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MATROIDS WHOSE GROUND SETS ARE DOMAINS OF FUNCTIONS

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Abstract

From an integer-valued function f we obtain, in a natural way, a matroid M_f on the domain of f. We show that the class $\mathfrak{M}$ of matroids so obtained is closed under restriction, contraction, duality, truncation and elongation, but not under direct sum. We give an excluded-minor characterization of $\mathfrak{M}$ and show that $\mathfrak{M}$ consists precisely of those transversal matroids with a presentation in which the sets in the presentation are nested. Finally, we show that on an n-set there are exactly 2^n members of $\mathfrak{M}$.

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1. Introduction

From any function f: $E \to Z$, where E is a finite set and Z is the set of integers, we obtain a function with domain 2^E whose value at $A \subseteq E$ is $\max\{f(a) \mid a \in A\}$ if A is non-empty and $\min\{f(a) \mid a \in E\}$ otherwise. This function is semimodular and increasing and as in Chapter 7 of Crapo and Rota (1970) we obtain a matroid M_f whose independent sets are exactly the subsets I of E such that $\max\{f(a) \mid a \in J\} \geqslant |J|$ for all non-empty subsets J of I. This paper investigates the class $\mathfrak{M}$ of matroids obtained in this way. Firstly, we show that $\mathfrak{M}$ consists exactly of those matroids, all of whose minors are free or have a unique minimal non-trivial flat. Secondly, we give an excluded minor characterisation of $\mathfrak{M}$. In obtaining this we prove $\mathfrak{M}$ closed under duality. Finally, we show that the members of $\mathfrak{M}$ are transversal and we use a result of Welsh (1969) to count the members of M.

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In general, we follow Welsh (1976) for matroid terminology. The ground set of a matroid M will be denoted by $E(M)$ or just E. If $T \subseteq E$, we shall sometimes write $M \setminus T$ and M/T for, respectively, the restriction and contraction of M to $E \setminus T$. The rank and closure of T in M will be denoted by $\mathrm{rk}(T)$ and $\sigma(T)$ respectively, and the subscript "cont" will be added to distinguish the rank and closure in a contraction of M. A flat F in M is *non-trivial* provided F is dependent. We call F a *non-trivial extension* of a flat H if $H \subseteq F$ and $F \setminus H$ is a non-trivial flat in M/H; otherwise, F is called a *free extension* of H. Except where otherwise stated, if $|E| = n$, we will identify E with the set $\{1, 2, \ldots, n\}$ in such a way that if $i < j$, then $f(i) \leqslant f(j)$.

We use the following properties of $\mathfrak{M}$.

LEMMA 1. *For any member M_f of $\mathfrak{M}$,*

(i) *$I = \{i_1, i_2, \ldots, i_s\}$, with $i_1 < i_2 < \cdots < i_s$, is independent in M_f exactly when $f(i_r) \geqslant r$ for all $r = 1, 2, \ldots, s$;*

(ii) *$I = \{i_1, i_2, \ldots, i_s\}$, with $i_1 < i_2 < \cdots < i_s$, is independent in M_f exactly when $|I \cap \{1, 2, \ldots, r\}| \leqslant f(r)$ for all $r = 1, 2, \ldots, n$;*

(iii) *$C = \{c_1, c_2, \ldots, c_s\}$, with $c_1 < c_2 < \cdots < c_s$, is a circuit in M_f exactly when $s - 1 \geqslant f(c_r) \geqslant \min\{r, s-1\}$ for all $r = 1, 2, \ldots, s$;*

(iv) *for each e in a minimal non-trivial flat F in M_f, $f(e) \leqslant \mathrm{rk}(F)$.*

PROOF. (i) If i_r is the maximal element of a subset J of I, then $r \geqslant |J|$ and so, if $f(i_r) \geqslant r$, then $\max\{f(x) \mid x \in J\} \geqslant |J|$. Conversely, if I is independent, $\{i_1, i_2, \ldots, i_r\} \subseteq I$ ensures $f(i_r) \geqslant r$.

(ii) For any k, let r_0 be the minimum r for which $\{1, 2, \ldots, r\} \cap I = \{i_1, i_2, \ldots, i_k\}$. Then $i_k = r_0$ and, if $k \leqslant f(r_0)$, we have $f(i_k) = f(r_0) \geqslant k$ and I is independent by (i). Conversely, if I is independent then, by (i), $f(i_r) \geqslant r \geqslant |I \cap \{1, 2, \ldots, r\}|$.

(iii) As $s - 1 \geqslant f(c_r) \geqslant \min\{r, s-1\}$ for all $r = 1, 2, \ldots, s$, we have $\max\{f(x) \mid x \in C\} = s - 1$ and so C is dependent. But any non-empty subset $P \subset C$ of size r contains an element $c \geqslant c_r$ and so $\max\{f(x) \mid x \in P\} \geqslant r$. Hence each proper subset of C is independent. Conversely, if C is a circuit, then $\{c_1, c_2, \ldots, c_r\}$ is independent for $r < s$ and so, by (i), $f(c_r) \geqslant r$, but as C is dependent $f(c_r) \leqslant f(c_s) < s$.

(iv) As e is in some circuit $C' \subseteq F$, we have by (iii), $f(e) \leqslant |C'| - 1 = \mathrm{rk}(C') \leqslant \mathrm{rk}(F)$.

2. Characterisation by flats

We denote by $\mathfrak{M}'$ the class of matroids having the property that each minor is either a free matroid or has a unique minimal non-trivial flat.

LEMMA 2. *Each member M of $\mathfrak{M}'$ has a finite chain $\sigma(\varnothing) = F_0 \subset F_1 \subset \cdots \subset F_k \subseteq E$ of flats where either F_{i+1} is the unique minimal non-trivial extension of F_i or $F_{i+1} = E$, the latter holding when F_i has no non-trivial extensions. Each flat in M is a direct sum of some F_i and a free matroid.*

PROOF. Unless M is free it has a unique minimal non-trivial flat. If $F_0 = \sigma(\varnothing)$ is empty we write F_1 for this minimal non-trivial flat, otherwise it is F_0. Let us suppose that the chain $\sigma(\varnothing) = F_0 \subset F_1 \subset \cdots \subset F_i$ exists as required. Then either there is a unique minimal non-trivial extension F_{i+1} of F_i or not. If not, either E is a free extension of F_i, or M has two minimal non-trivial extensions H, H' of F_i. In the latter case, since $H \cap H'$ is a flat in M, both $H \setminus H'$ and $H' \setminus H$ are flats in $M/(H \cap H')$. But H and H' are non-trivial extensions of F_i in M and $M/(H \cap H') \in \mathfrak{M}'$, so there is a unique minimal non-trivial flat of $M/(H \cap H')$ contained in both $H \setminus H'$ and $H' \setminus H$, contradicting $(H \setminus H') \cap (H' \setminus H) = \varnothing$. Thus we inductively obtain the required chain of flats. For any flat F in M there is a maximal $i \leqslant k$ such that $F_i \subseteq F$. Then F is a free extension of F_i, that is, a direct sum of F_i and the free matroid $M \mid (F \setminus F_i)$.

We prove $\mathfrak{M} \supseteq \mathfrak{M}'$ by characterising the circuits of members of $\mathfrak{M}'$. In the next two results, the flats $F_0, F_1, \ldots, F_k$ are as specified in the preceding lemma.

LEMMA 3. *For any $M \in \mathfrak{M}'$ the circuits contained in F_i but not in F_{i-1} are exactly the sets C satisfying $|C| = \mathrm{rk}(F_i) + 1$ and $|C \cap F_j| \leqslant \mathrm{rk}(F_j)$ for all $j < i$.*

PROOF. Again we proceed by induction. Either $F_0 = \varnothing$ or each element of F_0 is a loop C satisfying $|C| = 1 = \mathrm{rk}\, F_0 + 1$. Now suppose the circuits contained in F_j but not F_{j-1} are as prescribed for all $j < i$. If C is a circuit contained in F_i but not F_{i-1}, then $\sigma(C)$ is a non-trivial flat, every element of which is in a circuit. From the previous lemma, $\sigma(C) = F_j$, for some j. Consequently $\sigma(C) = F_i$, and $|C| = \mathrm{rk}(F_i) + 1$. For $j < i$, if $C \cap F_j \neq C$, then $C \cap F_j$ is independent and so $|C \cap F_j| = \mathrm{rk}(C \cap F_j) \leqslant \mathrm{rk}(F_j)$. Thus every circuit of M is of the specified form. Conversely, suppose C is contained in F_i but not F_{i-1}, $|C| = \mathrm{rk}(F_i) + 1$ and $|C \cap F_j| \leqslant \mathrm{rk}(F_j)$ for all $j < i$. As $C \subseteq F_i$ it is dependent and so contains a circuit C'. If $C' \subseteq F_j$ for some $j < i$, $|C'| = |C' \cap F_j| \leqslant |C \cap F_j| \leqslant \mathrm{rk}(F_j)$. Thus $|C'| \neq \mathrm{rk}(F_j) + 1$, contradicting the proven property of any such circuit. Hence $|C'| = \mathrm{rk}(F_i) + 1 = |C|$, and $C = C'$, a circuit. We have inductively characterised all circuits contained in some F_i. But E is a free extension of F_k and so any circuit in E is also in F_k.

LEMMA 4. $\mathfrak{M} \supseteq \mathfrak{M}'$.

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PROOF. For $M \in \mathfrak{M}'$, letting $F_{-1} = \varnothing$, we define an appropriate function on the underlying set E of M as follows:

$$f(e) = \begin{cases} \mathrm{rk}(F_i) & \text{if } e \in F_i \setminus F_{i-1} \text{ for some } i \geq 0, \\ \mathrm{rk}(E) & \text{if } e \in E \setminus F_k. \end{cases}$$

We prove $M_f = M$ by considering the circuits in both. If C is a circuit in M then, for some $i \geq 0$, $C \subseteq F_i$, $C \not\subseteq F_{i-1}$, $|C| = \mathrm{rk}(F_i) + 1$ and $|C \cap F_j| \leq \mathrm{rk}(F_j)$ for all $j < i$. Let $C = \{c_1, c_2, \ldots, c_s\}$ with $c_1 < c_2 < \cdots < c_s$. Then for all r, $f(c_r) \leq f(c_s) = \mathrm{rk}(F_i) = |C| - 1 = s - 1$. On the other hand, either $c_r \in F_i \setminus F_{i-1}$, for some $j < i$. In the first case, $f(c_r) = \mathrm{rk}(F_i) = s - 1$, and in the second case $f(c_r) = \mathrm{rk}(F_j) \geq |C \cap F_j| \geq r$. We conclude that $s - 1 \geq f(c_r) \geq \min\{r, s-1\}$ for all $r = 1, 2, \ldots, s$ and so, by Lemma 1(iii), C is a circuit in M_f. Conversely, if C is a circuit in M_f the above pair of inequalities hold for each $c_r \in C$, and so $f(c_s) = s - 1 = \mathrm{rk}(F_i)$, say. Then for all $j < i$, $C \cap F_j = \{c_r \mid f(c_r) \leq \mathrm{rk}(F_j)\} \subseteq \{c_r \mid r \leq \mathrm{rk}(F_j)\}$, giving $|C \cap F_j| \leq \mathrm{rk}(F_j)$. But $s = |C| = \mathrm{rk}(F_i) + 1$. Hence C is a circuit in M.

In view of Lemma 4, to prove $\mathfrak{M}' = \mathfrak{M}$ it suffices to show that $\mathfrak{M}$ is closed with respect to taking minors and that each $M_f \in \mathfrak{M}$ is a free matroid or has a unique minimal non-trivial flat.

LEMMA 5. *Each $M_f \in \mathfrak{M}$ is a free matroid or has a unique minimal non-trivial flat.*

PROOF. Let H and H' be distinct minimal non-trivial flats in M_f with $\mathrm{rk}(H) \leq \mathrm{rk}(H')$. For $e \in H \setminus H'$, by Lemma 1(iv), $f(e) \leq \mathrm{rk}(H)$. For any maximal independent subset I of H', $I \cup e$ is independent. But $\max\{f(x) \mid x \in I \cup e\} \leq \max\{\mathrm{rk}(H'), \mathrm{rk}(H)\} = \mathrm{rk}(H') < |I \cup e|$, contradicting the independence of $I \cup e$. Thus $H \subseteq H'$ and $H = H'$.

LEMMA 6. *Any restriction of a member of $\mathfrak{M}$ is also in $\mathfrak{M}$.*

PROOF. Clearly if $f\colon E \to Z$ defines M_f, then $f|_T$ defines $M_f | T$.

In order to show $\mathfrak{M}$ closed with respect to taking contractions we prove $\mathfrak{M}$ closed under duality. We call a function $f\colon E \to Z$ a *standard function* if $f(1) = 0$ or 1, and $0 \leq f(r+1) - f(r) \leq 1$ for all $r = 1, 2, \ldots, n-1$.

LEMMA 7. *Any matroid M_f is defined by a standard function.*

PROOF. Define

$$g(1) = \begin{cases} 1 & \text{if } f(1) \geqslant 1, \\ 0 & \text{otherwise}, \end{cases} \qquad g(r+1) = \begin{cases} g(r)+1 & \text{if } f(r+1) > g(r), \\ g(r) & \text{otherwise}, \end{cases}$$

for $r = 1, 2, \ldots, n-1$. Using induction on r, commencing with r_0, the least r for which $f(r) \geqslant 0$, we see that $f(r) \geqslant g(r)$. Consequently any independent set $I = \{i_1, i_2, \ldots, i_s\}$ in M_g has $i_1 \geqslant r_0$ and so $f(i_r) \geqslant g(i_r) \geqslant r$, ensuring I independent in M_f. Conversely, suppose I is independent in M_f. Then $f(i_1) \geqslant 1$ and so if either $i_1 = 1$ or $i_1 > 1$ we have $g(i_1) \geqslant 1$. Assuming $g(i_r) \geqslant r$ we consider $g(i_{r+1})$. Either $g(i_{r+1}) > g(i_{r+1} - 1)$ and $g(i_{r+1} - 1) \geqslant g(i_r) \geqslant r$ giving $g(i_{r+1}) \geqslant r+1$, or $g(i_{r+1}) = g(i_{r+1} - 1)$ and $f(i_{r+1}) \leqslant g(i_{r+1} - 1)$ giving $g(i_{r+1}) = f(i_{r+1}) \geqslant r + 1$. In both cases we have $g(i_r) \geqslant r$ for all $r = 1, 2, \ldots, s$ by induction. Consequently I is independent in M_g.

LEMMA 8. *$(M_f)^*$ is in $\mathfrak{M}$.*

PROOF. We may assume f is a standard function. Let $m = \mathrm{rk}(E)$. Then $m = f(n)$. We prove that if $f^*: E \to Z$ is defined by $f^*(1) = n - m$, $f^*(r+1) = n - m + f(r) - r$, for all $r = 1, 2, \ldots, n-1$, then $(M_f)^* = M_{f^*}$. Now let B be an m-element subset of E. Then it is routine to check that each statement in the following list is equivalent to its predecessor. The equivalence of (v) and (vi) uses the fact that $f(n) = m$ and $f^*(1) = n - m$, and the equivalence of (vi) and (vii) uses Lemma 1 and the fact that f^* is monotonic non-increasing.

(i) B is a base of M_f;

(ii) $|B \cap \{1, 2, \ldots, r\}| \leqslant f(r)$ for all $r = 1, 2, \ldots, n$;

(iii) $|B \cap \{r+1, r+2, \ldots, n\}| \geqslant m - f(r)$ for all $r = 1, 2, \ldots, n$;

(iv) $|B \cap \{n-r+1, n-r+2, \ldots, n\}| \geqslant m - f(n-r)$ for all $r = 0, 1, \ldots, n-1$;

(v) $|(E \setminus B) \cap \{n-r+1, n-r+2, \ldots, n\}| \leqslant r - m + f(n-r)$ for all $r = 0, 1, \ldots, n-1$;

(vi) $|(E \setminus B) \cap \{n-r+1, n-r+2, \ldots, n\}| \leqslant f^*(n-r+1)$ for all $r = 1, 2, \ldots, n$;

(vii) $E \setminus B$ is a base of M_{f^*}.

LEMMA 9. *Any contraction of a member of $\mathfrak{M}$ is in $\mathfrak{M}$.*

PROOF. $M_f \cdot T = (M_f{}^* \mid T)^*$.

THEOREM 10. $\mathfrak{M} = \mathfrak{M}'$.

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3. Excluded minor characterisation

We characterise $\mathfrak{M}'$, and hence $\mathfrak{M}$, by its excluded minors. For $k = 2, 3, \ldots$, consider a set E which is the disjoint union of two k-element subsets E_1 and E_2 and put $\mathcal{C} = \{E_1, E_2\} \cup \{C \mid C \not\supset E_1, C \not\supset E_2, C \subset E, |C| = k+1\}$.

LEMMA 11. *For each* $k = 2, 3, \ldots, \mathcal{C}$ *is the collection of circuits of a matroid* N^k *on* E.

PROOF. Consider any two distinct members C_1, C_2 of $\mathcal{C}$ with a common element e. Then $|(C_1 \cup C_2) \setminus e| \geqslant k+1$ and so $(C_1 \cup C_2) \setminus e$ contains a member of $\mathcal{C}$.

LEMMA 12. $N^k \notin \mathfrak{M}'$.

PROOF. Both E_1 and E_2 are minimal non-trivial flats.

THEOREM 13. $\mathfrak{M}'$ *is the class of matroids having no minor isomorphic to* N^k *for* $k = 2, 3, \ldots$.

PROOF. Suppose that M is not in $\mathfrak{M}'$ but every proper minor of M is in $\mathfrak{M}'$. Then M has two minimal non-trivial flats E_1 and E_2, say. If $E \neq E_1 \cup E_2$, choose $e \in E \setminus (E_1 \cup E_2)$ and consider $M \setminus e$. In this restriction both E_1 and E_2 are still minimal non-trivial flats, contradicting the choice of M. Thus $E = E_1 \cup E_2$.

We now show that each of E_1 and E_2 is a circuit of M. If E_1 is not, then M has a circuit $C \subset E_1$. Choose $e \in E_1 \setminus C$ and consider the contraction M/e. Again $E_1 \setminus e$ and $E_2 \setminus e$ are minimal non-trivial flats in M/e, contradicting our choice of M. Thus E_1, and similarly E_2, is a circuit of M.

We now prove E_1 and E_2 are disjoint. If not, choose $e \in E_1 \cap E_2$. In M/e both $E_1 \setminus e$ and $E_2 \setminus e$ are flats and circuits, and so are minimal non-trivial flats. Thus $E_1 \setminus e = E_2 \setminus e$ ensuring $E_1 = E_2$, contradicting our initial choice of E_1 and E_2. So $E = E_1 \cup E_2$.

Next we prove $|E_1| = |E_2|$. Suppose to the contrary that $|E_1| < |E_2|$, that is, $\mathrm{rk}(E_1) < \mathrm{rk}(E_2)$. Choosing $e \in E_2$ we consider the contraction M/e. In this contraction $E_2 \setminus e$ is a circuit and a flat and so is a minimal non-trivial flat of M/e. Also $\sigma_{\mathrm{cont}}(E_1) = \sigma(E_1 \cup e) \setminus e$ is a non-trivial flat in M/e. Thus we have $E_2 \setminus e \subseteq \sigma_{\mathrm{cont}}(E_1)$. Now $\mathrm{rk}_{\mathrm{cont}}(E_2 \setminus e) = \mathrm{rk}(E_2) - 1 \geqslant \mathrm{rk}(E_1)$ and $\mathrm{rk}_{\mathrm{cont}}(E_1) = \mathrm{rk}(E_1 \cup e) - 1 = \mathrm{rk}(E_1)$, since E_1 is a flat in M. Hence $\mathrm{rk}_{\mathrm{cont}}(E_2 \setminus e) \geqslant \mathrm{rk}_{\mathrm{cont}}(E_1)$. Thus $E_2 \setminus e = \sigma_{\mathrm{cont}}(E_1) = \sigma(E_1 \cup e) \setminus e$, ensuring that, in M, E_2 contains E_1. From this contradiction we can assume $|E_1| \geqslant |E_2|$; similarly $|E_2| \geqslant |E_1|$, giving $|E_1| = |E_2| = k$, say, for some $k > 1$.

It now remains only to prove that the other circuits in M are exactly the subsets of E of size $k + 1$ which contain neither E_1 nor E_2. By supposing that we initially specified E_1 as a non-trivial flat of minimal rank in M we deduce that each circuit in M has at least k elements. Suppose that C is a third circuit of size k in M, then $C \cap E_1 \neq \varnothing \neq C \cap E_2$. Hence $\sigma(C)$ is a minimal non-trivial flat of rank $k - 1$ in M and $E_1 \neq \sigma(C)$. But on proceeding as before with $\sigma(C)$ in place of E_2 we show $\sigma(C) \cap E_1 = \varnothing$, contradicting $C \cap E_1 \neq \varnothing$. So each circuit other than E_1 or E_2 has at least $k + 1$ elements. We need only show $\mathrm{rk}(M) = k$ to prove all $(k + 1)$-element subsets of E dependent and the circuits are as specified. Choosing $e \in E_2$ and considering the contraction M/e, as above, we have $E_2 \setminus e \subseteq \sigma(E_1 \cup e) \setminus e$, ensuring $E_2 \subseteq \sigma(E_1 \cup e)$ and so $E_1 \cup e$ spans M, giving $\mathrm{rk}(M) = \mathrm{rk}(E_1 \cup e) = \mathrm{rk}(E_1) + 1 = k$. Consequently $M = N^k$, for some $k > 1$.

In the preceding section it was shown that $\mathfrak{M}$ is closed under restriction, contraction and duality. It is straightforward to check that, in addition, $\mathfrak{M}$ is closed under truncation and hence also under elongation. However, $\mathfrak{M}$ is not closed under direct sum, for, although all uniform matroids are in $\mathfrak{M}$, the direct sum of two uniform matroids each having rank and corank at least one has N^2 as a minor and so is not in $\mathfrak{M}$. We now show that $\mathfrak{M}$ is a sub-class of the class of transversal matroids.

THEOREM 14. *A matroid M is in $\mathfrak{M}$ if and only if M is the transversal matroid $M[(A_i \mid i \in \{1,2,\ldots,m\})]$ of a family $(A_i \mid i \in \{1,2,\ldots,m\})$ of subsets of a set E where $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_m$.*

PROOF. If M is transversal having a presentation of the specified type, then define

$$f(j) = \begin{cases} 0 & \text{if } j \notin A_1, \\ i & \text{if } j \in A_i \setminus A_{i+1} \text{ for } i \in \{1,2,\ldots,m-1\}, \\ m & \text{if } j \in A_m. \end{cases}$$

It is routine to check that M_f is equal to M. Conversely, if $M_f \in \mathfrak{M}$, let $A_i = \{j \in E \mid f(j) \geqslant i\}$. Then again one can easily check that M_f is $M[(A_i \mid i \in \{1,2,\ldots,\mathrm{rk}(M)\})]$.

As $\mathfrak{M}$ is closed under duality, one can use the Ingleton-Piff construction (see, for example, Welsh (1976), page 221) with the preceding result to obtain a simple representation of a member of $\mathfrak{M}$ as a strict gammoid. Moreover, if $M \cong M_f$ where f is a standard function, it is not difficult to show that M^* is isomorphic to the fundamental transversal matroid associated with the cobase B of M^* where

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$B = \{i_1, i_2, \ldots, i_{\mathrm{rk}(M)}\}$ with $f(i_j) = j$ for all $j = 1, 2, \ldots, \mathrm{rk}(M)$. Thus $\mathfrak{M}$ is a sub-class of the class of fundamental transversal matroids.

Welsh (1969) gave a lower bound on the number of transversal matroids on an n-set S by constructing exactly 2^n non-isomorphic transversal matroids on S. It is straightforward to check that the union over all positive integers n of these sets of matroids is precisely the class $\mathfrak{M}$. Hence, by Theorems 1 and 2 of Welsh (1969), we have that on an n-set there are precisely 2^n non-isomorphic members of $\mathfrak{M}$ and of these exactly $\binom{n}{r}$ have rank r.

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