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LINEAR QUOTIENTS AND MULTIGRADED SHIFTS OF BOREL IDEALS

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Abstract

We investigate whether the property of having linear quotients is inherited by ideals generated by multigraded shifts of a Borel ideal and a squarefree Borel ideal. We show that the ideal generated by the first multigraded shifts of a Borel ideal has linear quotients, as do the ideal generated by the kth multigraded shifts of a principal Borel ideal and an equigenerated squarefree Borel ideal for each k. Furthermore, we show that equigenerated squarefree Borel ideals share the property of being squarefree Borel with the ideals generated by multigraded shifts.

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1. Introduction

Syzygies of a multigraded module are of great interest from various point of view (see, for example, [2–4, 7, 12]). We shall consider monomial ideals as multigraded modules and ask which properties are shared between the ideal itself and the ideals generated by its multigraded shifts. More precisely, let $S = k[x_1, \dots, x_n]$ be the polynomial ring in the variables $x_1, \dots, x_n$ over a field k. We consider this ring with its natural multigrading and refer to a monomial and its multidegree interchangeably, that is, $S(\mathbf{x}^{\mathbf{a}})$ denotes the free S-module with one generator of multidegree $\mathbf{x}^{\mathbf{a}}$. Suppose that $I \subseteq S$ is a monomial ideal with minimal multigraded free resolution

$$\mathbf{F} : 0 \to F_p \to \cdots \to F_1 \to F_0,$$

where

$$F_k = \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} S(\mathbf{x}^{\mathbf{a}})^{\beta_{k,\mathbf{a}}}.$$

Consider the ideal $J_k(I) = (\{\mathbf{x}^{\mathbf{a}} \mid \beta_{k,\mathbf{a}} \neq 0\})$ generated by the kth multigraded shifts of I. We ask which properties are preserved when transferring from I to $J_k(I)$.

It is possible to construct a monomial ideal with a given resolution (see, for example, [1, 6, 13, 15]). However, we will see that there may be no ideal with a specified property which has a given set of kth multigraded shifts.

We study these questions for Borel ideals and squarefree Borel ideals. Along with a variety of significant properties of Borel ideals, a result by Galligo [9] states that if k is of characteristic zero, the generic initial ideal of a graded ideal is always a Borel ideal. So, inheriting important homological properties by generic initial ideals makes Borel ideals more interesting. Multigraded shifts of these ideals were studied in Eliahou and Kervaire [5].

Borel ideals, as well as their squarefree analogue, squarefree Borel ideals, have the important property of having linear quotients. We investigate whether this property is inherited by the ideals generated by the multigraded shifts. Miller and Sturmfels [14, Theorem 2.18] showed that the ideal generated by the first multigraded shifts of an equigenerated Borel ideal has linear resolution. We will show in Proposition 3.2 that this ideal has linear quotients. We present an example in Section 3 to show that the same statement does not necessarily hold for the ideal generated by the kth shifts if $2 \leq k \leq \mathrm{pd}(I)$, the projective dimension of I. However, in Theorems 3.4 and 4.3, we show that if I is a principal Borel ideal or a squarefree Borel ideal, then the ideal generated by the kth multigraded shifts has linear quotients for $k = 0, \ldots, \mathrm{pd}(I)$. Furthermore, Proposition 4.1 shows that the property of being squarefree Borel is inherited by ideals generated by the multigraded shifts whenever I is equigenerated.

2. Preliminaries

In this section, we explain some terminology and facts that we shall use later.

Throughout, $S = k[x_1, \ldots, x_n]$ denotes a polynomial ring over a field k of characteristic zero with its natural multigrading, where $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$ is the unique monomial with multidegree $(a_1, \ldots, a_n)$. A monomial and its multidegree will be used interchangeably.

Let $u = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be a monomial in S. The degree of u in x_i, denoted by $\deg_i u$, is defined to be α_i. We denote the segments $x_1^{\alpha_1} \cdots x_{i-1}^{\alpha_{i-1}}$ and $x_1^{\alpha_1} \cdots x_i^{\alpha_i}$ of u respectively by $u_{<i}$ and $u_{\leq i}$. The final segments $u_{>i}$ and $u_{\geq i}$ of u are defined similarly. Set $\min u = \min\{i : \alpha_i \neq 0\}$ and $\max u = \max\{i : \alpha_i \neq 0\}$. By abuse of notation, if $\min u = i_0$ and $\max u = j_0$, we denote the variable x_{i_0} by $\min u$ and the variable x_{j_0} by $\max u$. We call a sequence $x_{i_1}, \ldots, x_{i_k}$ of variables *admissible* for the monomial u if $i_1, \ldots, i_k$ are pairwise distinct and $1 \leq i_t < \max u$ for each t. If u is a squarefree monomial, an admissible sequence $x_{i_1}, \ldots, x_{i_k}$ for u is called *squarefree admissible* when none of the elements of the sequence divides u, that is, $ux_{i_1} \cdots x_{i_k}$ is a squarefree monomial. As usual, the notation $x_{i_1} \cdots \hat{x}_t \cdots x_{i_k}$ shows a removed variable x_t from a product $x_{i_1} \cdots x_t \cdots x_{i_k}$.

An operation which sends the monomial u to a monomial $(u/x_j)x_i$ is called a *Borel move* if x_j divides u and $i < j$. When u is a squarefree monomial, such a Borel move is called a *squarefree Borel move* on u if the monomial $(u/x_j)x_i$ is also squarefree. If $v \in S$ is also a monomial, then $u : v$ denotes the monomial $u/\gcd(u, v)$.

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Let $I \subseteq S$ be a monomial ideal. We denote its minimal set of monomial generators by $G(I)$ and its projective dimension by $\mathrm{pd}(I)$. Suppose that

$$\mathbf{F} : 0 \to F_p \to \cdots \to F_1 \to F_0$$

is the minimal multigraded free resolution of I with

$$F_k = \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} S(\mathbf{x}^{\mathbf{a}})^{\beta_{k,\mathbf{a}}}.$$

The set of *kth multigraded shifts* of I is

$$B_k = \{\mathbf{x}^{\mathbf{a}} \mid \beta_{k,\mathbf{a}} \neq 0\}.$$

The ideal generated by B_k will be denoted by $J_k(I)$ or simply by J_k.

The monomial ideal I is called a *Borel ideal* if it is closed under Borel moves. A subset B of the Borel ideal I is called its *Borel generator* if I is the smallest Borel ideal containing B. The ideal I is said to be a *principal Borel ideal* if it has a Borel generator of cardinality one. The ideal I is a *squarefree Borel ideal* if it is a squarefree monomial ideal and closed under squarefree Borel moves. In [8], some important invariants of Borel ideals and squarefree Borel ideals were studied by applying their Borel generators.

Let $I \subseteq S$ be a Borel ideal. By [5, Theorem 2.1], the minimal multigraded free resolution $\mathbf{F}$ of I can be described as follows: the basis of the multigraded free S-module F_k in homological degree k of $\mathbf{F}$ is formed by monomials $ux_{i_1} \cdots x_{i_k}$, where $u \in G(I)$ and $x_{i_1}, \ldots, x_{i_k}$ is an admissible sequence for u.

Let $J \subseteq S$ be a squarefree Borel ideal. Suppose that $I \subseteq S$ is the smallest Borel ideal containing J with minimal multigraded free resolution $\mathbf{F}$ as described above. The ideal J is generated by $\{u \in G(I) : u \text{ divides } x_1 \cdots x_n\}$. By [10, Theorem 2.1] and [11, Lemma 4.4.1], the minimal multigraded free resolution of J is the subcomplex $\mathbf{G}$ of $\mathbf{F}$ which is generated by those multihomogeneous basis elements that divide $x_1 \cdots x_n$. In other words, the basis of the free S-module G_k in homological degree k consists of $ux_{i_1} \cdots x_{i_k}$, where $u \in G(J)$ and $x_{i_1}, \ldots, x_{i_k}$ is a squarefree admissible sequence for u. See also [1, Theorem 2.1] for an explicit description of the chain maps of this resolution.

A monomial ideal $I \subseteq S$ is said to have linear quotients if there exists an ordering $u_1, \ldots, u_r$ of the elements of $G(I)$ such that, for each $i = 1, \ldots, r-1$, the colon ideal $(u_1, \ldots, u_i) : (u_{i+1})$ is generated by a subset of $\{x_1, \ldots, x_n\}$.

3. Borel ideals

In this section, we study whether the ideals generated by multigraded shifts of a Borel ideal inherit the property of having linear quotients.

For ease of reference, we formulate the following lemma.

Lemma 3.1. *Let $I \subseteq S$ be a monomial ideal and u a monomial in $G(I)$. Suppose that $u' = ux_i/x_j \in I$ for some distinct i and j. Then there exists $\tilde{u} \in G(I)$ such that $\tilde{u} \mid u'$ and*

$$\deg_i \tilde{u} = \deg_i u' = \deg_i u + 1.$$

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Let I be an equigenerated Borel ideal. By [14, Theorem 2.18], the ideal generated by the first multigraded shifts of I has linear resolution. In the next result, we show that this ideal has linear quotients.

Proposition 3.2. *Let $I \subseteq S$ be an equigenerated Borel ideal. Then the ideal J_1 generated by its first multigraded shifts has linear quotients.*

Proof. Let $G(J_1) = \{w_1, \ldots, w_r\}$, where $w_1 > \cdots > w_r$ in the lexicographic order induced by the ordering $x_1 > x_2 > \cdots > x_n$ of variables. Consider two distinct elements $w_i = ux_{i_0}$ and $w_j = vx_{j_0}$ in J_1, where $u, v \in G(I)$, $i_0 < \max u$ and $j_0 < \max v$. Suppose that $i < j$ and

$$w_i : w_j = x_{\ell_1} \cdots x_{\ell_p}, \tag{3.1}$$

where $\ell_1 \leq \ell_2 \leq \cdots \leq \ell_p$ and $p \geq 2$. We show that there exists $w_\ell \in G(J_1)$ such that $\ell < j$ and $w_\ell : w_j = x_{\ell_t}$ for some $t \in \{1, \ldots, p\}$.

If $\ell_1 < j_0$, then the element $w_\ell = vx_{\ell_1} \in G(J_1)$ is the desired element. Indeed, $vx_{\ell_1} \in G(J_1)$ by the description of multigraded shifts given in Section 2 and because J_1 is equigenerated.

Now assume that $j_0 \leq \ell_1$. Then $\deg(w_i) = \deg(w_j)$ and, by (3.1), $\deg(w_i : w_j) \geq 2$. So, $\deg(w_j : w_i) \geq 2$. On the other hand, $(w_i)_{<\ell_1} = (w_j)_{<\ell_1}$ because $w_i >_{lex} w_j$ and $w_i : w_j = x_{\ell_1} \cdots x_{\ell_p}$. Now, by the assumption that $j_0 \leq \ell_1$ and since $\deg(w_j : w_i) \geq 2$ and $(w_i)_{<\ell_1} = (w_j)_{<\ell_1}$, it follows that $\deg(v_{>\ell_1}) \geq 2$. As a consequence,

$$\ell_1 < \max\left(\frac{v}{\max v} x_{\ell_1}\right).$$

So, $w_\ell = ((v/\max v)x_{\ell_1})x_{j_0} \in G(J_1)$ is the desired element. Here $(v/\max v)x_{\ell_1} \in G(I)$ because I is an equigenerated Borel ideal. □

The following example shows that if I is a nonequigenerated Borel ideal, the ideal $J_1(I)$ does not have linear quotients in general.

Example 3.3. Let I be the Borel ideal in $k[x_1, x_2, x_3, x_4]$ whose Borel generator is $\{x_1x_2, x_2^4, x_1x_3^2x_4^2\}$, that is,

$$I = \langle x_1^2, x_1x_2, x_2^4, x_1x_3^4, x_1x_3^3x_4, x_1x_3^2x_4^2 \rangle.$$

Consider the monomials $w = (x_2^4) \cdot x_1$ and $w' = (x_1x_3^2x_4^2) \cdot x_2$ in the minimal system of monomial generators of $J_1(I)$. Then $w : w' = x_2^3$ but there exists no monomial $w'' \in G(J_1(I))$ such that $w'' : w' = x_2$. On the other hand, $w' : w = x_3^2x_4^2$ but there exists no monomial $w'' \in G(J_1(I))$ such that $w'' : w = x_3$ or x_4. Consequently, the ideal $J_1(I)$ does not have linear quotients with respect to any ordering of the minimal system of monomial generators.

Theorem 3.4. *Let $I \subseteq S$ be a principal Borel ideal. Then the ideal J_k generated by the kth multigraded shifts has linear quotients for $k = 0, \ldots, \mathrm{pd}(I)$.*

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Proof. Arrange the elements of $G(J_k)$ in decreasing order $x_1 > x_2 > \cdots > x_n$ with respect to the lexicographical order. Consider two distinct elements $w_i = ux_{i_1}x_{i_2}\cdots x_{i_k}$ and $w_j = vx_{j_1}x_{j_2}\cdots x_{j_k}$ in J_k, where $u, v \in G(I)$, $i_1 < \cdots < i_k < \max u$ and $j_1 < \cdots < j_k < \max v$. Suppose that $w_i >_{lex} w_j$ and

$$w_i : w_j = x_{\ell_1}\cdots x_{\ell_p}, \tag{3.2}$$

where $\ell_1 \le \ell_2 \le \cdots \le \ell_p$ and $p \ge 2$. Using an inductive argument, it is enough to show that there exists $w \in G(J_k)$ with the following properties:

(i) $w >_{lex} w_j$;
(ii) the monomial $w : w_j$ divides $w_i : w_j$;
(iii) $\deg(w : w_j) < \deg(w_i : w_j)$.

By (3.2) and since $w_i >_{lex} w_j$, the monomials w_i and w_j have the same segment before ℓ_1, that is,

$$(w_i)_{<\ell_1} = (w_j)_{<\ell_1}. \tag{3.3}$$

If the degree of the segment $\tilde{v} = v_{>\ell_1}$ is at least two, then

$$w = \Big(\frac{v}{\min(\tilde{v})}x_{\ell_1}\Big)(x_{j_1}\cdots x_{j_k})$$

is the desired element with $w : w_j = x_{\ell_1}$. So, for the rest of the proof, we assume that

$$\deg(v_{>\ell_1}) = 1. \tag{3.4}$$

The elements w_i and w_j have the same degree. So, by (3.2), $\deg(w_j : w_i) = p > 1$. On the other hand, $(w_j)_{\le\ell_1}$ divides $(w_i)_{\le\ell_1}$ by (3.3) and $\deg(v_{>\ell_1}) = 1$. Hence, for some $j_s \in \{j_1, \ldots, j_k\}$,

$$\ell_1 < j_s < \max v \quad \text{and} \quad x_{j_s} \nmid w_i. \tag{3.5}$$

Recall that I is a principal Borel ideal. So, there exists a monomial $m \in G(I)$ such that I is the smallest Borel ideal containing m. As a result, for each i with $\max u < i \le \max m$, the monomial $(u/\max u)x_i$ is again an element of $G(I)$, although it is not obtained from a Borel move. It follows that whenever $\max u < \max v$, we have $u' = (u/\max u)\max v \in G(I)$. Thus, $w'_i = u'x_{i_1}\cdots x_{i_k} \in G(J_k)$. Moreover, $w'_i : w_j$ divides $w_i : w_j$ and $w'_i >_{lex} w_j$ (by comparing the segments $(w'_i)_{\le\ell_1}$ and $(w_j)_{\le\ell_1}$). Replacing w_i with w'_i if needed, for the rest of the proof we may assume that

$$\max v \le \max u. \tag{3.6}$$

On the other hand, if $\ell_p \in \{i_1, \ldots, i_k\}$, say $\ell_p = i_t$, then $w = ux_{i_1}\cdots\hat{x}_{i_t}\cdots x_{i_k}x_{j_s}$ is the desired element, where j_s is the element with the properties given in (3.5). Here one has $w >_{lex} w_j$ because of the equality of the segments $w_{<\ell_1} = (w_i)_{<\ell_1} = (w_j)_{<\ell_1}$ and the fact that $\deg_{\ell_1} w = \deg_{\ell_1} w_i > \deg_{\ell_1} w_j$. So, for the rest of the proof, we also assume that

$$\ell_p \notin \{i_1, \ldots, i_k\}. \tag{3.7}$$

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We consider two cases in both of which the desired element w is obtained by replacing ℓ_p with j_s in w_i. So, by the same argument on the segments of w and w_j before ℓ_1 and their degree in ℓ_1, we will have $w >_{\text{lex}} w_j$.

Case 1. $\ell_p = \max u$. Set $q = \max\{i_1, \ldots, i_k, j_s\}$. Since $j_s < \max v$ and $\max v \leq \max u$ by (3.6), we obtain $u' = (u/\max u)x_q \in G(I)$ by a Borel move. Furthermore, $x_{j_s} \notin \{i_1, \ldots, i_k\}$ by (3.5). Hence, $\{x_{i_1}, \ldots, x_{i_k}, x_{j_s}\} \setminus \{x_q\}$ gives an admissible sequence for u'. Consequently,

$$w = u' \frac{x_{i_1} \cdots x_{i_k} x_{j_s}}{x_q} \in G(J_k)$$

is the desired element, obtained by replacing $\ell_p = \max u$ in w_i with j_s.

Case 2. $\ell_p < \max u$. Then $\deg u_{>\ell_1} \geq 2$ by (3.7) and because x_{ℓ_p} is a distinct variable and $\max u$ divides u. On the other hand, u and v have the same degree and $\deg(v_{>\ell_1}) = 1$ by (3.4). Hence, $(\deg v_{<\ell_1}) > (\deg u_{<\ell_1})$. But $(w_i)_{<\ell_1} = (w_j)_{<\ell_1}$ by (3.3), so there exists $i_t \in \{i_1, \ldots, i_k\}$ such that $i_t < \ell_1$ and $\deg_{i_t} u < \deg_{i_t} v$. By a Borel move, set

$$u' = \frac{u}{x_{\ell_p}} x_{i_t} \in G(I).$$

Since we have assumed that $\ell_p < \max u$, it follows that $\max u' = \max u$,

$$x_{i_1}, \ldots, \hat{x}_{i_t}, \ldots, x_{i_k}, x_{j_s}$$

is an admissible sequence for u' and

$$w = u' x_{i_1} \cdots \hat{x}_{i_t} \cdots x_{i_k} x_{j_s}$$

is the desired element. □

The following example shows that if I is an equigenerated Borel ideal, but not principal, then the ideals generated by multigraded shifts do not need to have linear quotients with respect to any ordering of the generators.

Example 3.5. Let $I \subseteq k[x_1, \ldots, x_7]$ be a Borel ideal with Borel generator

$$B = \{x_1 x_2 x_7, x_3 x_4^2\}.$$

Then $w = (x_1 x_2 x_7)(x_3 x_4 x_5)$ and $w' = (x_3 x_4^2)(x_1 x_2 x_3)$ belong to $J_3(I)$ and

$$w : w' = x_5 x_7 \quad \text{and} \quad w' : w = x_3 x_4.$$

For each $w'' \in J_3(I)$,

$$w'' : w' \neq x_5 \quad \text{and} \quad w'' : w' \neq x_7.$$

So, $J_3(I)$ does not have linear quotients with respect to any ordering of the minimal system of monomial generators in which w appears earlier than w'. It can be also seen that for each $w'' \in J_3(I)$,

$$w'' : w \neq x_3 \quad \text{and} \quad w'' : w \neq x_4.$$

Thus, $J_3(I)$ also does not have linear quotients when w' appears earlier than w in the ordering of $G(J_3(I))$.

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4. Squarefree Borel ideals

In this section, we investigate when the properties of being squarefree Borel and having linear quotients are inherited by the ideals generated by multigraded shifts of a squarefree Borel ideal.

Proposition 4.1. *Let $I \subseteq S$ be an equigenerated squarefree Borel ideal. Then the ideal J_k generated by the kth multigraded shifts of I is also a squarefree Borel ideal for $k = 0, \ldots, \mathrm{pd}(I)$.*

Proof. Let $w = ux_{i_1} \cdots x_{i_k} \in G(J_k)$, where $u \in G(I)$ and $i_1 < \cdots < i_k < \max u$. Suppose that $x_j \,|\, w$. For each ℓ with $\ell < j$ and $x_\ell \nmid w$, we show that $(w/x_j)x_\ell \in J_k$. We distinguish two cases.

Case 1. $x_j \,|\, u$. From the description of the multigraded shifts of squarefree Borel ideals in Section 2, $j \neq i_k$ because w is a squarefree monomial. Set

$$t = \max\{\ell, i_k\} \quad \text{and} \quad \bar{t} = \min\{\ell, i_k\}.$$

Then $t \neq \bar{t}$ because the fact that $x_\ell \nmid w$ means that $\ell \neq i_k$. If $j > i_k$, then $x_{i_1}, \ldots, x_{i_{k-1}}, x_{\bar{t}}$ is a squarefree admissible sequence for $u' = (u/x_j)x_t \in G(I)$. Hence,

$$\frac{w}{x_j}x_\ell = u' x_{i_1} \cdots x_{i_{k-1}} x_{\bar{t}} \in J_k.$$

If $j < i_k$, then $j < \max u$ and so $\max u = \max((u/x_j)x_\ell)$. Therefore, $x_{i_1}, \ldots, x_{i_k}$ is a squarefree admissible sequence for $u'' = (u/x_j)x_\ell \in G(I)$ and

$$\frac{w}{x_j}x_\ell = u'' x_{i_1} \cdots x_{i_k} \in J_k.$$

Case 2. $x_j \nmid u$. In this case, $j \in \{i_1, \ldots, i_k\}$, say $j = i_t$, and we have a squarefree admissible sequence $x_{i_1}, \ldots, \hat{x}_{i_t}, \ldots, x_{i_k}, x_\ell$. Hence,

$$\frac{w}{x_j}x_\ell = ux_{i_1} \cdots \hat{x}_{i_t} \cdots x_{i_k} x_\ell \in J_k. \qquad \square$$

The following example shows that if a squarefree ideal I is not equigenerated, then the ideals generated by its multigraded shifts do not need to be squarefree Borel.

Example 4.2. Consider the squarefree Borel ideal

$$I = (x_1x_2, x_1x_3x_4, x_1x_3x_5, x_2x_3x_4, x_2x_3x_5) \subseteq k[x_1, \ldots, x_5].$$

Since x_4 is a squarefree admissible sequence for $x_2x_3x_5$, the monomial

$$w = (x_2x_3x_5)(x_4)$$

belongs to $J_1(I)$. But $(w/x_3)x_1 \notin J_1(I)$ because x_3 divides every monomial in $J_1(I)$. So, $J_1(I)$ is not a squarefree Borel ideal.

Theorem 4.3. *Let $I \subseteq S$ be a squarefree Borel ideal. Then the ideal J_k generated by the kth multigraded shifts of I has linear quotients for $k = 0, \dots, \mathrm{pd}(I)$.*

Proof. Let $G(J_k) = \{w_1, \dots, w_r\}$, where $i < j$ implies that either (i) $\deg(w_i) < \deg(w_j)$ or (ii) $\deg(w_i) = \deg(w_j)$ and $w_i >_{lex} w_j$. Here the lexicographical order is induced by the ordering $x_1 > x_2 > \cdots > x_n$.

Consider two distinct elements $w_i = ux_{i_1} \cdots x_{i_k}$ and $w_j = vx_{j_1} \cdots x_{j_k}$ in the minimal system of monomial generators of J_k, where $u, v \in G(I)$, $1 \le i_1 < \cdots < i_k < \max u$ and $1 \le j_1 < \cdots < j_k < \max v$. Recall from Section 2 that J_k is a squarefree monomial ideal. Suppose that $i < j$ and

$$w_i : w_j = x_{\ell_1} \cdots x_{\ell_p},$$

where $\ell_1 \le \ell_2 \le \cdots \le \ell_p$ and $p \ge 2$. We show that there exists $w_\ell \in G(J_k)$ such that $l < j$ and $w_\ell : w_j = x_{\ell_t}$ for some $t \in \{1, \dots, p\}$.

Since w_i and w_j are squarefree monomials and $w_i : w_j = x_{\ell_1} \dots x_{\ell_p}$, we see that x_{ℓ_1} does not divide w_j. In particular, $\ell_1 \neq \max v$. We consider two cases.

Case 1. $\ell_1 < \max v$. As discussed above, x_{ℓ_1} does not divide w_j and consequently

$$\ell_1 \notin \{j_1, \dots, j_k\}.$$

If $\ell_1 < j_k$, we take w to be the squarefree monomial $w = vx_{j_1} \cdots x_{j_{k-1}} x_{\ell_1}$, so that $w \in J_k$. By Lemma 3.1, we can choose an element $w_\ell \in G(J_k)$ as required such that $w_\ell \mid w$ and $\deg_{\ell_1} w_\ell = \deg_{\ell_1} w$. Otherwise, if $j_k < \ell_1$, by a squarefree Borel move consider

$$v' = \frac{v}{\max v} x_{\ell_1} \in I. \tag{4.1}$$

Here $x_{\ell_1} \nmid v$ because w_j is a squarefree monomial and $w_i : w_j = x_{\ell_1} \cdots x_{\ell_p}$. By Lemma 3.1, there exists $\tilde{v} \in G(I)$ such that $\tilde{v} \mid v'$ and $\deg_{\ell_1} \tilde{v} = \deg_{\ell_1} v'$. So, x_{ℓ_1} divides $\tilde{v}$. In particular, $x_{j_1} \cdots x_{j_k}$ is a squarefree admissible sequence for $\tilde{v}$ because we have assumed that $j_k < \ell_1$. We set

$$w = \tilde{v} x_{j_1} \cdots x_{j_k} \in J_k$$

and choose $w_\ell \in G(J_k)$ such that $w_\ell \mid w$. One has either (i) $\deg w_\ell < \deg w_j$ or (ii) $\deg w_\ell = \deg w_j$ and in this case $w_\ell >_{lex} w_j$ by (4.1). So, w_ℓ appears earlier than w_j in the ordering of elements of $G(J_k)$ described in the first part of the proof. Moreover, x_{ℓ_1} divides w_ℓ (otherwise, w_ℓ is an element of degree less than w_j which divides w_j; a contradiction to the fact that $w_j \in G(J_k)$). Hence, we also have $w_\ell : w_j = x_{\ell_1}$.

Case 2. $\max v < \ell_1$. Since w_i and w_j are elements of $G(J_k)$, w_j does not divide w_i. So, there exists $s < \ell_1$ such that $\deg_{x_s} w_i < \deg_{x_s} w_j$. In particular,

$$x_s \nmid w_i$$

because w_i and w_j are squarefree monomials. Set

$$t = \max\{i_k, s\} \quad \text{and} \quad \bar{t} = \min\{i_k, s\}.$$

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Consider $u' = (u/\max u)x_t \in I$ and, via Lemma 3.1, choose a monomial $\tilde{u} \in G(I)$ such that $\tilde{u} \,|\, u'$ and $x_t \,|\, \tilde{u}$. Then $x_{i_1}, \ldots, x_{i_{k-1}}, x_{\bar{t}}$ is a squarefree admissible sequence for $\tilde{u}$ and consequently $w = \tilde{u}x_{i_1} \cdots x_{i_{k-1}} x_{\bar{t}} \in J_k$. Choose an element $\tilde{w}_\ell \in G(J_k)$ for which $\tilde{w}_\ell \,|\, w$. The same argument as used in Case 1 shows that the monomial $\tilde{w}_\ell$ appears earlier than w_j in the ordering of the elements of $G(J_k)$. Furthermore, $\max v < \ell_1 \leq \max u$, so $\max v < \max u$ and $\ell_p = \max u$. In particular,

$$\tilde{w}_\ell : w_j \text{ divides } x_{\ell_1} \cdots x_{\ell_{p-1}}.$$

Now proceeding by induction on p completes the proof. □

EXAMPLE 4.4. Let $I \subseteq k[x_1, \ldots, x_5]$ be the squarefree Borel ideal with Borel generator $\{x_2, x_3x_5\}$, that is,

$$I = (x_1, x_2, x_3x_4, x_3x_5).$$

Then the ideals $J_0(I) = I$ and

$$J_1(I) = (x_1x_2, x_1x_3x_4, x_1x_3x_5, x_2x_3x_4, x_2x_3x_5, x_3x_4x_5)$$

with respect to the given ordering have linear quotients. From the proof of Theorem 4.3, minimal generators with lower degree appear earlier or, in case of equality of degrees, those generators which are lexicographically greater appear earlier. Also,

$$J_2(I) = (x_1x_2x_3x_4, x_1x_2x_3x_5, x_1x_3x_4x_5, x_2x_3x_4x_5),$$

$J_3(I) = (x_1x_2x_3x_4x_5)$ and $J_k(I) = (0)$ whenever $k > 3$. Their generators are ordered as specified and they have linear quotients with respect to these orderings.

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