{Hom}_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M, N/\Gamma_{\mathfrak{a}}(N)))$$

yields $\operatorname{Hom}_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M, N)) \in \mathcal{S}$ whenever $\operatorname{Hom}_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M, N/\Gamma_{\mathfrak{a}}(N))) \in \mathcal{S}$. Thus we can replace N by $N/\Gamma_{\mathfrak{a}}(N)$ and, without loss of generality, assume that $\Gamma_{\mathfrak{a}}(N) = 0$; and hence x is a nonzero devisor on N. Next, consider the exact sequence

$$0 \to N \xrightarrow{x} N \to N/xN \to 0$$

which induces the long exact sequence

$$\cdots \to H^i_{\mathfrak{a}}(M,N) \xrightarrow{x} H^i_{\mathfrak{a}}(M,N) \to H^i_{\mathfrak{a}}(M,N/xN) \to H^{i+1}_{\mathfrak{a}}(M,N) \xrightarrow{x} \cdots .$$

Now we may use the above sequence in conjunction with the hypothesis to deduce that $H^i_{\mathfrak{a}}(M,N/xN) \in \mathcal{S}$ for all $i < n-1$. Also it is easy to see that f-grad $(\mathfrak{a},\mathfrak{b},N/xN) =$ f-grad $(\mathfrak{a},\mathfrak{b},N) - 1$. Therefore, by induction, $\operatorname{Hom}_R(R/\mathfrak{b}, H^{n-1}_{\mathfrak{a}}(M,N/xN)) \in \mathcal{S}$. Next, we use the exact sequence

$$0 \to H^{n-1}_{\mathfrak{a}}(M,N)/xH^{n-1}_{\mathfrak{a}}(M,N) \to H^{n-1}_{\mathfrak{a}}(M,N/xN) \to 0 :_{H^n_{\mathfrak{a}}(M,N)} x \to 0,$$

to obtain the exact sequence

$$\begin{aligned}\operatorname{Hom}_R(R/\mathfrak{b}, H^{n-1}_{\mathfrak{a}}(M,N/xN)) &\to \operatorname{Hom}_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M,N)) \\ &\to \operatorname{Ext}^1_R(R/\mathfrak{b}, H^{n-1}_{\mathfrak{a}}(M,N)/xH^{n-1}_{\mathfrak{a}}(M,N))\end{aligned}$$

which in turn, by Lemma 2.1, yields $\operatorname{Hom}_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M,N)) \in \mathcal{S}$. This completes the inductive step. Finally, since the hypothesis $\operatorname{Supp} N/\mathfrak{b}N \subseteq V(\mathfrak{a})$ implies f-grad $(\mathfrak{a},\mathfrak{b},N) = \infty$, the last assertion follows immediately from the first one. □

Let M be an R-module. M is called an FSF module if there is a finite submodule N of M such that the support of the quotient module M/N is finite. If M is an FSF module, then Ass M is finite and the category of FSF R-modules is a Serre subcategory of the category of R-modules [13, Proposition 2.2].

By applying the above theorem to the category of FSF R-modules we have the following corollary which recovers the main result of [13] which has been proved for ordinary local cohomology modules.

Corollary 2.3. *Let M, N be finite R-modules and let $n \in \mathbb{N}_0$ be such that either $H^i_{\mathfrak{a}}(M,N)$ is finite or* Supp $H^i_{\mathfrak{a}}(M,N)$ *is finite for all $i < n$. Then* Ass $H^n_{\mathfrak{a}}(M,N)$ *is finite.*

3. Finiteness properties of generalised local cohomology modules

Let M be a finite R-module. Following [4, Proposition 9.1.2] and [6, Lemma 3], the finiteness dimension $f_{\mathfrak{a}}(M)$ of M relative to $\mathfrak{a}$ is defined as follows:

$$\begin{aligned} f_{\mathfrak{a}}(M) &= \inf\{i \in \mathbb{N}_0 \mid H^i_{\mathfrak{a}}(M) \text{ is not finite}\} \\ &= \inf\{i \in \mathbb{N}_0 \mid \mathfrak{a} \not\subseteq \sqrt{(0 :_R H^i_{\mathfrak{a}}(M))}\}.\end{aligned}$$

As a generalisation, the $\mathfrak{b}$-finiteness dimension $f^{\mathfrak{b}}_{\mathfrak{a}}(M)$ of M relative to $\mathfrak{a}$ is defined by

$$f^{\mathfrak{b}}_{\mathfrak{a}}(M) = \inf\{i \in \mathbb{N}_0 \mid \mathfrak{b} \not\subseteq \sqrt{(0 :_R H^i_{\mathfrak{a}}(M))}\}.$$

We now extend this definition to generalised local cohomology modules.

Proposition 3.1. *Let M, N be finite R-modules and $n \in \mathbb{N}_0$. Then the following statements are equivalent:*

(i) $H^i_{\mathfrak{a}}(M, N)$ *is finite for all* $i < n$;
(ii) $\mathfrak{a} \subseteq \sqrt{(0 :_R H^i_{\mathfrak{a}}(M, N))}$ *for all* $i < n$.

Proof. (i) $\Rightarrow$ (ii) is obvious. For (ii) $\Rightarrow$ (i), we use induction on n. When $n = 1$, there is nothing to prove. Now let $n > 1$ and suppose that the result has been proved for smaller values of n. By the inductive assumption, $H^i_{\mathfrak{a}}(M, N)$ is finite for $i = 0, \ldots, n-2$. Also, by hypothesis, $\mathfrak{a}^r H^{n-1}_{\mathfrak{a}}(M, N) = 0$ for some $r \in \mathbb{N}$, so that, in view of Theorem 2.2, $0 :_{H^{n-1}_{\mathfrak{a}}(M,N)} \mathfrak{a}^r = H^{n-1}_{\mathfrak{a}}(M, N)$ is finite. This completes the induction. □

Definition 3.2. Let M and N be finite R-modules. We define the $\mathfrak{b}$-finiteness dimension $f^{\mathfrak{b}}_{\mathfrak{a}}(M, N)$ of M, N relative to $\mathfrak{a}$ by

$$f^{\mathfrak{b}}_{\mathfrak{a}}(M, N) = \inf\{i \in \mathbb{N}_0 \mid \mathfrak{b} \nsubseteq \sqrt{(0 :_R H^i_{\mathfrak{a}}(M, N))}\}.$$

Notice that, by Proposition 3.1,

$$f^{\mathfrak{a}}_{\mathfrak{a}}(M, N) = \inf\{i \in \mathbb{N} \mid H^i_{\mathfrak{a}}(M, N) \text{ is not finite}\}.$$

We denote $f^{\mathfrak{a}}_{\mathfrak{a}}(M, N)$ by $f_{\mathfrak{a}}(M, N)$.

For $y \in R$, set $S = \{y^n : n \geq 0\}$. In the next lemma, for an R-module M, we denote $S^{-1}M$ by M_y. The following two lemmas are needed in the proof of the next proposition.

Lemma 3.3. *Let* M, N *be finite* R*-modules and* $x \in R$. *Then we have the following long exact sequence*

$$\cdots \to H^i_{\mathfrak{a}+Rx}(M, N) \to H^i_{\mathfrak{a}}(M, N) \to H^i_{\mathfrak{a}R_x}(M_x, N_x) \to H^{i+1}_{\mathfrak{a}+Rx}(M, N) \to \cdots .$$

Proof. Let $E^\bullet$ be an injective resolution of N. Then $E^\bullet_x$ is an injective resolution of R_x-module N_x. The split exact sequence

$$0 \to \Gamma_{\mathfrak{a}+Rx}(E^\bullet) \to \Gamma_{\mathfrak{a}}(E^\bullet) \to \Gamma_{\mathfrak{a}}(E^\bullet_x) \to 0$$

of complexes [4, Lemma 8.1.1] induces the exact sequence

$$0 \to \operatorname{Hom}_R(M, \Gamma_{\mathfrak{a}+Rx}(E^\bullet)) \to \operatorname{Hom}_R(M, \Gamma_{\mathfrak{a}}(E^\bullet)) \to \operatorname{Hom}_R(M, \Gamma_{\mathfrak{a}}(E^\bullet_x)) \to 0$$

of complexes. On the other hand, we have the following natural isomorphism of complexes:

$$\begin{aligned}\operatorname{Hom}_R(M, \Gamma_{\mathfrak{a}}(E^\bullet_x)) &\cong \operatorname{Hom}_R(M, \operatorname{Hom}_{R_x}(R_x, \Gamma_{\mathfrak{a}R_x}(E^\bullet_x)))\\ &\cong \operatorname{Hom}_{R_x}(M \otimes_R R_x, \Gamma_{\mathfrak{a}R_x}(E^\bullet_x))\\ &\cong H^0_{\mathfrak{a}R_x}(M_x, E^\bullet_x).\end{aligned}$$

Hence the above exact sequence of complexes induces the following long exact sequence of homology modules:

$$\begin{aligned}\cdots \to H^i(H^0_{\mathfrak{a}+Rx}(M, E^\bullet)) \to H^i(H^0_{\mathfrak{a}}(M, E^\bullet)) \to H^i(H^0_{\mathfrak{a}R_x}(M_x, E^\bullet_x))\\ \to H^{i+1}(H^0_{\mathfrak{a}+Rx}(M, E^\bullet)) \to \cdots .\end{aligned}$$

This completes the proof. □

Lemma 3.4 (see [4, Lemma 9.1.1]). *Let $M \to N \to L$ be an exact sequence of R-modules such that $\mathfrak{a} \subseteq \sqrt{(0 :_R M)}$ and $\mathfrak{a} \subseteq \sqrt{(0 :_R L)}$. Then $\mathfrak{a} \subseteq \sqrt{(0 :_R N)}$.*

Proposition 3.5. *Let M, N, L, K be finite R-modules.*

(i) *Let R' be a second commutative ring and let $f : R \to R'$ be a flat homomorphism of rings. Then*

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M, N) \leq f_{\mathfrak{a}R'}^{\mathfrak{b}R'}(M \otimes_R R', N \otimes_R R').$$

In particular, if S is a multiplicatively closed subset of R, then

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M, N) \leq f_{S^{-1}\mathfrak{a}}^{S^{-1}\mathfrak{b}}(S^{-1}M, S^{-1}N).$$

(ii) $f_{\mathfrak{a}}^{\mathfrak{b}}(M, N) = f_{\mathfrak{a}}^{\sqrt{\mathfrak{b}}}(M, N) = f_{\sqrt{\mathfrak{a}}}^{\mathfrak{b}}(M, N) = f_{\sqrt{\mathfrak{a}}}^{\sqrt{\mathfrak{b}}}(M, N)$.

(iii) *Let $x \in R$. Then*

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M, N) = \inf\{f_{\mathfrak{a}+Rx}^{\mathfrak{b}}(M, N), f_{\mathfrak{a}R_x}^{\mathfrak{b}R_x}(M_x, N_x)\}.$$

(iv) *Let $\mathfrak{a} \subseteq \mathfrak{c}$. Then*

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M, N) \leq f_{\mathfrak{c}}^{\mathfrak{b}}(M, N) \quad \text{and} \quad f_{\mathfrak{b}}^{\mathfrak{c}}(M, N) \leq f_{\mathfrak{b}}^{\mathfrak{a}}(M, N).$$

(v) *Let $\mathfrak{b} \subseteq \mathfrak{c}$. Then*

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M, N) \cong f_{\mathfrak{a}}^{\mathfrak{b}}(M, N/\Gamma_{\mathfrak{c}}(N)).$$

In particular,

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M, N) \cong f_{\mathfrak{a}}^{\mathfrak{b}}(M, N/\Gamma_{\mathfrak{b}}(N)).$$

(vi) *Let $0 \to L \to M \to N \to 0$ be an exact sequence. Then*

$$f_{\mathfrak{a}}^{\mathfrak{b}}(K, L) \geq \inf\{f_{\mathfrak{a}}^{\mathfrak{b}}(K, M), f_{\mathfrak{a}}^{\mathfrak{b}}(K, N) + 1\},$$
$$f_{\mathfrak{a}}^{\mathfrak{b}}(K, M) \geq \inf\{f_{\mathfrak{a}}^{\mathfrak{b}}(K, L), f_{\mathfrak{a}}^{\mathfrak{b}}(K, N)\},$$
$$f_{\mathfrak{a}}^{\mathfrak{b}}(K, N) \geq \inf\{f_{\mathfrak{a}}^{\mathfrak{b}}(K, L) - 1, f_{\mathfrak{a}}^{\mathfrak{b}}(K, M)\}$$

and

$$f_{\mathfrak{a}}^{\mathfrak{b}}(L, K) \geq \inf\{f_{\mathfrak{a}}^{\mathfrak{b}}(M, K), f_{\mathfrak{a}}^{\mathfrak{b}}(N, K) - 1\},$$
$$f_{\mathfrak{a}}^{\mathfrak{b}}(M, K) \geq \inf\{f_{\mathfrak{a}}^{\mathfrak{b}}(L, K), f_{\mathfrak{a}}^{\mathfrak{b}}(N, K)\},$$
$$f_{\mathfrak{a}}^{\mathfrak{b}}(N, K) \geq \inf\{f_{\mathfrak{a}}^{\mathfrak{b}}(L, K) + 1, f_{\mathfrak{a}}^{\mathfrak{b}}(M, K)\}.$$

(vii) *Let* $\operatorname{Supp} M \subseteq \operatorname{Supp} N$. *Then*

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M, K) \geq f_{\mathfrak{a}}^{\mathfrak{b}}(N, K).$$

In particular,

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M, K) = f_{\mathfrak{a}}^{\mathfrak{b}}(N, K)$$

whenever $\operatorname{Supp} M = \operatorname{Supp} N$.

(viii) $f_{\mathfrak{a}}^{\mathfrak{b}}(M, N) \geq f_{\mathfrak{a}}^{\mathfrak{b}}(N)$.

(ix) *Let* $0 \to L \to M \to N \to 0$ *be an exact sequence. Then*

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M, K) = \inf\{f_{\mathfrak{a}}^{\mathfrak{b}}(L, K), f_{\mathfrak{a}}^{\mathfrak{b}}(N, K)\}.$$

(x) *There exists a prime ideal* $\mathfrak{p}$ *in* Min Supp M *such that* $f_{\mathfrak{a}}^{\mathfrak{b}}(M, N) = f_{\mathfrak{a}}^{\mathfrak{b}}(R/\mathfrak{p}, N)$, *and hence*

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M, N) = \inf\{f_{\mathfrak{a}}^{\mathfrak{b}}(R/\mathfrak{p}, N) \mid \mathfrak{p} \in \operatorname{Supp} M\}.$$

Proof. (i) If $\mathfrak{b} \subseteq \sqrt{(0 :_R H_{\mathfrak{a}}^i(M, N))}$, then $\mathfrak{b}^r H_{\mathfrak{a}}^i(M, N) = 0$ for some $r \in \mathbb{N}$. Therefore

$$\mathfrak{b}^r R' H_{\mathfrak{a}R'}^i(M \otimes_R R', N \otimes_R R') \cong \mathfrak{b}^r H_{\mathfrak{a}}^i(M, N) \otimes_R R' = 0.$$

(ii) Let $E^\bullet$ be an injective resolution of N. Then, in view of [18, Proposition 2.1],

$$H_{\mathfrak{a}}^i(M, N) = H^i(\operatorname{Hom}_R(M, \Gamma_{\mathfrak{a}}(E^\bullet))) = H^i(\operatorname{Hom}_R(M, \Gamma_{\sqrt{\mathfrak{a}}}(E^\bullet))) = H_{\sqrt{\mathfrak{a}}}^i(M, N).$$

(iii) This follows from Lemmas 3.3, 3.4 and (i).

(iv) It follows from the definition that $f_{\mathfrak{b}}^{\mathfrak{c}}(M, N) \le f_{\mathfrak{b}}^{\mathfrak{a}}(M, N)$. Also, since R is Noetherian, we can use (iii) to obtain $f_{\mathfrak{a}}^{\mathfrak{b}}(M, N) \le f_{\mathfrak{c}}^{\mathfrak{b}}(M, N)$.

(v) Since $\mathfrak{c}^n \Gamma_{\mathfrak{c}}(N) = 0$ for some $n \in \mathbb{N}$, we have $\mathfrak{c}^n H_{\mathfrak{a}}^i(M, \Gamma_{\mathfrak{c}}(N)) = 0$ for all $i \in \mathbb{N}_0$. Therefore $\mathfrak{b} \subseteq \mathfrak{c} \subseteq \sqrt{(0 :_R H_{\mathfrak{a}}^i(M, \Gamma_{\mathfrak{c}}(N)))}$ for all i. Now the exact sequence

$$0 \to \Gamma_{\mathfrak{c}}(N) \to N \to N/\Gamma_{\mathfrak{c}}(N) \to 0$$

induces the long exact sequence

$$\cdots \to H_{\mathfrak{a}}^i(M, \Gamma_{\mathfrak{c}}(N)) \to H_{\mathfrak{a}}^i(M, N) \to H_{\mathfrak{a}}^i(M, N/\Gamma_{\mathfrak{c}}(N)) \to H_{\mathfrak{a}}^{i+1}(M, \Gamma_{\mathfrak{c}}(N)) \to \cdots .$$

Therefore, by Lemma 3.4, $\mathfrak{b} \subseteq \sqrt{(0 :_R H_{\mathfrak{a}}^i(M, N))}$ if and only if

$$\mathfrak{b} \subseteq \sqrt{(0 :_R H_{\mathfrak{a}}^i(M, N/\Gamma_{\mathfrak{c}}(N)))}.$$

(vi) We may consider the long exact sequence

$$\cdots \to H_{\mathfrak{a}}^i(N, K) \to H_{\mathfrak{a}}^i(M, K) \to H_{\mathfrak{a}}^i(L, K) \to H_{\mathfrak{a}}^{i+1}(N, K) \to \cdots ,$$

which is obtained in [5, Lemma 2.4], and the long exact sequence

$$\cdots \to H_{\mathfrak{a}}^i(K, L) \to H_{\mathfrak{a}}^i(K, M) \to H_{\mathfrak{a}}^i(K, N) \to H_{\mathfrak{a}}^{i+1}(K, L) \to \cdots$$

and apply Lemma 3.4 to establish the assertion.

(vii) We prove, by induction on $r \in \mathbb{N}_0$, that, for any finite R-module M, if Supp $M \subseteq$ Supp N and $r \le f_{\mathfrak{a}}^{\mathfrak{b}}(N, K)$, then $r \le f_{\mathfrak{a}}^{\mathfrak{b}}(M, K)$. If $r = 0$ there is nothing to prove. Now suppose that $r > 0$ and assume that the assertion holds for smaller values of r. Suppose that Supp $M \subseteq$ Supp N and $r \le f_{\mathfrak{a}}^{\mathfrak{b}}(N, K)$. By Gruson's theorem [22, Theorem 4.1], there exists a chain

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

of submodules of M such that M_i/M_{i-1} is a homomorphic image of a direct sum of finitely many copies of N for all $i = 1, \ldots, n$. On the other hand, by (vi),

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M, K) \geq \inf\{f_{\mathfrak{a}}^{\mathfrak{b}}(M_1/M_0, K), \ldots, f_{\mathfrak{a}}^{\mathfrak{b}}(M_n/M_{n-1}, K)\}.$$

Therefore it is enough to prove that $r \leq f_{\mathfrak{a}}^{\mathfrak{b}}(M, K)$ in the case where $n = 1$. Now there exists an exact sequence

$$0 \to L \to N^{\alpha} \to M \to 0,$$

for some $\alpha \in \mathbb{N}$. Since $\operatorname{Supp} L \subseteq \operatorname{Supp} N$, the induction hypothesis implies that $r - 1 \leq f_{\mathfrak{a}}^{\mathfrak{b}}(L, K)$. Therefore, by (vi),

$$r \leq \inf\{f_{\mathfrak{a}}^{\mathfrak{b}}(L, K) + 1, f_{\mathfrak{a}}^{\mathfrak{b}}(N, K)\} \leq f_{\mathfrak{a}}^{\mathfrak{b}}(M, K).$$

(viii), (ix) and (x) are immediate by (vii). □

Next, we provide an example to show that the inequality in (vii) and (viii) may be strict.

Example 3.6. Let $(R, \mathfrak{m})$ be a Gorenstein local ring with dimension $d > 0$ and M be a finite R-module. Then $H_{\mathfrak{m}}^i(R) = E(R/\mathfrak{m})$ if $i = d$ and 0 otherwise. Further, $H_{\mathfrak{m}}^d(R) = E(R/\mathfrak{m})$ is not finite [4, Corollary 7.3.3], so $f_{\mathfrak{m}}(R) = d$. Now let $E^{\bullet}$ be a minimal injective resolution of R. Then

$$H_{\mathfrak{m}}^i(M, R) = H^i(\operatorname{Hom}_R(M, \Gamma_{\mathfrak{m}}(E^{\bullet}))) = \begin{cases} \operatorname{Hom}_R(M, E(R/\mathfrak{m})) & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$$

In particular,

$$H_{\mathfrak{m}}^i(R/\mathfrak{m}, R) = \begin{cases} R/\mathfrak{m} & \text{if } i = d, \\ 0 & \text{if } i \neq d, \end{cases}$$

and $f_{\mathfrak{m}}(R/\mathfrak{m}, R) = \infty > f_{\mathfrak{m}}(R)$. Moreover, this example shows that the following statements of Chu are not true.

(i) [5, Lemma 2.9]. Let N be a finitely generated R-module and M a nonzero cyclic R-module. Let t be a positive integer and let I be an ideal of R. If $H_I^i(N)$ is finitely generated for all $i < t$, then $H_I^t(N)$ is finitely generated if and only if $\operatorname{Hom}_R(M, H_I^t(N))$ is finitely generated.

(ii) [5, Proposition 2.10]. Let the situation be as in (i). Then $H_I^i(N)$ is finitely generated for all $i < t$ if and only if $H_I^i(M, N)$ is finitely generated for all $i < t$.

4. Faltings' local-global principle for the annihilation of generalised local cohomology modules

We say that the local-global principle for the annihilation of generalised local cohomology modules holds at level r if the statement

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M, N) > r \Leftrightarrow f_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) > r \quad \text{for all } \mathfrak{p} \in \operatorname{Spec}(R)$$

is true for every choice of ideals $\mathfrak{a}$, $\mathfrak{b}$ and every choice of finite R-modules M, N. Since $(H^i_{\mathfrak{a}}(M,N))_{\mathfrak{p}} \cong H^i_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$ for each $\mathfrak{p} \in \operatorname{Spec}(R)$, the above statement is equivalent to the statement

$$f^{\mathfrak{b}}_{\mathfrak{a}}(M,N) > r \Leftarrow f^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) > r \quad \text{for all } \mathfrak{p} \in \operatorname{Spec}(R).$$

We say that the local-global principle for the annihilation of generalised local cohomology modules holds (over the ring R) if the local-global principle for the annihilation of generalised local cohomology modules holds at level r for every $r \in \mathbb{N}_0$. The following lemma is needed in the proof of the next proposition.

Lemma 4.1 (see [3, Lemma 2.1] or [19, Lemma 3.1]). *Let M be an R-module such that the set Δ of all maximal members of* Ass M *is finite. Suppose that there exists a positive integer n such that $(\mathfrak{a}^n M)_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \Delta$. Then $\mathfrak{a}^n M = 0$.*

Proposition 4.2. *The local-global principle for the annihilation of generalised local cohomology modules holds at levels* 0, 1, 2.

Proof. Let $0 \le i \le 1$. Assume that M and N are finite R-modules and that $f^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) > i$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$. Since $H^0_{\mathfrak{a}}(M,N)$ is finite, by Corollary 2.3, we see that $\operatorname{Ass} H^i_{\mathfrak{a}}(M,N)$ is finite. Therefore there exists $n \in \mathbb{N}$ such that $(\mathfrak{b}R_{\mathfrak{p}})^n H^i_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = 0$ for all $\mathfrak{p} \in \operatorname{Ass} H^i_{\mathfrak{a}}(M,N)$. Hence $\mathfrak{b}^n H^i_{\mathfrak{a}}(M,N) = 0$ by Lemma 4.1; so the local-global principle for the annihilation of generalised local cohomology modules holds at levels 0, 1.

Now let $f^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) > 2$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$. The above argument shows that there exists $r \in \mathbb{N}$ such that $\mathfrak{b}^r H^i_{\mathfrak{a}}(M,N) = 0$ for $i = 0, 1$. Since $f^{\mathfrak{b}}_{\mathfrak{a}}(M,N) = f^{\mathfrak{b}}_{\mathfrak{a}}(M, N/\Gamma_{\mathfrak{b}}(N))$, we can assume without loss of generality that $\Gamma_{\mathfrak{b}}(N) = 0$; and so f-grad$(\mathfrak{a}, \mathfrak{b}, N) \ge 1$. Therefore, by Theorem 2.2, $H^1_{\mathfrak{a}}(M,N) = \operatorname{Hom}_R(R/\mathfrak{b}^r, H^1_{\mathfrak{a}}(M,N))$ is finite. Now, we can use Corollary 2.3 and Lemma 4.1 to obtain that $f^{\mathfrak{b}}_{\mathfrak{a}}(M,N) > 2$. □

The next theorem is concerned with the local-global principle for the annihilation of generalised local cohomology modules. The following lemma is of assistance in the proof of that theorem.

Lemma 4.3 [8, Theorem 3.1]. *Let M, N be finite R-modules and let $x_1, \ldots, x_n$ be an $\mathfrak{a}$-filter regular N-sequence in $\mathfrak{a}$. Then the following statements hold.*

(i) $H^i_{\mathfrak{a}}(M,N) \cong H^i_{(x_1,\ldots,x_n)}(M,N)$ *for all* $i < n$.

(ii) *If* proj $\dim_R(M) = d < \infty$ *and L is projective, then*

$$H^{i+n}_{\mathfrak{a}}(M \otimes_R L, N) \cong H^i_{\mathfrak{a}}(M, H^n_{(x_1,\ldots,x_n)}(L,N))$$

for all $i \ge d$.

Theorem 4.4. *Let M and N be finite R-modules and let $\mathfrak{b} \subseteq \mathfrak{a}$.*

(i) $f_{\mathfrak{a}}(M,N) \ge \inf\{f^{\mathfrak{b}}_{\mathfrak{a}}(M,N), \text{f-grad}\,(\mathfrak{a}, \mathfrak{b}, N) + 1\}$. *In particular,* $f_{\mathfrak{a}}(M,N) = f^{\mathfrak{b}}_{\mathfrak{a}}(M,N)$ *whenever* $f^{\mathfrak{b}}_{\mathfrak{a}}(M,N) \le \text{f-grad}\,(\mathfrak{a}, \mathfrak{b}, N) + 1$.

(ii) *Assume that* $r \leq \text{f-grad}\,(\mathfrak{a}, \mathfrak{b}, N) + 1$. *Then*

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M, N) > r \Leftrightarrow f_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) > r \quad \textit{for all } \mathfrak{p} \in \operatorname{Spec}(R).$$

(iii) *If* $\operatorname{Supp} N/\mathfrak{b}N \subseteq V(\mathfrak{a})$*, then, for all* $r \in \mathbb{N}_0$,

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M, N) > r \Leftrightarrow f_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) > r \quad \textit{for all } \mathfrak{p} \in \operatorname{Spec}(R).$$

(iv) *Faltings' local-global principle for the finiteness of generalised local cohomology modules holds, that is, for any positive integer r,* $H^i_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$ *is finite for all* $i \leq r$ *and for all* $\mathfrak{p} \in \operatorname{Spec}(R)$ *if and only if* $H^i_{\mathfrak{a}}(M, N)$ *is finite for all* $i \leq r$.

Proof. (i) Set $g = \text{f-grad}\,(\mathfrak{a}, \mathfrak{b}, N)$. If $f_{\mathfrak{a}}^{\mathfrak{b}}(M, N) \leq g + 1$, then, for any $i < f_{\mathfrak{a}}^{\mathfrak{b}}(M, N)$, we have $H^i_{\mathfrak{a}}(M, N) = H^i_{\mathfrak{b}}(M, N)$ by Lemma 4.3(i); and hence $\mathfrak{b} \subseteq \sqrt{(0 : H^i_{\mathfrak{b}}(M, N))}$. Then by Proposition 3.1, $H^i_{\mathfrak{a}}(M, N)$ is finite for all $i < f_{\mathfrak{a}}^{\mathfrak{b}}(M, N)$ and hence, by Proposition 3.5(iv), $f_{\mathfrak{a}}^{\mathfrak{b}}(M, N) = f_{\mathfrak{a}}(M, N)$.

Therefore we may assume that $f_{\mathfrak{a}}^{\mathfrak{b}}(M, N) > g + 1$. Using the same argument as above, we see that $H^i_{\mathfrak{a}}(M, N)$ is finite for all $i < g$. Therefore, by Theorem 2.2, $\operatorname{Hom}_R(R/\mathfrak{b}^{\alpha}, H^g_{\mathfrak{a}}(M, N))$ is finite for all $\alpha \in \mathbb{N}$. On the other hand, by hypothesis $\mathfrak{b}^{\alpha} H^g_{\mathfrak{a}}(M, N) = 0$ for some $\alpha \in \mathbb{N}$. Thus $H^g_{\mathfrak{a}}(M, N)$ is finite and $f_{\mathfrak{a}}(M, N) \geq g + 1$.

(ii) Suppose that $r \leq \text{f-grad}\,(\mathfrak{a}, \mathfrak{b}, N) + 1$ and $f_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) > r$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$. If $f_{\mathfrak{a}}^{\mathfrak{b}}(M, N) \leq r$, then by (i), $f_{\mathfrak{a}}(M, N) = f_{\mathfrak{a}}^{\mathfrak{b}}(M, N)$. So by Corollary 2.3, $\operatorname{Ass} H_{\mathfrak{a}}^{f_{\mathfrak{a}}^{\mathfrak{b}}(M,N)}(M, N)$ is finite. This is a contradiction in view of Lemma 4.1. Hence $f_{\mathfrak{a}}^{\mathfrak{b}}(M, N) > r$.

(iii) Suppose that $\operatorname{Supp} N/\mathfrak{b}N \subseteq V(\mathfrak{a})$. Then $\text{f-grad}\,(\mathfrak{a}, \mathfrak{b}, N) = \infty$. Thus (iii) is an immediate consequence of (ii).

(iv) This is immediate by (iii) and Proposition 3.1. □

5. Local-global principle for the Artinianness of generalised local cohomology modules

Let M be a finite R-module. In [20], Tang proved that, for any integer n, $H^i_{\mathfrak{a}}(M)$ is Artinian for all $i < n$ if and only if $H^i_{\mathfrak{a}}(M)_{\mathfrak{p}}$ is Artinian for all $i < n$ and for all $\mathfrak{p} \in \operatorname{Spec}(R)$. In Theorem 5.3, we establish the above result for generalised local cohomology modules. The corollary to the following theorem is needed in the proof of Theorem 5.3.

Theorem 5.1 [8, Theorem 4.2]. *Let* $\mathcal{M}$ *be the set of all finite subsets of* $\max(R)$. *Then*

$$\begin{aligned}
&\sup_{A \in \mathcal{M}} \text{f-grad}\left(\bigcap_{\mathfrak{m} \in A} \mathfrak{m}, \mathfrak{a} + \operatorname{Ann} M, N\right) \\
&\quad = \inf\{i \in \mathbb{N}_0 \mid H^i_{\mathfrak{a}}(M, N) \textit{ is not Artinian}\} \\
&\quad = \inf\{i \in \mathbb{N}_0 \mid \operatorname{Supp} H^i_{\mathfrak{a}}(M, N) \nsubseteq \max(R)\} \\
&\quad = \inf\{i \in \mathbb{N}_0 \mid \operatorname{Supp} H^i_{\mathfrak{a}}(M, N) \nsubseteq A \textit{ for all } A \in \mathcal{M}\}.
\end{aligned}$$

COROLLARY 5.2. *Let M and N be finite R-modules. Then*

$$\inf\{i \in \mathbb{N}_0 \mid H^i_{\mathfrak{a}}(M, N) \text{ is not Artinian}\} = \inf\{i \in \mathbb{N}_0 \mid \dim \operatorname{Ext}^i_R(M/\mathfrak{a}M, N) > 0\}.$$

PROOF. Let $n \in \mathbb{N}_0$. By Theorem 5.1, $H^i_{\mathfrak{a}}(M, N)$ is an Artinian R-module for all $i \leq n$ if and only if $n <$ f-grad $(\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_t, \mathfrak{a} + \operatorname{Ann} M, N)$ for some maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$ of R. Also, by the facts mentioned at the beginning of Section 2, this is equivalent to $\operatorname{Supp} \operatorname{Ext}^i_R(M/\mathfrak{a}M, N) \subseteq \{\mathfrak{m}_1, \ldots, \mathfrak{m}_t\}$ for some maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$ of R and for all $i \leq n$. □

THEOREM 5.3. *Let M, N be finite R-modules and let n be a positive integer. Then the following statements are equivalent.*

(i) *$H^i_{\mathfrak{a}}(M, N)$ is Artinian for all $i < n$.*
(ii) *$H^i_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$ is Artinian for all $i < n$ and for all $\mathfrak{p} \in \operatorname{Spec}(R)$.*

PROOF. It is clear that $\dim \operatorname{Ext}^i_R(M/\mathfrak{a}M, N) = 0$ for all $i < n$ if and only if $\dim \operatorname{Ext}^i_R(M/\mathfrak{a}M, N)_{\mathfrak{p}} = 0$ for all $i < n$ and for all prime ideals $\mathfrak{p}$ of R. Therefore the assertion follows from Corollary 5.2. □

COROLLARY 5.4. *Let M, N be finite R-modules and let n be a positive integer. Then the following statements are equivalent.*

(i) *$H^i_{\mathfrak{a}}(M, N)$ has finite length for all $i < n$.*
(ii) *$H^i_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$ has finite length for all $i < n$ and for all $\mathfrak{p} \in \operatorname{Spec}(R)$.*

PROOF. This is immediate by Theorems 5.3 and 2.2. □

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