19
Static solutions

In this chapter a study of static solutions to the vacuum Einstein field equations from the point of view of conformal methods is undertaken. Static and, more generally, stationary solutions provide valuable physical and mathematical intuition concerning the behaviour of solutions to the Einstein field equations. Static solutions describe the exterior region of time-independent, non-rotating, isolated bodies. Accordingly, they provide an interesting class of solutions to analyse the structure of spatial infinity; see Chapter 20. In addition, some particular static solutions (the Schwarzschild spacetime) are expected to describe the asymptotic state of the evolution dictated by the Einstein field equations. From a mathematical point of view, the results discussed in this chapter are of particular interest as they lie at the interface of classical potential theory, conformal geometry and general relativity. Throughout this chapter, the focus is restricted to the asymptotic region of an asymptotically flat static spacetime. Several of the key results for static spacetimes admit a suitable stationary counterpart; the interested reader is referred to the literature for further details. These generalisations of the theory are much more technically involved than the original static version and they will not be considered here.

19.1 The static field equations

For a ***static spacetime*** it will be understood a solution to the Einstein field equations $\boldsymbol{Ric}[\tilde{\boldsymbol{g}}] = 0$ endowed with a hypersurface orthogonal Killing vector $\boldsymbol{\xi}$ which, in a suitable asymptotic region, is timelike. Using coordinates $(t, \underline{y}) = (t, y^\alpha)$ adapted to this Killing vector, one has that $\boldsymbol{\xi} = \boldsymbol{\partial}_t$. As $\boldsymbol{\xi}$ is hypersurface orthogonal, then there exists a function $v = v(\underline{y})$ such that

$$\boldsymbol{\xi}^\flat = \tilde{\boldsymbol{g}}(\boldsymbol{\xi}, \cdot) = v^2 \mathbf{d}t.$$

Thus, $v^2 = \tilde{\boldsymbol{g}}(\boldsymbol{\partial}_t, \boldsymbol{\partial}_t)$ is the square of the norm of $\boldsymbol{\xi}$. It follows that the hypersurfaces of constant coordinate t define a foliation of the spacetime. In what

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follows, it will be convenient to consider a frame $\{\boldsymbol{e_a}\}$ adapted to the static Killing vector and set $\boldsymbol{e_0}$ to be parallel to $\boldsymbol{\xi}$; that is, one has $\boldsymbol{\xi} = v\boldsymbol{e_0}$. The spatial part of the frame, $\{\boldsymbol{e_i}\}$, spans the tangent bundle of the hypersurfaces of constant t. Without loss of generality, the spatial frame can be parallely propagated along the direction of $\boldsymbol{e_0}$ so that using the definition of the connection coefficients one has that $\tilde{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{a}}{}_{\boldsymbol{i}} = 0$. Let $\{\omega^{\boldsymbol{a}}\}$ be the associated coframe. One readily finds that $\boldsymbol{\omega}^{\boldsymbol{0}} = v\mathbf{d}t$. It follows from this discussion that the metric $\tilde{\boldsymbol{g}}$ takes the form

$$\tilde{\boldsymbol{g}} = v^2\mathbf{d}t\otimes\mathbf{d}t + \tilde{\boldsymbol{h}}, \qquad v = v(\underline{y}) > 0, \qquad \tilde{\boldsymbol{h}} = \tilde{h}_{\alpha\beta}(\underline{y})\mathbf{d}y^\alpha\otimes\mathbf{d}y^\beta, \tag{19.1}$$

where $\tilde{\boldsymbol{h}}$ denotes a (negative definite) Lorentzian metric on the hypersurfaces of constant time coordinate.

Derivation of the static equations

The equations satisfied by the fields v and $\tilde{\boldsymbol{h}}$ appearing in the metric (19.1) can be deduced using the frame formalism introduced in the previous paragraphs. Observing that $\xi_{\boldsymbol{0}} \equiv \langle\boldsymbol{\xi}^\flat, \boldsymbol{e_0}\rangle = v$ one concludes that $\xi_{\boldsymbol{a}} = v\delta_{\boldsymbol{a}}{}^{\boldsymbol{0}}$. It follows that the ***Killing equation***

$$\tilde{\nabla}_{\boldsymbol{a}}\xi_{\boldsymbol{b}} + \tilde{\nabla}_{\boldsymbol{b}}\xi_{\boldsymbol{a}} = 0$$

takes the form

$$\big(\tilde{\nabla}_{\boldsymbol{a}}v\delta_{\boldsymbol{b}}{}^{\boldsymbol{0}} + \tilde{\nabla}_{\boldsymbol{b}}v\delta_{\boldsymbol{a}}{}^{\boldsymbol{0}}\big) + v\big(\tilde{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{0}}{}_{\boldsymbol{b}} + \tilde{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{0}}{}_{\boldsymbol{a}}\big) = 0.$$

As v is time independent, one concludes from setting $\boldsymbol{a}, \boldsymbol{b} = \boldsymbol{0}$ that $\tilde{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{0}}{}_{\boldsymbol{0}} = 0$. Setting $\boldsymbol{a} = \boldsymbol{i}$ and $\boldsymbol{b} = \boldsymbol{j}$ one finds that $\tilde{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{0}}{}_{\boldsymbol{j}} + \tilde{\Gamma}_{\boldsymbol{j}}{}^{\boldsymbol{0}}{}_{\boldsymbol{i}} = 0$ so that from the definition of the extrinsic curvature, Equation (2.45), one concludes that $K_{\boldsymbol{ij}} = 0$; that is, the surfaces of constant coordinate t are *time symmetric*. Accordingly, the Einstein constraint Equations (11.13a) and (11.13b) reduce to the condition

$$r[\tilde{\boldsymbol{h}}] = 0. \tag{19.2}$$

A further condition can be obtained from the equation

$$\tilde{\nabla}_{\boldsymbol{0}}\big(\tilde{\nabla}_{\boldsymbol{a}}\xi_{\boldsymbol{b}} + \tilde{\nabla}_{\boldsymbol{b}}\xi_{\boldsymbol{a}}\big) = 0.$$

Commuting covariant derivatives and using that the Killing equation implies

$$\tilde{\nabla}_{\boldsymbol{0}}\xi_{\boldsymbol{b}} = -\tilde{\nabla}_{\boldsymbol{b}}\xi_{\boldsymbol{0}} = -\tilde{\nabla}_{\boldsymbol{b}}v$$

one finds that

$$\tilde{\nabla}_{\boldsymbol{a}}\tilde{\nabla}_{\boldsymbol{b}}v + v\tilde{R}_{\boldsymbol{0a0b}} = 0.$$

From the last equation, using the Gauss-Codazzi identity, Equation (2.47), one concludes that

$$\tilde{\nabla}_{\boldsymbol{i}}\tilde{\nabla}_{\boldsymbol{j}}v = v\tilde{r}_{\boldsymbol{ij}} \tag{19.3}$$

where $r_{\boldsymbol{ij}}$ denotes the components with respect to $\{\boldsymbol{e_i}\}$ of the Ricci tensor of the 3-metric $\tilde{\boldsymbol{h}}$.

Equations (19.2) and (19.3) provide the required ***static Einstein field equations*** for the fields v and $\tilde{\boldsymbol{h}}$. After some further slight manipulations they can be rewritten in tensorial form as

$$\Delta_{\tilde{\boldsymbol{h}}} v = 0, \tag{19.4a}$$

$$\tilde{r}_{ij} = \frac{1}{v}\tilde{D}_i\tilde{D}_j v, \tag{19.4b}$$

where $\tilde{D}_i$ and $\tilde{r}_{ij}$ denote, respectively, the Levi-Civita connection and the Ricci tensor of the 3-metric $\tilde{\boldsymbol{h}}$. In what follows, a pair $(v, \tilde{\boldsymbol{h}})$ solving the ***static equations*** (19.4a) and (19.4b) will be called a ***static solution***. A static solution, expressed in terms of $\tilde{\boldsymbol{h}}$-harmonic coordinates is analytic; see Müller zu Hagen (1970).

Observe that discarding the field v, a solution to the static equations gives rise to a solution to the time-symmetric Einstein constraints. This dual perspective of static solutions as a spacetime and as time-symmetric initial data for a spacetime will be used often. The context will dictate the appropriate point of view. Equations (19.4a) and (19.4b) can be regarded as a three-dimensional analogue of the Einstein field equations in which the curvature is coupled to a *fictitious matter field* described by v. This interpretation also holds for other *symmetry reductions* of the vacuum Einstein field equations, say, axial symmetry; see, for example, Geroch (1971a, 1972a).

Asymptotic conditions and the Licnerowicz theorem

Of special interest are static solutions describing the asymptotic region of isolated systems. For simplicity, it will be assumed that $\tilde{\mathcal{S}}$ has a single asymptotic region in which coordinates $\underline{y} = (y^\alpha)$ can be found such that

$$v = 1 - \frac{m}{|y|} + O_k(|y|^{-(1+\varepsilon)}), \tag{19.5a}$$

$$\tilde{h}_{\alpha\beta} = -\left(1 + \frac{2m}{|y|}\right)\delta_{\alpha\beta} + O_k(|y|^{-(1+\varepsilon)}), \tag{19.5b}$$

as $|y| \to \infty$ where $m \neq 0$ denotes the Arnowitt-Deser-Miser (ADM) mass and $\varepsilon > 0$. The notation O_k has been described in the Appendix to Chapter 11. The above decay conditions can be deduced from more primitive assumptions which make no reference to asymptotic flatness; see Reiris (2014a,b). In order to describe an isolated system – say, the exterior of a star – Equations (19.4a) and (19.4b) need to be supplemented with suitable boundary conditions at an interior boundary $\partial\tilde{\mathcal{S}}$ – say, the surface of a star. An analysis of this type has been carried out by Reula (1989) and Miao (2003). The role played by boundary conditions in the determination of static solutions is nicely exhibited in the case where $\tilde{\mathcal{S}} \approx \mathbb{R}^3$. In this case, it follows from Equation (19.4a) by integration by parts that

$$0 = \int_{\tilde{\mathcal{S}}} v \Delta_{\tilde{\boldsymbol{h}}} v \mathrm{d}\mu = -\int_{\tilde{\mathcal{S}}} \tilde{D}^i v \tilde{D}_i v \mathrm{d}\mu,$$

so that $\tilde{D}_i v = 0$ on $\tilde{\mathcal{S}}$. This implies that v has to be constant on $\tilde{\mathcal{S}}$. Moreover, using (19.5a) one concludes that $v = 1$. Substituting into Equation (19.4a) one finds that $\tilde{r}_{ij} = 0$ so that $\tilde{\boldsymbol{h}}$ must be flat – recall that in three dimensions the curvature is fully determined by the Ricci tensor. Consequently, in order to have static solutions other than the Minkowski solution one needs hypersurfaces $\tilde{\mathcal{S}}$ with a non-trivial topology or with some inner boundary $\partial\tilde{\mathcal{S}}$. This result is usually known as ***Licnerowicz's theorem***.

19.1.1 The conformal static field equations

In the remainder of this chapter, *the discussion of static solutions will be restricted to a suitable asymptotic region where the decay conditions* (19.5a) *and* (19.5b) *hold.* Accordingly, it is convenient to make use of the definition of *asymptotically Euclidean and regular manifolds* given in Section 11.6.2. Hence, one considers a function Ω on $\mathcal{S} \equiv \tilde{\mathcal{S}} \cup \{i\}$ with $\Omega \in C^2(\tilde{\mathcal{S}}) \cap C^\infty(\tilde{\mathcal{S}})$, $\Omega > 0$ on $\tilde{\mathcal{S}}$ which conformally extends $\tilde{\boldsymbol{h}}$ to a smooth metric

$$\boldsymbol{h} \equiv \Omega^2 \tilde{\boldsymbol{h}} \qquad \text{on } \mathcal{S},$$

in such a way that

$$\Omega = 0, \qquad D_i\Omega = 0, \qquad D_iD_j\Omega = -2h_{ij}, \qquad \text{at } i. \tag{19.6}$$

In order to exploit the above conformal setting, it is convenient to rewrite the static Equations (19.4a) and (19.4b) in terms of fields satisfying regular equations in a neighbourhood of i. The procedure of constructing a system of ***regular conformal static equations*** is similar in spirit to the one carried out in Chapter 8 to obtain the conformal field equations. The key idea is to identify quantities which in the conformally rescaled picture are both suitably regular and which satisfy equations that are formally regular at i. In this spirit, the equation obtained from combining the static Equation (19.4b) with the transformation law of the three-dimensional Ricci tensor, Equation (5.16a), should be read not as a differential condition for the components of a conformally rescaled metric but rather as differential equations involving second derivatives of a quantity associated to the conformal factor. Similar considerations need to be taken into account when attempting to construct a *conformal equation* for the scalar field v. Using the transformation law for the Yamabe operator, Equation (11.23), one obtains

$$\left(\Delta_{\boldsymbol{h}} - \frac{1}{8} r[\boldsymbol{h}]\right)(\Omega^{-1/2} v) = 0.$$

This equation is formally singular at i unless it is possible to tie the behaviour of Ω with that of v. Alternatively, one could try to find a regular equation for a quantity which indirectly allows one to gain knowledge about v. These ideas are explored in the following subsections.

Fixing the conformal gauge

The standard approach to obtain a set of regular conformal static field equations relies on a specific choice of conformal gauge which explicitly prescribes the conformal factor Ω in terms of the norm of the static Killing vector v; see, for example, Beig and Simon (1980a) and Friedrich (1988, 2004, 2007). In the following, the approach taken in the last two references will be followed. A general version of the conformal static equations which retains the whole conformal freedom has been given in Friedrich (2013).

It can be verified that the conditions (19.6) expressed in terms of physical coordinates $\underline{y} = (y^\alpha)$ require Ω to behave like $1/|y|^2$ as $|y| \to 0$. This observation suggests, in turn, considering a conformal factor of the form

$$\Omega = \left(\frac{1-v}{m}\right)^2. \tag{19.7}$$

As will be seen in Section 19.2, this is not the only possible way of fixing the conformal freedom. The choice in Equation (19.7) fixes the value of the Ricci scalar of the conformal metric $\boldsymbol{h}$. This can be seen from the transformation law of the Yamabe operator, Equation (11.23), by setting $u = \Omega^{1/2}$ and making the replacements $\phi \mapsto \Omega^{1/2}$, $\boldsymbol{h}' \mapsto \boldsymbol{h}$, $\boldsymbol{h} \mapsto \tilde{\boldsymbol{h}}$ so that, on the one hand, one has

$$\mathbf{L}_{\boldsymbol{h}}[1] = \Omega^{-5/2}\mathbf{L}_{\tilde{\boldsymbol{h}}}\left(\frac{1-v}{m}\right) = -\frac{1}{m}\Omega^{-5/2}\Delta_{\tilde{\boldsymbol{h}}} v = 0,$$

while, on the other hand,

$$\mathbf{L}_{\boldsymbol{h}}[1] = \left(\Delta_{\boldsymbol{h}} - \frac{1}{8}r[\boldsymbol{h}]\right)[1] = -\frac{1}{8}r[\boldsymbol{h}].$$

Hence, one concludes that $r[\boldsymbol{h}] = 0$.

A decomposition of the conformal factor

Following the general discussion of Section 11.6.3, one has that the conformal factor Ω satisfies

$$\left(\Delta_{\boldsymbol{h}} - \frac{1}{8}r[\boldsymbol{h}]\right)(\Omega^{-1/2}) = 0, \qquad \text{on } \tilde{\mathcal{S}},$$

and $|x|\Omega^{-1/2} \to 1$ as $|x| \to 0$. Here, and in what follows, let $\underline{x} = (x^\alpha)$ denote some coordinates in a neighbourhood $\mathcal{U} \subset \mathcal{S}$ with $x^\alpha(i) = 0$. Close to i one has the representation

$$\Omega^{-1/2} = \zeta^{-1/2} + W, \tag{19.8}$$

with ζ, W smooth – confront this decomposition with the discussion in Section 11.6.4. In particular, one has

$$\left(\Delta_{\boldsymbol{h}} - \frac{1}{8}r[\boldsymbol{h}]\right)W = 0, \qquad W(i) = \frac{m}{2}, \tag{19.9}$$

and

$$\left(\Delta_{\boldsymbol{h}} - \frac{1}{8}r[\boldsymbol{h}]\right)(\zeta^{-1/2}) = 4\pi\delta[i]. \tag{19.10}$$

From this last equation it follows that

$$\zeta(i) = 0, \qquad D_i\zeta(i) = 0, \qquad D_iD_j\zeta(i) = -2h_{ij}(i). \tag{19.11}$$

One has that ζ is essentially the Green function of the Yamabe operator and describes the local geometry in a neighbourhood of i, while W encodes information of a global nature – in particular, its ADM mass m. Accordingly, the functions ζ and W will be called, respectively, the ***massless part*** and ***mass part*** of the conformal factor Ω. Given a conformal metric $\boldsymbol{h}$, *the decomposition* (19.8) *is unique*. Moreover, using the so-called ***Hadamard's parametrix construction***, it can be shown that ζ and W are analytic if $\boldsymbol{h}$ is analytic; see Friedrich (1998c, 2004) for further details about this last statement and Garabedian (1986) for the underlying PDE theory. In particular, for the choice of the conformal factor (19.7) it follows that the parametrisation (19.8) takes the form

$$\zeta = \frac{1}{\mu}\left(\frac{1-v}{1+v}\right)^2, \qquad W = \frac{m}{2}, \qquad \mu \equiv \frac{m^2}{4}. \tag{19.12}$$

It can be verified that the function ζ satisfies the asymptotic conditions (19.11) and that W is, indeed, a solution of Equation (19.9).

Using the chain rule to rewrite Equation (19.10) as an equation for $\Delta_{\boldsymbol{h}}\zeta$ and taking into account the asymptotic conditions (19.11), one finds that

$$2\zeta\varsigma = D_i\zeta D^i\zeta, \qquad \text{with } \varsigma \equiv \frac{1}{3}\Delta_{\boldsymbol{h}}\zeta, \tag{19.13}$$

which is a regular equation in a suitable neighbourhood of i. In particular, it can be verified that $\varsigma(i) = -2$. Equation (19.13) is the analogue of the conformal Einstein field Equation (8.24). It encodes a regularised version of the transformation equation for the Ricci scalar. As will be seen in the following, it can be interpreted as a constraint which is automatically satisfied if other equations hold.

Equations for the curvature

To exploit the fact that one is working with a gauge for which $r[\boldsymbol{h}] = 0$, it is convenient to introduce an $\boldsymbol{h}$-tracefree tensor s_{ij} such that

$$r_{ij}[\boldsymbol{h}] = s_{ij}.$$

Recalling that in three dimensions the Riemann curvature tensor $r^i{}_{jkl}$ is fully determined by the Ricci tensor, it is then natural to interpret the tensors r_{ij} and s_{ij} as describing, respectively, the ***geometric*** and ***algebraic three-dimensional curvatures***; see Section 8.3.1 for further discussion on these notions in the context of the conformal Einstein field equations. If the zero quantity

$$\Xi_{ij} \equiv r_{ij}[\boldsymbol{h}] - s_{ij}$$

vanishes, then the three-dimensional (contracted) Bianchi identity takes the form

$$D^i s_{ij} = 0. \tag{19.14}$$

The fields ζ, ς and s_{ij} can be used to obtain a regular version of the formally singular transformation law for the Ricci tensor; see Equation (5.16a). Rewriting derivatives of the conformal factor Ω as derivatives of ζ one obtains

$$S_{ij} \equiv D_i D_j \zeta - \varsigma h_{ij} + \zeta(1-\mu\zeta)s_{ij} = 0. \tag{19.15}$$

Equation (19.15) will be read as a differential equation for ζ. To close the system one needs differential equations for ς and s_{ij}. Suitable equations can be obtained from the integrability conditions

$$D^i S_{ij} = 0, \qquad D_{[k}S_{i]j} + \frac{1}{2}D^l S_{l[k}h_{i]j} = 0, \tag{19.16}$$

encoding the three-dimensional second Bianchi identity in contracted and uncontracted form, respectively. The identities (19.16) can be verified through a direct computation and introducing the *zero quantities*

$$\begin{aligned} S_i &\equiv D_i\varsigma + (1-\mu\zeta)s_{ij}D^j\zeta = 0, \\ H_{kij} &\equiv (1-\mu\zeta)D_{[k}s_{i]j} - \mu\big(2D_{[k}\zeta s_{i]j} + D^l\zeta h_{l[k}s_{i]j}\big). \end{aligned}$$

It follows from a further computation that $S_i = 0$ and $H_{kij} = 0$ are equivalent to the integrability conditions (19.16). The condition $H_{kij} = 0$ can be read as an expression for the *Cotton tensor*

$$b_{jki} \equiv D_{[k}r_{i]j} - \frac{1}{4}D_{[k}r h_{i]j} = 2D_{[k}l_{i]j},$$

where l_{ij} denotes the three-dimensional Schouten tensor. In the remainder of this chapter it is often more convenient to work with the dualised version of b_{jki}, $b_{ij} \equiv \frac{1}{2}b_{ikl}\epsilon_j{}^{kl}$. A computation shows that

$$b_{ij} = \frac{\mu}{1-\mu\zeta}\left(s_{li}\epsilon_j{}^{kl}D_k\zeta - \frac{1}{2}s_{lm}\epsilon_{ji}{}^l D^m\zeta\right).$$

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Summary: the conformal static equations

In what follows, the conditions

$$\Xi_{ij} = 0, \qquad S_i = 0, \qquad S_{ij} = 0, \qquad H_{kij} = 0 \tag{19.17}$$

will be known as the ***conformal static equations***. They provide an overdetermined system of differential conditions for the fields h_{ij}, s_{ij}, ζ and ς. As will be seen, the equations in (19.17) imply an elliptic system for the components of the various ***conformal fields***.

Remark. A direct computation yields the identity

$$D_i\big(2\zeta\varsigma - D_k\zeta D^k\varsigma\big) = 2\zeta S_i - 2S_{ik}D^k\zeta.$$

Thus, if $S_i = 0$ and $S_{ij} = 0$, then $2\zeta\varsigma - D_k\zeta D^k\varsigma$ is a constant. Evaluating at i and using the known values of the various fields at this point, one concludes that the expression in brackets must vanish. This argument shows that Equation (19.13) plays the role of a constraint. Hence, it has not been included in the list (19.17).

19.1.2 Spinorial version of the equations

To write the spinorial version of the conformal static equations, let $s_{ABCD} = s_{(ABCD)}$ denote the spinorial counterpart of the trace-free tensor s_{ij}. The Bianchi identity (19.14) takes the form

$$D^{AB}s_{ABCD} = 0.$$

In terms of the spinor s_{ABCD} the equation $H_{kij} = 0$ takes, after exploiting the antisymmetry in the pair $_{ki}$, the simple form

$$D_A{}^Q s_{BCDQ} = \frac{2\mu}{1-\mu\zeta} s_{Q(ABC}D_{D)}{}^Q\zeta. \tag{19.18}$$

The spinorial transcription of equations $S_i = 0$ and $S_{ij} = 0$ is completely direct. When working with spinors, the equation $r_{ij} = s_{ij}$ is replaced by the Cartan structure equations, Equations (2.41) and (2.42), for the 3-geometry with an ***algebraic 3-curvature*** given by s_{ij}. These structure equations provide, respectively, differential conditions for the coefficients of a frame and for the associated spin connection coefficients; see, for example, Friedrich (2007). It will often be convenient to express the various spinorial fields and the associated equations in terms of their components (e.g. $s_{\boldsymbol{ABCD}}$) with respect to some spin dyad $\{\epsilon_{\boldsymbol{A}}{}^A\}$.

For later use, let b_{ABCDEF} denote the spinorial counterpart of the Cotton tensor b_{ijk}. Exploiting the antisymmetry in the indices $_{jk}$, one obtains the decomposition

$$b_{ABCDEF} = b_{ABCE}\epsilon_{DF} + b_{ABDF}\epsilon_{CE}, \qquad b_{ABCD} \equiv D^Q{}_{(A}s_{BCD)Q}. \tag{19.19}$$

Consequently, one has the symmetry $b_{ABCD} = b_{(ABCD)}$. Moreover, it can be verified that

$$D^{AB} b_{ABCD} = 0.$$

In what follows, b_{ABCD} will be referred to as the ***Cotton spinor***.

19.2 Analyticity at infinity

The conformal static Equations (19.17) allow one to show that, under some basic regularity assumptions, there exist coordinates in a neighbourhood of the point at infinity for which all the conformal fields are analytic. This result brings to the forefront the inherent ellipticity of the equations and constitutes the foundation of any further analysis of static solutions from a conformal perspective. The result was originally proven by Beig and Simon (1980a). In the following, an adaptation of this result will be given. One has:

Theorem 19.1 (***analyticity of static solutions at infinity***) *Let* $(v, \tilde{\boldsymbol{h}})$ *denote a solution to the static Equations* (19.4a) *and* (19.4b) *such that* Ω *as defined by Equation* (19.7) *satisfies the conditions* (19.6) *with* $h_{\alpha\beta} = \Omega^2 \tilde{h}_{\alpha\beta}$ *the components of a* $C^{4,\alpha}$ *metric for some coordinates* $\underline{x} = (x^\alpha)$ *in a neighbourhood of* i. *Then there exist coordinates* $\underline{x}' = (x'^\alpha)$ *defined in a neighbourhood of* i *such that* $h'_{\alpha\beta}$, ζ', ς' *and* $s'_{\alpha\beta}$ *are analytic.*

Remark. The regularity assumptions in this result are expressed in terms of Hölder spaces; see the Appendix to this chapter.

Proof The proof exploits the fact that the *Ricci operator of a Riemannian metric expressed in harmonic coordinates is elliptic* – the Lorentzian counterpart of this observation has been discussed in the Appendix to Chapter 13. The general theory of elliptic equations – see, for example, Garabedian (1986) – shows that it is always possible to find a neighbourhood of i in which the equations

$$\Delta_h x'^\alpha = 0 \tag{19.20}$$

have a solution $x'^\alpha = x'^\alpha(\underline{x})$. The coefficients of the differential operator in Equation (19.20) consist of $h_{\alpha\beta}$ and its derivatives so that they are of class $C^{3,\alpha}$. The general theory of elliptic partial differential equations (PDEs) shows that solutions of second-order elliptic equations gain two derivatives with respect to the coefficients of the equation. Accordingly, one has that $x'^\alpha = x'^\alpha(\underline{x})$ is $C^{5,\alpha}$. This regularity is sufficient to invert the coordinates. Taking into account the transformation law of the metric tensor under change of coordinates,

$$h'_{\alpha\beta} = \frac{\partial x^\gamma}{\partial x'^\alpha} \frac{\partial x^\delta}{\partial x'^\beta} h_{\gamma\delta},$$

it follows that $h'_{\alpha\beta}$ is $C^{4,\alpha}$. Similarly, the field ζ' can be verified to be $C^{4,\alpha}$, while ς' and $s'_{\alpha\beta}$ are $C^{2,\alpha}$. To conclude the proof, one needs to construct a system

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of elliptic equations for the various fields. In the remainder of the proof it is assumed that all the fields are expressed in terms of the coordinates $\underline{x}'$, and the primes will be dropped from the expressions. Let $\gamma_\alpha{}^\beta{}_\gamma$ denote the Christoffel symbols of the metric $\boldsymbol{h}$ and denote by $\gamma^\beta \equiv h^{\alpha\gamma}\gamma_\alpha{}^\beta{}_\gamma$ the associated ***contracted Christoffel symbols***. A discussion analogous to that of the hyperbolic reduction of the Einstein field equations in generalised wave coordinates – see the discussion in the Appendix of Chapter 13 – shows that

$$r_{\alpha\beta} - D_{(\alpha}\gamma_{\beta)} = s_{\alpha\beta}$$

is an elliptic equation for the components $h_{\alpha\beta}$ of the metric $\boldsymbol{h}$ in the coordinates (x^α) if $s_{\alpha\beta}$ are known. To close the system one considers the equations

$$S^\alpha{}_\alpha = 0, \qquad D^\alpha S_\alpha = 0, \qquad D^\gamma H_{\gamma\alpha\beta} = 0.$$

Using the Bianchi identity (19.14) and the conformal static field equations to remove all the second derivatives of the conformal fields which are not Laplacians, one obtains a system of the form

$$\Delta_{\boldsymbol{h}}\big(h_{\alpha\beta},\, \zeta,\, \varsigma,\, s_{\alpha\beta}\big) = \mathbf{F}(h_{\alpha\beta}, \zeta, \varsigma, s_{\alpha\beta}, D_\gamma h_{\alpha\beta}, D_\alpha\zeta, D_\alpha\varsigma, D_\gamma s_{\alpha\beta}), \qquad (19.21)$$

with $\mathbf{F}$ an analytic vector-valued function of its entries. Despite having a Laplacian operator on the left-hand side, it is a priori not clear that the system (19.21) is elliptic as $\Delta_{\boldsymbol{h}}$ applied to $h_{\alpha\beta}$ and $s_{\alpha\beta}$ gives rise to further second-order derivatives of $h_{\alpha\beta}$ which come from derivatives of the Christoffel symbols. To verify ellipticity one needs to compute the determinant of the symbol of (19.21). A calculation shows that this determinant is, in fact, proportional to $(h^{\alpha\beta}\xi_\alpha\xi_\beta)^{13}$ so that one, indeed, has an elliptic system as $h_{\alpha\beta}$ are the components of a Riemannian metric; compare the definition of ellipticity in Section 11.2. The general theory of the regularity of solutions of elliptic systems shows that if one has a $C^{2,\alpha}$ solution to the above equation, then it must, in fact, be analytic in $\mathcal{U}$; a discussion of this result is given in Morrey (1958). $\square$

Remarks

(i) The original proof in Beig and Simon (1980a) was carried out in a conformal gauge obtained from writing the static metric (19.1) in the form

$$\tilde{\boldsymbol{g}} = e^{2U}\mathbf{d}t \otimes \mathbf{d}t - e^{-2U}\hat{h}_{\alpha\beta}\mathbf{d}y^\alpha \otimes \mathbf{d}y^\beta,$$

where U is a scalar field and $\hat{h}_{\alpha\beta}$ denote the components of a Riemannian 3-metric. Their analysis shows that $\omega \equiv (U/m)^2$ and $\boldsymbol{h}' \equiv \omega^2\hat{\boldsymbol{h}}$ are analytic in $\boldsymbol{h}'$-harmonic coordinates. This gauge and the one used to prove Theorem 19.1 can be related by letting $\Omega' \equiv \omega e^U$. One has by analogy to Equation (19.8) the split $\Omega'^{-1/2} = \zeta'^{-1/2} + W'$ with

$$\zeta' = \frac{\omega}{\cosh^2 U/2}, \qquad W' = \frac{m\sinh U/2}{U}.$$

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It can be verified that the conformal metrics $\boldsymbol{h}$ and $\boldsymbol{h}'$ are related to each other via $\boldsymbol{h} = \vartheta^4 \boldsymbol{h}'$ with $\vartheta \equiv 2W'/m$.

(ii) Kennefick and O'Murchadha (1995) have shown that the smoothness assumption on the conformal metric made in Theorem 19.1 can be deduced from weaker differentiability and decay conditions on the physical 3-metric $\tilde{\boldsymbol{h}}$.

(iii) Theorem 19.1 can be further strengthened by considering $\boldsymbol{h}$-normal coordinates based on i. It can be verified that the coordinate transformation relating the analytic coordinate system $\underline{x}'$ and normal coordinates is also analytic.

A remark concerning the notion of analyticity at i

As a consequence of the analytic behaviour ensured by Theorem 19.1 one has that, for example, the field ζ can be expressed in a suitably small neighbourhood $\mathcal{U}$ of i as a convergent series of the form

$$\zeta = \sum_{p=2}^{\infty} \zeta_{\alpha_2 \cdots \alpha_p} x^{\alpha_2} \cdots x^{\alpha_p}, \qquad \zeta_{\alpha_2 \cdots \alpha_p} \in \mathbb{R}. \tag{19.22}$$

The other conformal fields have similar expansions. An alternative description of the above expansion can be obtained by introducing ***polar coordinates***. Accordingly, one defines

$$\rho^2 \equiv |x|^2 = \delta_{\alpha\beta} x^\alpha x^\beta, \qquad \rho^\alpha \equiv \frac{x^\alpha}{|x|}. \tag{19.23}$$

The ***unit position vector*** ρ^α can be parametrised by means of some coordinates $\theta = (\theta^{\mathcal{A}})$ on the 2-sphere $\mathbb{S}^2$ so that one can write $\rho^\alpha = \rho^\alpha(\theta)$. In what follows it will be assumed, for convenience, that the coordinates $(\theta^{\mathcal{A}})$ are analytic functions of the original coordinates $\underline{x}$ – clearly, the coordinates $(\theta^{\mathcal{A}})$ cannot cover the whole of $\mathbb{S}^2$. This fact will not play a role in the sequel. In terms of the coordinates $(\rho, \theta^{\mathcal{A}})$ the expansion (19.22) takes the form

$$\zeta = \sum_{p=2}^{\infty} \zeta_{\alpha_2 \cdots \alpha_p} \rho^{\alpha_2} \cdots \rho^{\alpha_p} \rho^p.$$

Accordingly, ζ is also an analytic function of the coordinates $(\rho, \theta^{\mathcal{A}})$. Decomposing the product $\rho^{\alpha_2} \cdots \rho^{\alpha_p}$ (which depends only on the angular coordinates) into symmetric, trace-free terms one obtains the usual expansion in terms of spherical harmonics Y_{lm}. This computation can be conveniently performed in space spinors; see, for example, Torres del Castillo (2003).

Remark. *Not every analytic function of $(\rho, \theta^{\mathcal{A}})$ is an analytic function of the associated Cartesian coordinates.* The standard counterexample for this is the radial coordinate ρ as defined by Equation (19.23), whose second derivative with respect to the coordinates (x^α) is singular at i. To have analyticity with respect to the coordinates (x^α) one needs the *right combination of spherical harmonics and powers of* ρ.

A particular case of the above discussion concerns the conformal factor Ω. From Equation (19.8) it follows that

$$\Omega = \frac{\zeta}{(1+\zeta^{1/2}W)^2}.$$

A direct computation taking into account the asymptotic conditions (19.11) shows that while Ω is C^2 at i, it will fail to be of class C^3 unless $W=0$ – which, in the present gauge, means that $m=0$. Thus, in general, the conformal factor Ω is not analytic in the harmonic coordinates (x^α) even if ζ is analytic. It is, nevertheless, analytic in ρ.

19.2.1 A spacetime conformal completion of static solutions

Theorem 19.1 is a statement about the conformal structure of hypersurfaces of a canonical foliation of a static spacetime. Thus, it is of natural interest to analyse the consequences of this property from a spacetime perspective. Intuitively, one expects that the nice conformal properties of the leaves of the foliation will lead to a good spacetime conformal behaviour. As the spatial conformal factor is not analytic with respect to the harmonic coordinates $\underline{x}=(x^\alpha)$, one cannot expect analyticity of a spacetime conformal extension in terms of these coordinates. Instead, one looks for extensions which are analytic in the associated radial coordinate.

Following Remark (iii) after Theorem 19.1, it is assumed that the harmonic coordinates $\underline{x}=(x^\alpha)$ are $\boldsymbol{h}$-normal and centred at i. Writing $\theta=(\theta^{\mathcal{A}})$, one has that for $\theta=\theta_\star$ fixed and $\mathrm{s}\in[0,\mathrm{s}_\star)$ for $\mathrm{s}_\star\geq 0$ suitably small, $x^\alpha(\mathrm{s})=\mathrm{s}\rho^\alpha(\theta_\star)$ describes a geodesic passing through i. A function $f:\mathcal{U}\to\mathbb{R}$ evaluated along one of these geodesics will be denoted by $f(\mathrm{s}\rho^\alpha)$. From $x^\alpha=\rho\,\rho^\alpha$ it follows that

$$\mathbf{d}x^\alpha=\rho^\alpha\mathbf{d}\rho+\mathbf{d}\rho^\alpha=\rho^\alpha\mathbf{d}\rho+\rho\partial_{\mathcal{A}}\rho^\alpha\mathbf{d}\theta^{\mathcal{A}},$$

so that, using the normal coordinates condition $h_{\alpha\beta}x^\alpha=-\delta_{\alpha\beta}x^\alpha$, one concludes that

$$\boldsymbol{h}=-\mathbf{d}\rho\otimes\mathbf{d}\rho+\rho^2\boldsymbol{k}, \tag{19.24}$$

where

$$\boldsymbol{k}\equiv h_{\alpha\beta}\mathbf{d}\rho^\alpha\otimes\mathbf{d}\rho^\beta=h_{\alpha\beta}\partial_{\mathcal{A}}\rho^\alpha\partial_{\mathcal{B}}\rho^\beta\mathbf{d}\theta^{\mathcal{A}}\otimes\mathbf{d}\theta^{\mathcal{B}}$$

corresponds to the metric of the two-dimensional surfaces of constant ρ. In particular, one has that $\boldsymbol{k}|_{\rho=0}=-\boldsymbol{\sigma}$ – the negative definite standard metric of $\mathbb{S}^2$.

Putting together the discussion of the previous paragraph and recalling that $\boldsymbol{h}=\Omega^2\tilde{\boldsymbol{h}}$, one finds that the static metric (19.1) can be rewritten as

$$\tilde{\boldsymbol{g}}=v^2\mathbf{d}t\otimes\mathbf{d}t-\Omega^{-2}\mathbf{d}\rho\otimes\mathbf{d}\rho+\Omega^{-2}\rho^2\boldsymbol{k}. \tag{19.25}$$

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The claim is now that the conformal metric $\boldsymbol{g} \equiv \Xi^2 \tilde{\boldsymbol{g}}$ with

$$\Xi = \Omega^{1/2}$$

gives rise to a conformal extension of the static spacetime which is *as regular as one can possibly expect*, that is, analytic in the coordinates $(\rho, \theta^{\mathcal{A}})$. From Equation (19.25) one has that

$$\boldsymbol{g} = \Omega v^2 \mathbf{d}t \otimes \mathbf{d}t - \Omega^{-1} \mathbf{d}\rho \otimes \mathbf{d}\rho + \rho^2 \Omega^{-1} \boldsymbol{k}. \qquad (19.26)$$

Recalling that $\Omega = O(\rho^2)$ and $v = O(1)$, one finds that while the first and third terms of the above metric are regular, the second one is singular. This singularity is a coordinate artefact which can be removed by considering the *null coordinate*

$$u \equiv t + \int_{\rho}^{\rho_\star} \frac{\mathrm{d}s}{v(s\rho^\alpha)\Omega(s\rho^\alpha)},$$

for fixed ρ^α and $\rho_\star > 0$. Observe, in particular, that as a consequence of the behaviour of Ω near i one has that $u \to -\infty$ as $\rho \to 0$. The differential of the null coordinate u is given by

$$\mathbf{d}u = \mathbf{d}t - \frac{1}{v\Omega} \mathbf{d}\rho + \boldsymbol{\lambda},$$

where

$$\boldsymbol{\lambda} \equiv \lambda_{\mathcal{A}} \mathbf{d}\theta^{\mathcal{A}}, \qquad \lambda_{\mathcal{A}} \equiv \int_{\rho}^{\rho_\star} \partial_{\mathcal{A}} \left(\frac{1}{v(s\rho^\alpha)\Omega(s\rho^\alpha)} \right) \mathrm{d}s.$$

Substituting the above expressions into the conformal metric (19.26) yields

$$\begin{aligned} \boldsymbol{g} = {} & \Omega v^2 \mathbf{d}u \otimes \mathbf{d}u + v(\mathbf{d}u \otimes \mathbf{d}\rho + \mathbf{d}\rho \otimes \mathbf{d}u) - \Omega v^2 (\mathbf{d}u \otimes \boldsymbol{\lambda} + \boldsymbol{\lambda} \otimes \mathbf{d}u) \\ & - v(\boldsymbol{\lambda} \otimes \mathbf{d}\rho + \mathbf{d}\rho \otimes \boldsymbol{\lambda}) + \Omega v^2 \boldsymbol{\lambda} \otimes \boldsymbol{\lambda} + \rho^2 \Omega^{-2} \boldsymbol{k}, \end{aligned}$$

which is regular whenever $\Omega = 0$. Moreover, following the discussion of Section 19.2, the various metric coefficients are analytic in the coordinates $(\rho, \theta^{\mathcal{A}})$. The conformal representation of static spacetimes given above shows that *static spacetimes admit a smooth conformal extension which includes a portion of null infinity.* However, this description is not suitable for a spacetime discussion of spatial infinity. This issue will be elaborated in Chapter 20. The discussion of this section can be extended to include stationary spacetimes; for a discussion of the required considerations, see Dain (2001b).

19.3 A regularity condition

As an application of the results on the analyticity of solutions to the conformal static equations at i, in this section a proof is given of a property of the conformal structure of static solutions which plays a central role in the discussion of Chapter 20. The analysis of this section is best carried out in spinors and is adapted from Beig (1991).

Before stating the main result of this section it is convenient to discuss some ancillary consequences of the conformal static equations. In what follows, all the spinors are expressed in terms of their components with respect to some spin dyad $\{\epsilon_{\boldsymbol{A}}{}^{A}\}$ associated to the frame $\{\boldsymbol{e}_{\boldsymbol{i}}\}$ corresponding to the particular realisation of harmonic $\boldsymbol{h}$-normal coordinates at i.

Lemma 19.1 (***behaviour of the symmetrised derivatives of*** ζ ***at*** i) *A solution to the conformal static equations satisfies*

$$D_{(\boldsymbol{A}_p\boldsymbol{B}_p}D_{\boldsymbol{A}_{p-1}\boldsymbol{B}_{p-1}}\cdots D_{\boldsymbol{A}_1)\boldsymbol{B}_1}\zeta(i)=0.$$

Proof For the cases $p=0,\,1,\,2$, the result follows from a direct computation using the conditions in (19.11), observing that $h_{(\boldsymbol{ABCD})}=0$. For higher order derivatives, the result follows by induction, using that the spinorial version of the equation associated to the zero quantity S_{ij} is given by

$$D_{\boldsymbol{AB}}D_{\boldsymbol{CD}}\zeta=\varsigma h_{\boldsymbol{ABCD}}+(\mu-1)\zeta s_{\boldsymbol{ABCD}}, \tag{19.27}$$

and using that $D_{\boldsymbol{EF}}h_{\boldsymbol{ABCD}}=0$ and $h_{(\boldsymbol{ABC})\boldsymbol{D}}=0$. □

Remark. Lemma 19.1 implies, in particular, that

$$D_{(\boldsymbol{A}_p\boldsymbol{B}_p}D_{\boldsymbol{A}_{p-1}\boldsymbol{B}_{p-1}}\cdots D_{\boldsymbol{A}_1\boldsymbol{B}_1)}\zeta(i)=0.$$

The main result of this section is the following:

Proposition 19.1 (***behaviour of the derivatives of the Cotton spinor at*** i) *A solution to the conformal static equations satisfies*

$$D_{(\boldsymbol{A}_p\boldsymbol{B}_p}D_{\boldsymbol{A}_{p-1}\boldsymbol{B}_{p-1}}\cdots D_{\boldsymbol{A}_1\boldsymbol{B}_1}b_{\boldsymbol{ABCD})}(i)=0, \qquad p=0,\,1,\,2,\ldots. \tag{19.28}$$

The original proofs of this result were given independently by Friedrich (1988) and Beig (1991).

Proof The proof of this result follows from considering Equation (19.18) in the form

$$(1-\mu\zeta)b_{\boldsymbol{ABCD}}=2\mu s_{\boldsymbol{Q}(\boldsymbol{ABC}}D_{\boldsymbol{D})}{}^{\boldsymbol{Q}}\zeta. \tag{19.29}$$

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Using the conditions in (19.11) one obtains $b_{\boldsymbol{ABCD}}(i) = 0$. Repeated differentiation and symmetrisation of Equation (19.29) yields

$$
\begin{aligned}
(1-\mu\zeta) & D_{(\boldsymbol{A}_p\boldsymbol{B}_p} D_{\boldsymbol{A}_{p-1}\boldsymbol{B}_{p-1}} \cdots D_{\boldsymbol{A}_1\boldsymbol{B}_1} b_{\boldsymbol{ABCD})} \\
& - p\mu D_{(\boldsymbol{A}_p\boldsymbol{B}_p} \zeta D_{\boldsymbol{A}_{p-1}\boldsymbol{B}_{p-1}} \cdots D_{\boldsymbol{A}_1\boldsymbol{B}_1} b_{\boldsymbol{ABCD})} \\
& \quad + \cdots - \mu D_{(\boldsymbol{A}_p\boldsymbol{B}_p} D_{\boldsymbol{A}_{p-1}\boldsymbol{B}_{p-1}} \cdots D_{\boldsymbol{A}_1\boldsymbol{B}_1} \zeta b_{\boldsymbol{ABCD})} \\
& \qquad = D_{(\boldsymbol{A}_p\boldsymbol{B}_p} \cdots D_{\boldsymbol{A}_1\boldsymbol{B}_1} s_{|\boldsymbol{Q}|\boldsymbol{ABC}} D_{\boldsymbol{D})}{}^{\boldsymbol{Q}} \zeta \\
& \qquad\quad + \cdots + s_{\boldsymbol{Q}(\boldsymbol{ABC}} D_{\boldsymbol{A}_p\boldsymbol{B}_p} \cdots D_{\boldsymbol{A}_1\boldsymbol{B}_1} D_{\boldsymbol{D})}{}^{\boldsymbol{Q}} \zeta.
\end{aligned}
$$

Using Lemma 19.1 it follows that every term in the above expression, save for the first one in the left-hand side, vanishes when evaluated at i. This yields the desired result. □

Remark. Condition (19.28) has been called in Friedrich (1988), for reasons to be elaborated in Chapter 20, the ***radiativity condition***. In Friedrich (1998c) it has been given the name ***regularity condition***. In tensorial notation Equation (19.28) takes the form

$$
D_{\{\alpha_p} D_{\alpha_{p-1}} \cdots D_{\alpha_1} b_{\beta\gamma\}}(i) = 0 \qquad p = 0,\, 1,\, 2, \ldots.
$$

Conformal transformation properties

Let ω denote a smooth function defined in a neighbourhood of i satisfying $\omega(i) \neq 0$. From the conformal transformation properties of the Cotton tensor – see Equation (5.19) – it follows that under the rescaling $\boldsymbol{h} \mapsto \boldsymbol{h}' = \omega^2 \boldsymbol{h}$ the Cotton spinor satisfies

$$
b'_{\boldsymbol{ABCD}} = \omega^{-1} b_{\boldsymbol{ABCD}}.
$$

Thus, $b'_{\boldsymbol{ABCD}}(i) = 0$ if $b_{\boldsymbol{ABCD}}(i) = 0$. Using the transformation law of the connection one finds that $D'_{\boldsymbol{A}_1\boldsymbol{B}_1} b'_{\boldsymbol{ABCD}}(i) = D_{\boldsymbol{A}_1\boldsymbol{B}_1} b_{\boldsymbol{ABCD}}(i)$ as the correction terms associated to the transition tensor involve $b_{\boldsymbol{ABCD}}(i) = 0$. Hence, $D'_{(\boldsymbol{A}_1\boldsymbol{B}_1} b'_{\boldsymbol{ABCD})}(i) = 0$. Proceeding inductively one concludes that

$$
D'_{(\boldsymbol{A}_p\boldsymbol{B}_p} D'_{\boldsymbol{A}_{p-1}\boldsymbol{B}_{p-1}} \cdots D'_{\boldsymbol{A}_1\boldsymbol{B}_1} b'_{\boldsymbol{ABCD})}(i) = 0, \qquad p = 0,\, 1,\, 2,\, \ldots.
$$

Consequently, *the regularity condition* (19.28) *is conformally invariant*. This conformal invariance allows the following reading of Proposition 19.1: *the conformal class of a 3-metric satisfying the static equations cannot be arbitrary*. More precisely, condition (19.28) is a necessary condition for a metric $\boldsymbol{h}$ to belong to the conformal class of a static metric.

19.4 Multipole moments

In Newtonian gravity time-independent gravitational fields are characterised by a sequence of multipole moments. It is desirable to have a similar characterisation for time-independent solutions to the Einstein field equations describing isolated bodies. One of the advantages of the conformal approach to static spacetimes

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is that it allows a geometric formulation of the notion of multipole moments. Following the original treatment in Geroch (1970a,b) one defines a sequence of tensor fields $\{P, P_i, P_{i_1 i_2}, \ldots\}$ in a neighbourhood $\mathcal{U}$ of i via the recursive relations

$$
\begin{aligned}
&P \equiv \Omega^{-1/2}(1-v),\\
&P_i \equiv D_i P,\\
&P_{i_2 i_1} \equiv D_{\{i_2} P_{i_1\}} - \frac{1}{2} P r_{i_2 i_1},\\
&P_{i_{p+1}\cdots i_1} \equiv D_{\{i_{p+1}} P_{i_p \cdots i_1\}} - \frac{1}{2} p(2p-1) P_{\{i_{p+1}\cdots i_3} r_{i_2 i_1\}}, \qquad p = 2, 3, \ldots .
\end{aligned}
$$

The particular form of the lower order correction terms in the definition of the tensors $P_{i_{p+1}\cdots i_1}$ has been chosen so as to ensure conformal invariance of the definition of multipole moments to be given below – this observation follows from a tedious computation which will not be further elaborated here. The multipole moments of a static solution are then obtained by evaluating the above tensors at i. To this end, choose a smooth coordinate system $\underline{x} = (x^\alpha)$ on $\mathcal{U}$ and denote by P_α, $P_{\alpha_2\alpha_1}, \ldots$ the components of the tensors with respect to these coordinates and define the ***multipole moments of the static solution*** with respect to the coordinates $\underline{x}$ to be the sequence $\{m, m_\alpha, m_{\alpha_2\alpha_1}, \ldots\}$ with

$$
m \equiv P(i), \qquad m_{\alpha_p\cdots\alpha_1} \equiv P_{\alpha_p\cdots\alpha_1}(i), \qquad p = 1, 2, 3 \ldots .
$$

For a given p, the 2^p quantities $m_{\alpha_p\cdots\alpha_1}$ are called the ***2^p-poles***. In particular, m is the ***monopole*** (the mass) and m_α is the ***dipole moment***. As the multipole moments are expressed as the value of a tensor at a point, it follows that, under a coordinate transformation $\underline{x}' = (x'^\alpha(\underline{x}))$, the multipole moments transform as

$$
m' = m, \qquad m'_\alpha = A_\alpha{}^\beta m_\beta, \qquad m'_{\alpha_p\cdots\alpha_1} = A_{\alpha_p}{}^{\beta_p}\cdots A_{\alpha_1}{}^{\beta_1} m_{\beta_p\cdots\beta_1}, \tag{19.30}
$$

where $(A_\alpha{}^\beta)$ are the components of 3×3 invertible real matrices; that is, $(A_\alpha{}^\beta) \in GL(3,\mathbb{R})$. Observe that the monopole is invariant under a change of coordinates. For the particular choice of the conformal factor given by Equation (19.7) one has that $P = m$ so that $D_i P = 0$ and accordingly $m_\alpha = 0$; in other words, in the conformal gauge determined by (19.7) one is automatically in the *centre of mass*.

The properties of the multipole expansions in Newtonian gravity raise the question of to what extent the general relativistic multipole moments determine a solution to the static equations, and vice versa. The construction described in the previous paragraph can be thought of as mapping a static solution $(v, \tilde{\boldsymbol{h}})$ to the collection of multipoles $\{m, m_\alpha, m_{\alpha_2\alpha_1}, \ldots\}$. Now, two collections of multipoles $\{m, m_\alpha, m_{\alpha_2\alpha_1}, \ldots\}$ and $\{m', m'_\alpha, m'_{\alpha_2\alpha_1}, \ldots\}$ are said to be ***equivalent*** if there exists $(A_\alpha{}^\beta) \in GL(3,\mathbb{R})$ such that the relations in (19.30) hold. In Beig and Simon (1980a) the following has been proved:

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Theorem 19.2 (***multipole theorem***) *If two static solutions* $(v, \tilde{\boldsymbol{h}})$ *and* $(v', \tilde{\boldsymbol{h}}')$ *lead to multipole sequences which are equivalent, then the static solutions are isometric in a neighbourhood of* i.

Although a detailed proof of the above theorem will not be provided, it is of interest to discuss the basic underlying ideas. The fundamental problem is the following: given a sequence of multipoles $\{m, m_\alpha, m_{\alpha_2\alpha_1}, \ldots\}$, *how can one reconstruct the pair* $(v, \tilde{\boldsymbol{h}})$ *solving the static equations?* To answer this question one first employs an inductive argument which relies on the definition of the multipole moments, the conformal static equations and the commutator of covariant derivatives to show that the values of the fields ζ, s, $s_{\alpha\beta}$ and any of their covariant derivatives at the point i can be expressed in terms of the multipole moments. Thus, one can compute the Taylor expansions (in harmonic $\boldsymbol{h}$-normal coordinates) of these fields around i. From the general theory of Taylor expansions one knows that the expansions are unique. Moreover, it is a classical result of Riemannian geometry that the sequence

$$\{r_{\alpha\beta\gamma\delta}(i),\, D_\eta r_{\alpha\beta\gamma\delta}(i),\, D_{\eta_2\eta_1} r_{\alpha\beta\gamma\delta}(i),\, \ldots\}$$

determines, in a unique way, the Taylor expansion of the components of the metric $h_{\alpha\beta}$ – again, in $\boldsymbol{h}$-normal coordinates (x^α) centred at i; see, for example, Günther (1975). A final argument shows that applying the above procedure to two equivalent sequences of multipoles leads to two metrics which are isometric.

Now, given any set of multipole moments subject to the appropriate convergence condition, it is natural to expect that there exists a static solution having precisely those multipole moments. In other words, the sequence of multipoles characterises (in a suitable) unique manner the static spacetime. As a result of the analyses in Friedrich (2007) and Herberthson (2009) one has the following:

Theorem 19.3 (***sufficient conditions on the sequence of multipoles for the existence of a static solution***) *Let* $\{m, m_\alpha, m_{\alpha_2\alpha_1}, \ldots\}$ *denote the components of a sequence of real valued, totally symmetric trace-free tensors at the origin of* $\mathbb{R}^3$ *expressed in terms of Cartesian coordinates* $\underline{x} = (x^\alpha)$. *If constants* M, $C > 0$ *can be found such that*

$$|m_{\alpha_p \cdots \alpha_1}| \leq \frac{p!M}{C^p}, \tag{19.31}$$

then there exists a static, asymptotically flat spacetime having the multipole moments $\{m, m_\alpha, m_{\alpha_2\alpha_1}, \ldots\}$.

The proof of the above result goes beyond the scope of this book. Again, only the basic underlying ideas are briefly discussed. The starting point of the analysis is to exploit the analyticity of the solutions to the conformal static equations provided by Theorem 19.1 to implement a ***complex analytic extension*** of the whole setting. More precisely, the fields $h_{\alpha\beta}$, ζ, ς, $s_{\alpha\beta}$ can be extended

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near i by analyticity into the complex domain and regarded as *holomorphic* (i.e. complex analytic) fields on a complex analytic manifold $\mathcal{S}_{\mathbb{C}}$. Restricting $\mathcal{S}_{\mathbb{C}}$ to be a sufficiently small neighbourhood of i one can use similarly extended normal coordinates $\underline{x} = (x^\alpha)$ centred at i to define an analytic system of coordinates on $\mathcal{S}_{\mathbb{C}}$ which identifies the latter with an open neighbourhood of the origin in $\mathbb{C}^3$. The original manifold $\mathcal{S}$ is then a three-dimensional real analytic submanifold of $\mathcal{S}_{\mathbb{C}}$. Under the analytic extension the main differential geometric concepts and formulas remain valid. In particular, the extended fields, to be denoted again by $h_{\alpha\beta}$, ζ, s, $s_{\alpha\beta}$, satisfy the conformal static vacuum field equations on $\mathcal{S}_{\mathbb{C}}$. In order to provide a geometric perspective of the problem, one considers the function $\Gamma \equiv \delta_{\alpha\beta}x^\alpha x^\beta$ on $\mathcal{S}$ which extends to a holomorphic function on $\mathcal{S}_{\mathbb{C}}$ satisfying the equation $h^{\alpha\beta}D_\alpha\Gamma D_\beta\Gamma = -4\Gamma$. While restricted to $\mathcal{S}$, the function Γ vanishes only at i. On $\mathcal{S}_{\mathbb{C}}$ its set of zeros is a two-dimensional complex submanifold of $\mathcal{S}_{\mathbb{C}}$,

$$\mathcal{N}_i \equiv \{p \in \mathcal{S}_{\mathbb{C}} \,|\, \Gamma(p) = 0\},$$

the so-called ***complex null cone at*** i. This cone is generated by the complex null geodesics through i; see Figure 19.1. The analogy between the (spacetime) conformal field equations and the conformal static field equations discussed in Section 19.1.1 suggests the formulation of a characteristic initial value problem for the conformal static field equations on the null cone $\mathcal{N}_i$. The formulation of this characteristic initial value problem requires the determination of suitable initial data. An argument involving the idea of *exact sets of fields* – see Penrose and Rindler (1984) – allows one to show that the basic data for this characteristic problem are is given by the sequence of fields

$$\big\{s_{\alpha\beta}(i),\, D_{\{\alpha_1}s_{\alpha\beta\}}(i),\, \ldots,\, D_{\{\alpha_p\cdots\alpha_1}s_{\alpha\beta\}}(i), \ldots\big\}.$$

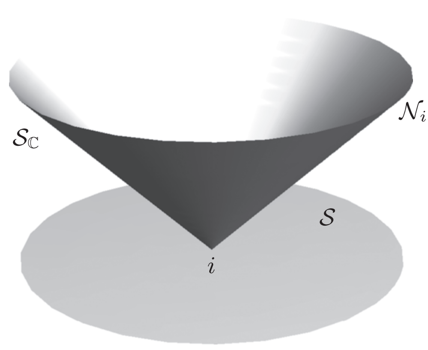

Figure 19.1 Schematic representation of the complex null cone through i, $\mathcal{N}_i$, as described in the main text.

The above ***null data*** can be obtained by repeated differentiation along the direction of the complex null generators of $\mathcal{N}_i$ of the components of $s_{\alpha\beta}$; see, for example, the discussion in Friedrich (2004, 2007).

Given the analyticity of the setting described in the previous paragraphs, one can make use of the *Cauchy-Kowalewskaya theorem* to discuss the existence of analytic solutions to this characteristic problem and to provide convergence conditions on the null data which ensure the existence of a solution; see the Appendix to this chapter. The convergence conditions thus obtained are similar to the ones in Equation (19.31) of Theorem 19.3 and, in particular, ensure the existence of a real static solution. This is the main result of Friedrich (2007). To obtain the convergence condition on the sequence of multipoles one needs to analyse the relation between the null data and the sequence of multipoles. Inspection shows that the null data and the multipole moments are in a one-to-one correspondence. This correspondence, however, is non-linear and implicit. The detailed analysis of this correspondence in Herberthson (2009) allows the transformation of the convergence conditions for the null data into convergence conditions for the sequence of multipoles.

19.5 Uniqueness of the conformal structure of static metrics

As a final application of the conformal static equations, the extent to which the conformal class of the 3-metric $\tilde{\boldsymbol{h}}$ determines a solution to the static equations will be analysed. This question was first analysed in Beig (1991) from where the main ideas of the analysis are adapted. An alternative discussion of some aspects of this problem is given in Friedrich (2008a,b).

The multipole Theorem 19.2 shows that a static solution is determined by its multipole moments. Thus, it is natural to try to relate the multipole moments to the conformal class of the metric $\boldsymbol{h}$. In what follows, for $p = 1, 2, 3, \ldots$ define

$$\beta_{\boldsymbol{A}_p\boldsymbol{B}_p\cdots\boldsymbol{A}_1\boldsymbol{B}_1\boldsymbol{A}_0\boldsymbol{B}_0} \equiv D^{\boldsymbol{Q}}{}_{(\boldsymbol{B}_p}D_{\boldsymbol{A}_{p-1}\boldsymbol{B}_{p-1}}\cdots D_{\boldsymbol{A}_1\boldsymbol{B}_1}b_{\boldsymbol{A}_0\boldsymbol{B}_0\boldsymbol{A}_p\boldsymbol{Q})}(i).$$

Using an inductive argument similar (albeit lengthier!) to the one leading to Proposition 19.1 one obtains the family of identities

$$\beta_{\boldsymbol{A}_p\boldsymbol{B}_p\cdots\boldsymbol{A}_1\boldsymbol{B}_1\boldsymbol{A}_0\boldsymbol{B}_0} = 6\mu D_{(\boldsymbol{A}_p\boldsymbol{B}_p}\cdots D_{\boldsymbol{A}_2\boldsymbol{B}_2}s_{\boldsymbol{A}_1\boldsymbol{B}_1\boldsymbol{A}_0\boldsymbol{B}_0)}(i), \tag{19.32}$$

for $p = 1, 2, 3, \ldots$ with $\mu = m^2/4$; see Equation (19.12). The above identities constitute the main tool for the reminder of the section. Observe that the tensorial counterpart of the symmetrised derivatives of the spinor enter directly in the definition of the multipole moments. The quantities $\beta_{\boldsymbol{A}_p\boldsymbol{B}_p\cdots\boldsymbol{A}_0\boldsymbol{B}_0}$ have good conformal properties. Recalling that under the rescaling $\boldsymbol{h}' = \omega^2\boldsymbol{h}$ with $\omega(i) \neq 0$, one has the transformation rules

$$b'_{\boldsymbol{ABCD}} = \omega^{-1}b_{\boldsymbol{ABCD}}, \qquad \epsilon'^{\boldsymbol{AB}} = \omega^{-1}\epsilon^{\boldsymbol{AB}}.$$

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It follows that

$$\beta'_{\boldsymbol{A}_p\boldsymbol{B}_p\cdots\boldsymbol{A}_0\boldsymbol{B}_0} = \omega^{-2}(i)\beta_{\boldsymbol{A}_p\boldsymbol{B}_p\cdots\boldsymbol{A}_0\boldsymbol{B}_0}.$$

Consider now two solutions $(\boldsymbol{h}, s_{ABCD}, \zeta, \varsigma)$ and $(\boldsymbol{h}', s'_{ABCD}, \zeta', \varsigma')$ to the conformal static equations such that

$$\boldsymbol{h}' = \omega^2 \boldsymbol{h},$$

and consider the question of under which circumstances will the above two solutions determine *the same physical static solution* $(v, \tilde{\boldsymbol{h}})$ – modulo isometries. The identities (19.32) show how the multipole moments of the two solutions are connected to each other. One then needs further conditions that allow one to constrain the relation between solutions further. In view of the conformal nature of the problem, the natural object to look for those extra conditions is the Cotton spinor. Proposition 19.1 and the identities (19.32) already provide information about some of the derivatives of $b_{\boldsymbol{ABCD}}$ at i. The only derivatives which have not yet been considered are *divergences* of the form $D^{\boldsymbol{PQ}}D_{(\boldsymbol{PQ}}D_{\boldsymbol{A}_p\boldsymbol{B}_p}\cdots D_{\boldsymbol{A}_1\boldsymbol{B}_1}b_{\boldsymbol{ABCD})}$. A direct computation using Equations (19.27) and (19.29) yields

$$D^{\boldsymbol{PQ}}b_{\boldsymbol{PQCD}}(i) = 0, \tag{19.33a}$$
$$D^{\boldsymbol{PQ}}D_{(\boldsymbol{PQ}}b_{\boldsymbol{ABCD})}(i) = 0, \tag{19.33b}$$
$$D^{\boldsymbol{PQ}}D_{(\boldsymbol{PQ}}D_{\boldsymbol{EF}}b_{\boldsymbol{ABCD})}(i) = 0. \tag{19.33c}$$

However, a lengthy computation reveals that

$$D^{\boldsymbol{PQ}}D_{(\boldsymbol{PQ}}D_{\boldsymbol{GH}}D_{\boldsymbol{EF}}b_{\boldsymbol{ABCD})}(i) = -24\mu D_{(\boldsymbol{GH}}s^{\boldsymbol{Q}}{}_{\boldsymbol{EFA}}(i)s_{\boldsymbol{BCD})\boldsymbol{Q}}(i).$$

Defining, for convenience, $O_{\boldsymbol{GHEFABCD}} \equiv D_{(\boldsymbol{GH}}s^{\boldsymbol{Q}}{}_{\boldsymbol{EFA}}(i)s_{\boldsymbol{BCD})\boldsymbol{Q}}(i)$, a computation using the definition of the quantities $\beta_{\boldsymbol{A}_p\boldsymbol{B}_p\cdots\boldsymbol{A}_0\boldsymbol{B}_0}$ allows the reexpression of this quantity in the form

$$O_{\boldsymbol{GHEFABCD}} = \frac{1}{36}\beta^{\boldsymbol{Q}}{}_{(\boldsymbol{GHFEA}}\beta_{\boldsymbol{BCD})\boldsymbol{Q}}. \tag{19.34}$$

The conformal transformation properties can be easily read from this last expression. Namely, one has that

$$O'_{\boldsymbol{GHEFABCD}} = \omega^{-5}(i)O_{\boldsymbol{GHEFABCD}}.$$

On the other hand, it can be checked that

$$D'^{\boldsymbol{PQ}}D'_{(\boldsymbol{PQ}}D'_{\boldsymbol{GH}}D'_{\boldsymbol{EF}}b'_{\boldsymbol{ABCD})}(i) = \omega^{-3}(i)D^{\boldsymbol{PQ}}D_{(\boldsymbol{PQ}}D_{\boldsymbol{GH}}D_{\boldsymbol{EF}}b_{\boldsymbol{ABCD})}(i).$$

From the above transformation rules, assuming that $O_{\boldsymbol{GHEFABCD}} \neq 0$ one concludes that

$$\omega^2(i) = \mu'/\mu.$$

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Observe that if $O_{GHEFABCD} = 0$, no conclusion can be extracted from the analysis. As a consequence of Equation (19.34), the requirement $O_{GHEFABCD} \neq 0$ is a condition on the conformal structure of the static solutions under consideration. If it holds, then using the identities (19.32) one concludes that the two solutions to the conformal static equations will have the same multipole moments if they have the same mass. Moreover, as a consequence of the multipole Theorem 19.2, they are isometric. The analysis of this section is summarised in the following theorem, first proven in Beig (1991):

Theorem 19.4 (***uniqueness of the conformal structure of static solutions***) *Two solutions to the conformal static equations with the same mass, lying in the same conformal class and satisfying $O_{GHEFABCD} \neq 0$ are isometric.*

The condition $O_{GHEFABCD} \neq 0$ can be seen to be violated if the two static solutions are axially symmetric about a common axis; see, for example, Beig (1991). As discussed in Friedrich (2008a) this is, in fact, the only possibility. More precisely, a static solution which admits a non-trivial rescaling leading to a new static solution must be axially symmetric and admit a conformal Killing vector. There exists a three-parameter family of such solutions. These have been explicitly found in Friedrich (2008b).

The Schwarzschild solution

A case of particular interest is when $\tilde{\boldsymbol{h}}$ is conformally flat. It then follows that $\beta_{\boldsymbol{A}_p\boldsymbol{B}_p\cdots\boldsymbol{A}_0\boldsymbol{B}_0} = 0$ for all p, and the only non-vanishing mass multipole is the mass m. Invoking, again, the multipole Theorem 19.2 it follows that for a given value of m there exists only one solution, up to isometries, with this property – namely, the Schwarzschild solution. An alternative derivation of the *uniqueness of the Schwarzschild spacetime among the class of conformally flat static solutions* which makes no use of the multipole theorem has been given in Friedrich (2004). In this analysis the conformal static equations are explicitly integrated along geodesics starting at i.

19.6 Characterisation of static initial data

An issue related to the questions discussed in the previous section concerns the characterisation of initial data for a static spacetime – this question will be of relevance in Chapter 20. More precisely, one is interested in the following question: *given a 3-metric $\boldsymbol{h}$, under which circumstances does there exist in the conformal class $[\boldsymbol{h}]$ another metric $\tilde{\boldsymbol{h}}$ which, together with some scalar v, constitutes a solution to the static equations?*

As in the rest of the chapter, the above question is restricted to a suitable neighbourhood of infinity. Proposition 19.1 shows that not every conformal class will contain a static metric. In other words, condition (19.28) is a *necessary*

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condition for a metric $\boldsymbol{h}$ to be conformal to a static metric. Now, condition (19.28) is not sufficient. The relations (19.33b) and (19.33c) show that there exist further conditions (in fact, an infinite hierarchy of them) on the conformal class, algebraically independent from (19.28), which need to be satisfied by a metric $\boldsymbol{h}$ in order to be conformal to a static metric. The *gap* between a conformal class of 3-metrics satisfying the regularity condition (19.28) and a conformal class containing a static metric has been analysed in detail in Friedrich (2013).

The level of detail required to discuss the main result of Friedrich (2013) goes well beyond the scope of this chapter, and only the key ideas are briefly mentioned. If a metric $\boldsymbol{h}$ is conformal to a metric $\boldsymbol{h}'$ solving the conformal static equations, then writing $\boldsymbol{h}' = \omega^2 \boldsymbol{h}$ for some suitable conformal factor ω, it is possible to rewrite the conformal static equations as a highly overdetermined system of differential equations for ω. To analyse the solvability of the conditions one needs to consider the associated integrability conditions. As already anticipated by (19.33b) and (19.33c), these integrability conditions give rise, in addition to (19.28), to restrictions on the conformal structure which take the form of an infinite hierarchy of differential conditions on the Cotton tensor at i. These conditions can be expressed in terms of requirements on a covector constructed from the 3-metric $\boldsymbol{h}$. An interesting feature of the analysis is that the overdetermined system involving the conformal factor ω is highly singular at i. For this system to have a solution, a hierarchy of regularity conditions need to be imposed on the singular part of the equation so that it admits a smooth extension to a neighbourhood of i – this is reminiscent of a procedure which arises in the construction of *radiative initial data sets* in Section 20.2. Remarkably, the required *regularity conditions* turn out to be nothing else but the conditions (19.28).

19.7 Further reading

A systematic analysis of time-independent solutions to the Einstein field equations is provided in Beig and Schmidt (2000). This reference provides an excellent point of entry to the extensive literature on static and stationary solutions in general relativity. A survey of the various approaches to define multipole moments for time-independent solutions to the Einstein field equations can be found in Quevedo (1990). An analysis of global aspects of static and stationary spacetimes can be found in Anderson (2000).

Several of the results discussed in this chapter admit a generalisation to the case of stationary solutions. The definitions of multipole moments given by Geroch (1970a,b) have been extended to the stationary case in Hansen (1974). The analyticity of solutions of the conformal stationary field equations has been analysed in Beig and Simon (1980b, 1981); see also Kundu (1981). However, in this case the 3-metric $\boldsymbol{h}$ of a surface of constant time will not be analytic; see Dain (2001b). Instead, the analyticity refers to the 3-metric $\boldsymbol{\gamma}$ of the ***quotient space*** obtained from identifying points on the spacetime lying on the same orbit of the stationary Killing vector. The analysis of the convergence conditions for null data of static solutions in Friedrich (2007) has been extended to the case of stationary solutions in Aceña (2009). An alternative analysis of multipole expansions of static solutions with the aim of obtaining convergence conditions on sequences of multipoles has been given in Bäckdahl and Herberthson (2005a,b, 2006) and Bäckdahl (2007).

The analysis of the conformal static equations by means of the complex null cone through i was first introduced in Friedrich (1988). Further extensions of this method have been given in Friedrich (2004, 2007, 2013).

Appendix 1: Hölder conditions

Given $0 < \alpha \leq 1$, a real valued function f on an open set $\mathcal{U} \subset \mathbb{R}^n$ is said to satisfy the ***Hölder condition*** with exponent α on $\mathcal{U}$ if there exists a non-negative constant C such that

$$|f(x) - f(y)| \leq C|x - y|^{\alpha}, \qquad \text{for all } x,\, y \in \mathcal{U}.$$

If the above is the case, one writes $f \in C^{0,\alpha}(\mathcal{U})$. The Hölder condition is a stronger notion of continuity; that is, a function satisfying the Hölder condition is continuous, but not all continuous functions satisfy the Hölder condition for some α. More generally, one says that $f \in C^{k,\alpha}(\mathcal{U})$ if all its derivatives up to order k satisfy the Hölder condition for a given α. The Hölder condition plays an important role in the regularity of solutions to elliptic PDEs; see, for example, Evans (1998) for further details.

Appendix 2: the Cauchy-Kowalewskaya theorem

The ***Cauchy-Kowalewskaya*** theorem asserts the local existence, in a neighbourhood of $t = 0$, of a real analytic solution $\mathbf{u}(t, \underline{x})$ to the quasilinear first-order initial value problem

$$\begin{aligned} &\partial_t \mathbf{u} = \mathbf{A}^{\alpha}(t, \underline{x}, \mathbf{u})\partial_{\alpha}\mathbf{u} + \mathbf{B}(t, \underline{x}, \mathbf{u}), \\ &\mathbf{u}(0, \underline{x}) = \mathbf{u}_{\star}(\underline{x}), \end{aligned}$$

where $\mathbf{A}^{\alpha}(t, \underline{x}, \mathbf{u})$, $\mathbf{B}(t, \underline{x}, \mathbf{u})$ and $\mathbf{u}_{\star}(\underline{x})$ are real analytic functions of their arguments; see, for example, Evans (1998) for further details. A discussion of the various approaches to prove this result can be found in Shinbrot and Welland (1976).