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Path Decompositions of Kneser and Generalized Kneser Graphs

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Abstract. Necessary and sufficient conditions are given for the existence of a graph decomposition of the Kneser Graph $KG_{n,2}$ and of the Generalized Kneser Graph $GKG_{n,3,1}$ into paths of length three.

1 Introduction

An H-decomposition of a graph $G = (V, E)$ is a pair (V, B), where B is a collection of edge-disjoint subgraphs of G, each isomorphic to H, whose edges partition $E(G)$. There is much in the literature concerning decompositions of various graphs. The subgraphs used to partition the edges of G are, for example, cycles [1,8,12] and paths [14], and G is most commonly a complete multipartite graph. More recently, stemming from statistical designs, G has been chosen to be the graph formed from a complete multipartite graph with multiplicity λ_2 by adding a copy of $\lambda_1 K_n$ to each part of size n and H is a 3-cycle, a 4-cycle, or a Hamilton cycle [2,5,6].

In this paper, we consider path decompositions in the case where G, the graph being decomposed, is a Kneser Graph or a Generalized Kneser Graph. The Kneser Graph $KG_{n,k}$ is the graph whose vertices are the k-element subsets of some set of n elements, in which two vertices are adjacent if and only if their intersection is empty. The Generalized Kneser Graph, $GKG_{n,k,r}$ is the graph whose vertices are the k-element subsets of some set of n elements, in which two vertices are adjacent if and only if they intersect in precisely r elements. The graph-decomposition problem of finding necessary and sufficient conditions for the existence of P_3-decompositions of $KG_{n,2}$ and $GKG_{n,3,1}$ is completely solved in Theorems 1 and 2 respectively, where P_i denotes a path of length i. An explicit construction is provided to find the relevant decompositions.

It is worth noting that Kneser graphs have attracted much interest in the years since Kneser first described them in 1955 [9]. For instance, Kneser Graphs are known to contain a Hamiltonian cycle if $n \geq 3k$ [4]. The current conjecture is that all Kneser Graphs are Hamiltonian if $n \geq 2k + 1$, with the exception of $KG_{5,2}$, which is the Petersen Graph. It has been shown computationally that all connected Kneser graphs with $n \leq 27$ except for the Petersen Graph are Hamiltonian [13]. Also, much interest has centered on solving the conjecture by Kneser that $\chi(KG_{n,k}) = n-2k+2$ whenever

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$n \geq 2k$ [3, 7, 9–11], where $\chi(G)$ is the chromatic number of G ($KG_{n,k}$ has no edges if $n < 2k$).

2 Building Blocks

Let $T_k(V)$ be the set of k-element subsets of the set V. Let $P_3 = (a, b, c, d)$ denote the path of length three with edge set $\{\{a, b\}, \{b, c\}, \{c, d\}\}$.

The following lemmas will be useful in the constructions to come.

Lemma 1 *There exists a P_3-decomposition of each of the following graphs:*

(i) $K_{2,3}$;
(ii) $K_{3,3}$;
(iii) $K_{n,3k}$ *for any* $n \geq 2$ *and* $k \geq 1$;
(iv) $H_4 = K_{3,3} - F$ *with bipartition* $\{\mathbb{Z}_3, \mathbb{Z}_6 \backslash \mathbb{Z}_3\}$ *of* $V(K_{3,3})$, *and where* $E(F) = \{\{i, i+3\} \mid i \in \mathbb{Z}_3\}$;
(v) $H_5 = H_4 \cup G'$, *where* $G' = (\mathbb{Z}_9 \backslash \mathbb{Z}_3, \{\{3,6\}, \{4,7\}, \{5,8\}\})$;
(vi) H_6, *the bipartite graph with bipartition* $\{T_2(\mathbb{Z}_4), \mathbb{Z}_4\}$ *of* $V(H_6)$ *and* $E(H_6) = \{\{a, b\} \mid b \notin a, a \in T(\mathbb{Z}_4), b \in \mathbb{Z}_4\}$;
(vii) $KG_{5,2}$ *(the Petersen Graph);*
(viii) H_8, *the bipartite graph with bipartition* $\{T_2(\mathbb{Z}_5), \mathbb{Z}_5\}$ *of* $V(H_8)$ *and* $E(H_8) = \{\{a, b\} \mid b \notin a, a \in T(\mathbb{Z}_5), b \in \mathbb{Z}_5\}$.

Proof (i) Define $K_{2,3}$ with bipartition $\{\mathbb{Z}_2, \mathbb{Z}_5 \backslash \mathbb{Z}_2\}$ of the vertex set $\mathbb{Z}_5$. Then

$$\big(\mathbb{Z}_5, \{(0,2,1,3), (3,0,4,1)\}\big)$$

is the required decomposition.

(ii) Define $K_{3,3}$ with bipartition $\{\mathbb{Z}_3, \mathbb{Z}_6 \backslash \mathbb{Z}_3\}$ of the vertex set $\mathbb{Z}_6$. Then

$$\big(\mathbb{Z}_6, \{(3,0,5,2), (1,3,2,4), (0,4,1,5)\}\big)$$

is the required decomposition.

(iii) Since $n \geq 2$, form a partition P of $\mathbb{Z}_n$ into sets of size 2 and 3, and a partition Q of $\mathbb{Z}_{n+3k} \backslash \mathbb{Z}_n$ into sets of size 3. For each $p \in P$ and $q \in Q$, let ($p \cup q$, $B_{p,q}$) be a P_3-decomposition of $K_{|p|,3}$ with bipartition $\{p, q\}$ of the vertex set. Then $(\mathbb{Z}_{n+3k}, \bigcup_{p \in P, q \in Q} B_{p,q})$ is the required P_3-decomposition of $K_{n,3k}$.

(iv) With bipartition $\{\mathbb{Z}_3, \mathbb{Z}_6 \backslash \mathbb{Z}_3\}$, $(\mathbb{Z}_6, \{(0,4,2,3), (0,5,1,3)\})$ is the required decomposition with $F = \{\{0,3\}, \{1,4\}, \{2,5\}\}$.

(v) $(\mathbb{Z}_9, \{(6,3,1,5), (7,4,2,3), (8,5,0,4)\})$ is the required decomposition.

(vi)

$$\big(V(H_6), \big\{(0, \{2,3\}, 1, \{0,2\}), (1, \{0,3\}, 2, \{1,3\}), (2, \{0,1\}, 3, \{0,2\}), (3, \{1,2\}, 0, \{1,3\})\big)$$

is the required decomposition.

(vii) Let $V(KG_{5,2}) = T_2(\mathbb{Z}_5)$. Then

$$\big(T_2(\mathbb{Z}_5), \big\{(\{i, i+1\}, \{i+2, i+3\}, \{i+1, i+4\}, \{i, i+3\}) \mid i \in \mathbb{Z}_5\big\}$$

reducing the sums modulo 5 is the required decomposition.

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(viii)

$$\big(V(H_8), \{(1,\{0,2\},3,\{0,1\}),(2,\{0,3\},4,\{0,2\}),\\ (2,\{0,4\},1,\{0,3\}),(0,\{1,2\},3,\{0,4\}),\\ (0,\{1,3\},4,\{1,2\}),(3,\{1,4\},2,\{1,3\}),\\ *(4,\{2,3\},0,\{1,4\}),(3,\{2,4\},1,\{2,3\}),\\ (1,\{3,4\},0,\{2,4\}),(4,\{0,1\},2,\{3,4\})\big)$$

is the required decomposition. ■

A graph G is said to have an Euler tour if there exists a closed walk in G that contains each edge of G exactly once.

The following is well known.

Lemma 2 *A connected simple graph G has an Euler tour if and only if the degree of every vertex in G is even.*

From this, we can easily obtain the following result.

Lemma 3 *If G is a connected bipartite simple graph in which the number of edges is divisible by three and all vertices have even degree, then G has a P_3-decomposition.*

Proof By Lemma 2, let $P = (v_0, v_1, \dots, v_e)$ be an Euler tour of G. Since G is bipartite, each set of three consecutive edges of P induce a P_3. Therefore, since $e = |E(G)|$ is divisible by three, $\big(V(G), \{(v_{3i}, v_{3i+1}, v_{3i+2}, v_{3i+3}) \mid i \in \mathbb{Z}_{e/3}\}\big)$ is a P_3-decomposition of G. ■

Lemma 4 *There exists a P_3-decomposition of each of the following graphs:*

(i) $GKG_{5,3,1}$ *(the Petersen Graph), and*
(ii) $GKG_{6,3,1}$.

Proof (i) $GKG_{5,3,1} = KG_{5,2}$ as can be seen by taking the complement of each vertex. The result follows from Lemma 1(vii).

(ii) Partition the vertices of $GKG_{6,3,1}$ into the following two types:

- Type 1: $T_3(\mathbb{Z}_5)$, and
- Type 2: $T_3(\mathbb{Z}_6)\backslash T_3(\mathbb{Z}_5)$.

Let G_1 be the subgraph induced by the Type 1 vertices, G_2 be the subgraph induced by the Type 2 vertices, and G_3 be the bipartite subgraph induced by the edges of the form $\{x, y\}$ where x is a Type 1 vertex and y is a Type 2 vertex. G_1 is clearly a $GKG_{5,3,1}$ and has a P_3-decomposition by (i). G_2 is isomorphic to $KG_{5,2}$ (all vertices share the element 5, so two are adjacent only if their other two elements are disjoint) and has a P_3-decomposition by Lemma 1(vii). G_3 is a bipartite graph that is 6-regular, so $|E(G_3)|$ is a multiple of three. To see that G_3 is connected, for each Type 1 vertex, $\{a, b, c\}$ in G_3 we display a path to each vertex of Type 2 as follows (where a, b, c, d and e are the distinct elements of $\mathbb{Z}_5$): $(\{a,b,c\},\{a,d,5\},\{b,c,d\},\{a,b,5\})$, $(\{a,b,c\},\{a,d,5\},\{a,b,e\},\{d,e,5\})$, and $(\{a,b,c\},\{a,d,5\})$. These account for

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all pairs of nonadjacent vertices in G_3, so G_3 is connected. Therefore, G_3 is a connected even regular bipartite graph with a multiple of three edges, so by Lemma 3, it also has a P_3-decomposition. The union of these three decompositions forms a P_3-decomposition of $GKG_{6,3,1}$. ∎

3 A P_3-Decomposition of $KG_{n,2}$

Theorem 1 *$KG_{n,2}$ is P_3-decomposable if and only if $n \neq 4$.*

Proof If $n \in \{1,2,3\}$, then $KG_{n,2}$ has no edges, so the result is vacuously true. Since $KG_{4,2}$ is a 1-factor on six vertices, it is clearly not P_3-decomposable. $KG_{5,2}$ is decomposable by Lemma 1(vii).

The remaining cases are proved by induction on n. So now assume that $KG_{w,2}$ is P_3-decomposable for all $w \leq n$ for some $n \geq 5$. It is shown that $G = KG_{n+1,2}$ is P_3-decomposable. Let $\epsilon \in \{0,1,2\}$ such that $\epsilon \equiv n \pmod 3$. Let $\big(T_2(\mathbb{Z}_n), B\big)$ be a P_3-decomposition of $KG_{n,2}$.

The subgraph of $KG_{n+1,2}$ induced by vertices in $T_2(\mathbb{Z}_{n+1})\backslash T_2(\mathbb{Z}_n)$ clearly has no edges, since they all share the element n. What remains to be shown is that the subgraph induced by the edges connecting vertices in $T_2(\mathbb{Z}_n)$ to vertices in $T_2(\mathbb{Z}_{n+1})\backslash T_2(\mathbb{Z}_n)$ has a P_3-decomposition.

Partition $\mathbb{Z}_n$ into $t = (n-\epsilon)/3$ sets: $S_i = \{3i, 3i+1, 3i+2\}$ for $i \in \mathbb{Z}_{t-1}$ and $S_{t-1} = \{i \mid n-3-\epsilon \leq i \leq n-1\}$. It is convenient to partition the old vertices, $T(\mathbb{Z}_n)$, into the following two types:

$$V_i = \big\{\{x,y\} \mid x,y \in S_i, x \neq y\big\} \quad \text{for } i \in \mathbb{Z}_t,$$
$$V_{i,j} = \big\{\{x,y\} \mid x \in S_i, y \in S_j\big\} \quad \text{for } 0 \leq i < j < t.$$

Further, partition the new vertices into t sets:

$$S'_i = \big\{\{x,n\} \mid x \in S_i\big\} \quad \text{for } i \in \mathbb{Z}_t.$$

All of the edges not involving vertices with elements in S_{t-1} are handled first. Of these edges, the edges that require special attention are those joining two vertices in $\big\{\{v,s\} \mid v \in V_i, s \in S'_i\big\}$ for some $i \in \mathbb{Z}_{t-1}$. For each $i \in \mathbb{Z}_{t-1}$, these edges induce a matching on six vertices, so they can't be decomposed into three paths in isolation. To decompose these edges, they are combined with edges joining two vertices in $\big\{\{v,s\} \mid v \in V_{0,i}, s \in S'_i\big\}$ for some $i \in \mathbb{Z}_{t-1}\backslash\{0\}$ to form 3-paths as described in the next two paragraphs, with $i = 1$ being an even more special case.

First, consider the bipartite subgraph G_0 of $KG_{n+1,2}$ induced by the edges joining the vertices in $V_0 \cup V_1 \cup V_{0,1}$ to the vertices in $S'_0 \cup S'_1$. Partition these edges as follows. The edges joining the vertices of $V_1 \cup \big\{\{0,x\} \mid x \in S_1\big\}$ to S'_1 induce a subgraph isomorphic to H_5, so by Lemma 1(v), there exists a P_3-decomposition of this subgraph. The edges joining the vertices of $V_0 \cup \big\{\{x,3\} \mid x \in S_0\big\}$ to S'_0 also form a subgraph isomorphic to H_5, so by Lemma 1(v), there exists a P_3-decomposition of this subgraph as well. Now, for each $k \in \{1,2\}$ consider the edges joining the vertices $\big\{\{k,x\} \mid x \in S_1\big\}$ to the vertices in S'_1. These edges induce a subgraph isomorphic to H_4, so by Lemma 1(iv), there exists a P_3-decomposition of this subgraph. Also, for each $k \in$

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$\{4,5\}$ the edges joining the vertices $\{\{x,k\} \mid x \in S_0\}$ to the vertices in S_0' induce a subgraph isomorphic to H_4, so by Lemma 1(iv), there exists a P_3-decomposition of this subgraph. The union of these sets of 3-paths produces a P_3-decomposition $(V(G_0), B_0')$ of most of G_0. The edges connecting V_1 to S_0' and connecting V_0 to S_1' occur in paths in $B_{1,0}'$ and $B_{0,1}'$, respectively as defined below.

Now, for each $i \in \{\mathbb{Z}_{t-1} \backslash \mathbb{Z}_2\}$, consider the bipartite subgraph G_i of $KG_{n+1,2}$ induced by the edges joining the vertices in $V_i \cup V_{0,i}$ to the vertices in $S_0' \cup S_i'$. The edges in G_i connecting the vertices of $V_i \cup \{\{0,x\} \mid x \in S_i\}$ to S_i' induce a subgraph isomorphic to H_5, thus it has a P_3-decomposition by Lemma 1(v). Now, for each $k \in \{1,2\}$, the edges joining the vertices in $\{\{k,x\} \mid x \in S_i\}$ to the vertices in S_i' induce a subgraph isomorphic to H_4, so there exists a P_3-decomposition of the subgraph by Lemma 1(iv). Further, for each $k \in S_i$, the edges connecting the vertices of $\{\{x,k\} \mid x \in S_0\}$ to the vertices of S_0' induce a subgraph isomorphic to H_4 which therefore has a decomposition by Lemma 1(iv). The union of these decompositions produce a P_3-decomposition, $(V(G_i), B_i')$, of most of G_i for each $i \in \{\mathbb{Z}_t \backslash \mathbb{Z}_2\}$. The remaining edges in G_i, namely those connecting V_i to S_0', are in paths in $B_{i,0}'$ as defined below.

For each $i \in \mathbb{Z}_{t-1}$ and for each $j \in \mathbb{Z}_{t-1} \backslash \{i\}$, the bipartite subgraph of G induced by the edges joining vertices in V_i to vertices in S_j' is isomorphic to $K_{3,3}$, so by Lemma 1(ii) there exists a P_3-decomposition $(V_i \cup S_j', B_{i,j}')$ of this subgraph.

For $1 < j < t-1$, the edges connecting vertices of $V_{0,1}$ to vertices of S_j' induce a $K_{9,3}$, and thus this graph has a P_3-decomposition $(V_{0,1} \cup S_j', B_{0,1,j})$ by Lemma 1(iii).

For each $i \in \mathbb{Z}_{t-1} \backslash \mathbb{Z}_2$ and for each $j \in \mathbb{Z}_{t-1} \backslash \{0,i\}$, the edges connecting vertices of $V_{0,i}$ with the vertices of S_j' induce a copy of $K_{9,3}$ so this graph has a P_3-decomposition, $(V_{0,i} \cup S_j', B_{0,i,j})$, by Lemma 1(iii). For $0 < i < j < t-1$ and $0 \le k < t-1$, consider the bipartite subgraph of G induced by edges joining vertices in $V_{i,j}$ to vertices in S_k'. This subgraph of G has a P_3-decomposition as follows:

(a) if $k \notin \{i,j\}$, then the subgraph is isomorphic to $K_{9,3}$, so it has a P_3-decomposition $(V_{i,j} \cup S_k', B_{i,j,k})$ by Lemma 1(iii);

(b) if $k = i$, then for each $y \in S_j$ the edges connecting the vertices $\{\{x,y\} \mid x \in S_i\}$ and $S_{k=i}'$ induce a subgraph isomorphic to H_4, which has a P_3-decomposition $(V_{i,j} \cup S_k', B_{i,j,k})$ by Lemma 1(iv);

(c) if $k = j$, then for each $x \in S_i$ the edges connecting the vertices $\{\{x,y\} \mid y \in S_j\}$ and $S_{k=j}'$ form a subgraph isomorphic to H_4, which has a P_3-decomposition $(V_{i,j} \cup S_k', B_{i,j,k})$ by Lemma 1(iv).

The only edges left to consider are all the edges which are incident with a vertex in $S_{t-1}' \cup V_{t-1} \cup V_{i,t-1}$, $i \in \mathbb{Z}_{t-1}$. The handling of thse edges depends on the value of ϵ.

First, consider the bipartite subgraph G_{t-1} of $KG_{n+1,2}$ induced by the edges joining the vertices in $V_{t-1} \cup V_{0,t-1}$ to the vertices in $S_0' \cup S_{t-1}'$. We consider each value of ϵ in turn.

For $\epsilon = 0$, the edges in G_{t-1} connecting the vertices of $V_{t-1} \cup \{\{0,x\} \mid x \in S_{t-1}\}$ to S_{t-1}' induce a subgraph isomorphic to H_5, thus it has a P_3-decomposition by Lemma 1(v). Now, for each $k \in \{1,2\}$, the edges joining the vertices in $\{\{k,x\} \mid x \in S_{t-1}\}$ to the vertices in S_{t-1}' induce a subgraph isomorphic to H_4, so there exists a

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P_3-decomposition by Lemma 1(iv). Further, for each $k \in S_{t-1}$, the edges connecting the vertices of $\big\{ \{x,k\} \mid x \in S_0 \big\}$ to the vertices of S_0' induce a subgraph isomorphic to H_4 which therefore has a decomposition by Lemma 1(iv). Lastly, the edges joining the vertices in v_{t-1} to the vertices in S_0' induce a subgraph isomorphic to $K_{3,3}$, and so has a P_3-decomposition be Lemma 1(ii). The union of these decompositions produce a P_3-decomposition, $\big(V(G_{t-1}), B_{t-1}' \big)$, of G_{t-1}.

For $\epsilon = 1$ or 2, the edges connecting vertices in V_{t-1} to S_{t-1}' induce a graph isomorphic to H_6 or H_8 respectively, which has a P_3-decomposition by Lemma 1(vi) ((viii) respectively). Regarding the edges connecting vertices in $V_{0,t-1}$ to S_{t-1}', for each $y \in S_{t-1}$ the edges connecting the vertices $\big\{ \{x,y\} \mid x \in S_0 \big\}$ and S_{t-1}' induce a subgraph isomorphic to $K_{3,3}$ if $\epsilon = 1$ and $K_{3,4}$ if $\epsilon = 2$, which has a P_3-decomposition by Lemma 1(iii) in both cases. For each $k \in S_{t-1}$ the edges connecting the vertices in $\big\{ \{x,k\} \mid x \in S_0 \big\}$ to the vertices in S_0' induce a subgraph isomorphic to H_4, so there exists a P_3-decomposition by Lemma 1(iv). Lastly, the edges joining the vertices in v_{t-1} to the vertices in S_0' induce a subgraph isomorphic to $K_{3+\epsilon,3}$, and so has a P_3-decomposition be Lemma 1(iii). The union of these decompositions produce a P_3-decomposition, $\big(V(G_{t-1}), B_{t-1}' \big)$, of G_{t-1}.

The rest of the edges are easier to decompose.

For each $i \in \mathbb{Z}_{t-1}$, the bipartite subgraph induced by the edges joining the vertices of V_i to the vertices of S_{t-1}' induce a graph isomorphic $K_{3,3+\epsilon}$, so it has a P_3-decomposition $(V_i \cup S_{t-1}', B_{i,t-1})$ by Lemma 1(iii).

For $0 \le i < j < t-1$, the bipartite subgraph induced by the edges joining the vertices of $V_{i,j}$ to the vertices of S_{t-1}' induce a graph isomorphic to $K_{9,3+\epsilon}$, so it has a P_3-decomposition $(V_{i,j} \cup S_{t-1}', B_{i,j,t-1})$ by Lemma 1(iii).

Finally, for $0 < i < t-1$ and $0 \le k < t$, consider the bipartite subgraph of G induced by edges joining vertices in $V_{i,t-1}$ to vertices in S_k'. This subgraph of G has a P_3-decomposition as follows.

(a) If $k \notin \{i, t-1\}$, then the subgraph is isomorphic to $K_{3(3+\epsilon),3}$, so it has a P_3-decomposition $(V_{i,t-1} \cup S_k', B_{i,t-1,k})$ by Lemma 1(iii).

(b) If $k = i$, then for each $y \in S_{t-1}$ the edges connecting the vertices $\{\{x,y\} \mid x \in S_i\}$ and $S_{k=i}'$ induce a subgraph isomorphic to H_4, which has a P_3-decomposition $(V_{i,t-1} \cup S_k', B_{i,t-1,k})$ by Lemma 1(iv).

(c) If $k = t-1$, then we have the following cases:
- Case $\epsilon = 0$: For each $x \in S_i$ the edges connecting the vertices $\{\{x,y\} \mid y \in S_j\}$ and $S_{k=j}'$ induce a subgraph isomorphic to H_4, which has a P_3-decomposition $(V_{i,j} \cup S_k', B_{i,j,k})$ by Lemma 1(iv).
- Case $\epsilon = 1$: For each $y \in S_{t-1}$ the edges connecting the vertices $\{\{x,y\} \mid x \in S_i\}$ and $S_{k=t-1}'$ induce a subgraph isomorphic to $K_{3,3}$, which has a P_3-decomposition $(V_{i,t-1} \cup S_k', B_{i,t-1,k})$ by Lemma 1(ii).
- Case $\epsilon = 2$: For each $y \in S_{t-1}$ the edges connecting the vertices $\{\{x,y\} \mid x \in S_i\}$ and $S_{k=t-1}'$ induce a subgraph isomorphic to $K_{4,3}$, which has a P_3-decomposition $(V_{i,t-1} \cup S_k', B_{i,t-1,k})$ by Lemma 1(iii).

This accounts for all new edges.

Let $B_1 = \bigcup_{i \in \mathbb{Z}_t} B'_i$, $B_2 = \bigcup_{0 \le i < j < t} B'_{i,j}$, and $B_3 = \bigcup_{0 \le i < j < t, k \in \mathbb{Z}_t} B'_{i,j,k}$. The required P_3-decomposition of G is given by $\big(V(G), B \cup B_1 \cup B_2 \cup B_3\big)$. ∎

4 A P_3-Decomposition of $GKG_{n,3,1}$

Before stating and proving the main result for $GKG_{n,3,1}$, a few definitions and a technical lemma are presented.

A digraph is an ordered quadruple $D = (V, E, t, h)$ where V is a set of vertices, E is a set of ordered pairs of vertices (each element of which is called an arc or directed edge), and $t, h: E \to V$ are functions defined by $t((u,v)) = u$ and $h((u,v)) = v$ for each arc $(u,v) \in E$ ($t(e)$ and $h(e)$ are called the tail and head of arc e, respectively). A complete digraph is a digraph in which $E = V \times V$.

A directed 2-factor of a digraph D is a spanning subdigraph F in which every vertex is the head of exactly one arc and the tail of exactly one arc of F.

Let $D = (V, E, t, h)$ be a digraph, let C be a set of colors, and for each $e \in E$, let $C_e \subseteq C$. A $(C_1, \dots, C_e)$-coloring of D is a function $c: E \to C$ such that if $e \in E$ then $c(e) \in C_e$ (this is known as a list arc-coloring). A list arc-coloring is said to be proper if no two adjacent arcs receive the same color. In the following lemma, the vertex set is $T_3(\mathbb{Z}_n)$, so we can refer to the intersection of two vertices (it is the intersection of two 3-element sets).

Lemma 5 *Let $D = \big(T_3(\mathbb{Z}_n), E, t, h\big)$ be a complete digraph. Let $C = \mathbb{Z}_n$ be a set of colors. For each $e \in E$, let $C_e = t(e) \cap h(e)$ (so possibly $C_e = \varnothing$). There exists a proper list arc-colored directed 2-factor of D.*

Proof Let D, C, and C_e be defined as stated in the lemma. Form a directed 2-factor, F, of D as follows.

First, form a partition, P, of the vertex set $T_3(\mathbb{Z}_n)$ so that two vertices $\{a, b, c\}$ and $\{x, y, z\}$ are in the same element of P if and only if $\{x, y, z\} = \{a + i, b + i, c + i\}$ for some $i \in \mathbb{Z}_n$ with the sums reduced modulo n. If n is not a multiple of three, then P contains $l = \frac{(n-1)(n-2)}{6}$ sets, each containing n elements. If n is a multiple of three, say $n = 3k$, then P contains $l = (\binom{3k}{3} - k)/3k = 3\binom{k}{2}$ sets of size n and one set of size k. In either case, let the elements of P of size n be $\{E_0, E_1, \dots, E_{l-1}\}$, and if n is a multiple of three then let $E_l = \big\{\{i, i+k, i+2k\} \mid i \in \mathbb{Z}_k\big\}$ be the single set of size k.

For $0 \le i < l$, among the vertices in E_i, let $e_i = \{0, a_i, b_i\}$ with $a_i < b_i$ be one that contains both zero and as small a nonzero element of $\mathbb{Z}_n$ as possible (two such vertices might exist, in which case either can be e_i). Let $d_i = \gcd(a_i, n)$. For $0 \le j < d_i$, and for $0 \le r < \frac{n}{d_i}$, define the arc $e_{i,r,j} = (\{ra_i + j, (r+1)a_i + j, ra_i + b_i + j\}, \{(r+1)a_i + j, (r+2)a_i + j, (r+1)a_i + b_i + j\})$ in D. Then for each $j \in \mathbb{Z}_{d_i}$, the subgraph $S_{i,j}$ of D induced by $\{e_{i,r,j} \mid 0 \le r < \frac{n}{d_i}\}$ is a directed cycle. Note that the both the tail and head of each arc $e_{i,r,j}$ contains the element $(r+1)a_i + j$; so let $c(e_{r,j}) = (r+1)a_i + j$. Clearly this coloring is proper since consecutive arc colors differ by a_i (mod n), where clearly $a_i < n$. If n is not a multiple of three, then $F = \bigcup_{i \in \mathbb{Z}_l, j \in \mathbb{Z}_{d_i}} S_{i,j}$ is a directed 2-factor that is properly list arc-colored as required.

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If n is a multiple of three, then F is a properly list arc-colored directed 2-factor that includes all of the vertices in D except for those in E_I. We now insert the k vertices in E_I into an already created directed cycle in F and then give a proper list arc-coloring to the modified cycle. Recall that $E_I = \{i, k+i, 2k+i\} \mid 0 \leq i < k\}$. Consider the colored directed cycle, C', in F containing the vertex $\{0,1,1+k\}$. Then $C' = (v'_0, v'_1, \dots, v'_{n-1})$ where for each $j \in \mathbb{Z}_n$, $v'_j = \{0+j, 1+j, 1+k+j\}$ and where the arc (v'_j, v'_{j+1}) is colored $j+1$. For $0 \leq j \leq k$, replace the arc $(\{0+j, 1+j, 1+k+j\}, \{1+j, 2+j, 2+k+j\})$ colored $i+j$ in F with the arcs $(\{0+j, 1+j, 1+k+j\}, \{1+j, 1+k+j, 1+2k+j\})$ colored $1+k+j$ and $(\{1+j, 1+k+j, 1+2k+j\}, \{1+j, 2+j, 2+k+j\})$ colored $1+j$. The resulting cycle is still properly list edge-colored since the only potential conflict is at the vertex $\{0,1,1+k\}$ which previously was incident with arcs colored 0 and 1 and now is incident with arcs colored 0 and $1+k$. ■

We are now ready to prove our second main result.

Theorem 2 *$G = GKG_{n,3,1}$ has a P_3-decomposition for all $n > 0$.*

Proof For $n \in \{1,2,3,4\}$, G has no edges, so the result is vacuously true. For $n = 5$, G has a P_3-decomposition by Lemma 4(i). For $n = 6$, G has a P_3-decomposition by Lemma 4(ii).

The remaining cases are proved by induction on n. So now assume that $GKG_{w,3,1}$ is P_3-decomposable for all $w \leq n$ for some $n \geq 6$. It is shown that $H = GKG_{n+2,3,1}$ is P_3-decomposable. Let $\big(T_3(\mathbb{Z}_n), B_0\big)$ be a P_3-decomposition of $G = GKG_{n,3,1}$. Partition the vertices of H as follows:

(i) $V_0 = T_3(\mathbb{Z}_n)$ (the vertices of G),
(ii) $V_1 = \big\{\{a, b, n\} \mid a, b \in \mathbb{Z}_n\big\}$,
(iii) $V_2 = \big\{\{a, b, n+1\} \mid a, b \in \mathbb{Z}_n\big\}$, and
(iv) $V_3 = \big\{\{a, n, n+1\} \mid a \in \mathbb{Z}_n\big\}$.

Consider the following subgraphs of H:

(i) H_0 is the subgraph induced by the vertices of V_0;
(ii) H_1 is the subgraph induced by the vertices of $V_1 \cup V_3$;
(iii) H_2 is the subgraph induced by the vertices of $V_2 \cup V_3$;
(iv) H_3 is the bipartite subgraph induced by the edges $\big\{\{x, y\} \mid x \in V_0, y \in V_1 \cup V_2\big\}$;
(v) H_4 is the bipartite subgraph induced by the edges $\big\{\{x, y\} \mid x \in V_1, y \in V_2\big\}$;
(vi) H_5 is the bipartite subgraph induced by the edges $\big\{\{x, y\} \mid x \in V_0, y \in V_3\big\}$.

These six subgraphs clearly partition the edges of H, so combining P_3-decompositions of each will create a P_3-decomposition of H itself.

Since $H_0 = G$, it has a decomposition $\big(T_3(\mathbb{Z}_n), B_0\big)$ by assumption.

Next, notice that in H_1 and H_2, all vertices share the element $x = n$ or $n+1$ respectively; so any two vertices, say $\{a, b, x\}$ and $\{c, d, x\}$, are adjacent if and only if $\{a, b\} \cap \{c, d\} = \varnothing$. So H_1 is clearly isomorphic to $KG_{n+1,2}$ with vertex set $\big\{v \backslash \{n\} \mid v \in V(H_1)\big\}$ and H_2 is isomorphic to $KG_{n+1,2}$ with vertex set $\big\{v \backslash \{n+1\} \mid v \in V(H_2)\big\}$. Therefore, H_1 and H_2 have P_3-decompositions $\big(V(H_1), B_1\big)$ and $\big(V(H_2), B_2\big)$ respectively by Theorem 1.

Next, consider the bipartite subgraph H_3. If $v \in V_0$, then $d_{H_3}(v) = 6\binom{n-3}{2}$, and if $v \in V_1 \cup V_2$, then $d_{H_3}(v) = 2\binom{n-2}{2}$, both of which are even. Also, $|E(H_3)| = 6\binom{n}{2}\binom{n-3}{2}$ which is clearly a multiple of three. Finally, to show H_3 is connected, for each $\{s,t,u\} \in V_0$ we display a path to each vertex in $V_1 \cup V_2$ as follows (where a, b, s, t, and u are distinct elements of $\mathbb{Z}_n$ and $x \in \{n, n+1\}$):

$$(\{s,t,u\},\{a,s,x\},\{b,s,u\},\{a,b,x\}),$$
$$(\{s,t,u\},\{a,s,x\},\{a,b,t\},\{s,t,x\}), \text{ and } (\{s,t,u\},\{a,s,x\}).$$

These account for all pairs of nonadjacent vertices in H_3, so H_3 is easily seen to be connected. Therefore, H_3 has a P_3-decomposition $\big(V(H_3), B_3\big)$ by Lemma 3. We also use Lemma 3 to find a P_3-decomposition of H_4 as the following shows. H_4 is a $2\binom{n-2}{2}$-regular bipartite graph, so all vertices have even degree. Also, $|E(H_4)| = 2\binom{n}{2}\binom{n-2}{2}$ which is a multiple of three. To see this, note $|E(H_4)|$ is the product of four consecutive integers (one of which must be a multiple of three) divided by two. Finally, to show that H_4 is connected, for each vertex $\{a,b,n\} \in V_1$ we display a path to each vertex in V_2 as follows (where a, b, s, and t are distinct elements of $\mathbb{Z}_n$):

$$(\{a,b,n\},\{a,t,n+1\},\{a,s,n\},\{a,b,n+1\}),$$
$$(\{a,b,n\},\{b,s,n+1\},\{a,s,n\},\{s,t,n+1\}), \text{ and } (\{a,b,n\},\{a,c,n+1\}).$$

These account for all pairs of nonadjacent vertices in H_4, so H_4 is easily seen to be connected. Therefore, H_4 has a P_3-decomposition $\big(V(H_4), B_4\big)$ by Lemma 3. Finally, consider H_5. Using Lemma 5, let F be a properly list arc-colored 2-factor of the complete digraph with vertex set V_0, with the set of colors $C = \mathbb{Z}_n$, and with lists of colors $(C_0, C_1, \dots, C_{|E|-1})$ defined by $C_e = t(e) \cap h(e)$ for each $e \in E$. Assume F has components $\{f_0, f_1, \dots, f_{m-1}\}$. For each $i \in \mathbb{Z}_m$, consider the directed cycle f_i of length l with $E(f_i) = \{e_0, e_1, \dots, e_{l-1}\}$ where $h(e_k) = t(e_{k+1})$ for $k \in \mathbb{Z}_l$ with additions done modulo l. Form the following 3-paths in H_5:

$$T_i = \Big\{ \big(t(e_j), \{c(e_j), n, n+1\}, h(e_j), \big\{ h(e_j) \backslash \{c(e_j), c(e_{j+1})\}, n, n+1 \big\} \big) \mid j \in \mathbb{Z}_l \Big\}$$

with subscript additions done modulo l. The edges in T_i exist in H_5 since C_e is a list of the shared elements of $t(e)$ and $h(e)$. H_5 has a P_3-decomposition $\big(V(H_5), B_5\big)$ where $B_5 = \bigcup_{i \in \mathbb{Z}_m} T_i$. To see that each edge in H_5 is in exactly one path in B_5, consider the edge $e = (\{a,b,c\},\{a,n,n+1\})$ in H_5. The vertex $\{a,b,c\}$ is in exactly one component, f_i, of F. Consider the two arcs, e_1 and e_2 in f_i such that $h(e_1) = \{a,b,c\}$ and $t(e_2) = \{a,b,c\}$. There are three possibilities.

(i) If $c(e_1) = a$, then e is in $\big(t(e_1), \{a,n,n+1\}, \{a,b,c\}, \big\{ \{b,c\} \backslash c(e_2), n, n+1 \big\} \big)$.
(ii) If $c(e_2) = a$, then let e_3 be the arc in f_i with $t(e_3) = h(e_2)$. Then e is in $\big(\{a,b,c\}, \{a,n,n+1\}, h(e_2), \big\{ h(e_2) \backslash \{a, c(e_3)\}, n, n+1 \big\} \big)$.
(iii) If $a \notin \{c(e_1), c(e_2)\}$, the e is in $\big(t(e_1), \{c(e_1), n, n+1\}, \{a,b,c\}, \{a,n,n+1\} \big)$.

Since F is a properly list arc-colored 2-factor, exactly one of the previous three cases holds. Thus every edge of H_5 is in exactly one path in B_5.

Let $B = \bigcup_{i \in \mathbb{Z}_6} B_i$. Then $\big(V(H), B\big)$ is the desired P_3-decomposition. ■

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