ON THE LAWS OF THE VARIETY $s\mathfrak{A}_e$

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A group is called an $s\mathfrak{A}$-group if it is locally finite and all its Sylow subgroups are abelian. Kovács [4] has shown that, for any positive integer e, the class $s\mathfrak{A}_e$ of all $s\mathfrak{A}$-groups of exponent dividing e is a (locally finite) variety. The proof of this relies on the fact that, for any e, there are only finitely many (isomorphism classes of) non-abelian finite simple groups in $s\mathfrak{A}_e$; and this is a consequence of deep results of Walter and others (see [6]). In [2], Christensen raised the finite basis question for the laws of the varieties $s\mathfrak{A}_e$. It is easy to establish the finite basis property for an $s\mathfrak{A}_e$ which contains no non-abelian finite simple group; and Christensen gave a finite basis for the laws of the variety $s\mathfrak{A}_{30}$, whose only non-abelian finite simple group is $PSL(2,5)$. Here we prove

THEOREM. *For any positive integer e, the variety $s\mathfrak{A}_e$ has a finite basis for its laws.*

The notation and terminology used below is that of Neumann [5].

For a fixed positive integer e, we write $\mathfrak{B} = s\mathfrak{A}_e$. Then, as usual, $\mathfrak{B}^{(n)}$ will denote the variety defined by the n-variable laws of $\mathfrak{B}$. If $G \in \mathfrak{B}^{(2)}$ then clearly G has exponent dividing e, and the Sylow subgroups of G are abelian. Since $\mathfrak{B}$ is locally finite, $\mathfrak{B}^{(n)}$ is finitely based for all n (see 51.54 of [5]), and to prove the theorem it suffices to show that, for some n, $\mathfrak{B}^{(n)}$ is locally finite.

It follows from VI.14.16 of Huppert [3] that there is a bound, d say, on the derived length of the finite soluble groups in $\mathfrak{B}$. Thus, since $\mathfrak{B}$ is locally finite, every soluble group of $\mathfrak{B}$ has derived length at most d. Let $\mathfrak{A}_e$ be the variety of all abelian groups of exponent dividing e and let $\mathfrak{S}$ be the product variety $\mathfrak{S} = \mathfrak{A}_e \cdots \mathfrak{A}_e$ of d copies of $\mathfrak{A}_e$. Then $\mathfrak{B} \wedge \mathfrak{S}$ is the variety of all soluble groups in $\mathfrak{B}$.

Suppose that G is a finite group in $\mathfrak{B}$. Then, by Theorem I of [6], G has a normal series

$$(1) \qquad \{1\} \leqq N \leqq H \leqq G$$

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in which G/H and N are soluble and H/N is a direct product of non-abelian finite simple groups. Let $\mathfrak{W}$ be the subvariety of $\mathfrak{V}$ generated by the class W of finite groups G of $\mathfrak{V}$ having a normal series of type (1) in which $H > N = \{1\}$. Then all finite groups of $\mathfrak{V}$ are in $\mathfrak{S}\mathfrak{W}$, and so, since $\mathfrak{V}$ is locally finite, $\mathfrak{V} \subseteq \mathfrak{S}\mathfrak{W}$. For each positive integer m, $\mathfrak{W}^{(m)}$ is finitely based. Thus $\mathfrak{S}\mathfrak{W}^{(m)}$ is finitely based, by d applications of 34.24 of [5]; and so, for some n, $\mathfrak{V}^{(n)} \subseteq \mathfrak{S}\mathfrak{W}^{(m)}$. If $\mathfrak{W}^{(m)}$ is locally finite then so also is $\mathfrak{S}\mathfrak{W}^{(m)}$ (since $\mathfrak{S}$ is locally finite), and consequently $\mathfrak{V}^{(n)}$ is locally finite. Thus to prove the theorem it suffices to show that, for some n, $\mathfrak{W}^{(n)}$ is locally finite.

As remarked above, $\mathfrak{V}$ contains only finitely many distinct isomorphism classes of non-abelian finite simple groups: let $\{G_1, \cdots, G_k\}$ be a set of representatives of these. We may assume that $k \geqq 1$, for otherwise $\mathfrak{W}$ is trivial and the result is clear. For $i = 1, \cdots, k$, let W_i denote the class of those finite groups G of $\mathfrak{V}$ having a normal series of type (1) in which $N = \{1\}$ and H is a non-trivial direct power of G_i. If G is in W and $H \leqq G$ is as before, then $H = H_1 \times \cdots \times H_k$ where, for $i = 1, \cdots, k$, H_i is a direct power of G_i and is normal in G. Thus G is a subdirect product of groups in $W_1 \cup \cdots \cup W_k$, and so

$$\mathfrak{W} = \mathfrak{W}_1 \vee \cdots \vee \mathfrak{W}_k,$$

where $\mathfrak{W}_i = \operatorname{var} W_i$, $i = 1, \cdots, k$.

If $\mathfrak{X}$ and $\mathfrak{Y}$ are varieties such that $\mathfrak{X}^{(l)}$ and $\mathfrak{Y}^{(m)}$ are locally finite, then also $\mathfrak{X}^{(l)}$ and $\mathfrak{Y}^{(m)}$ are finitely based. By Lemma 1.3 of [1], $(\mathfrak{X}^{(l)} \vee \mathfrak{Y}^{(m)})^{(n)}$ is locally finite for some n, and so $(\mathfrak{X} \vee \mathfrak{Y})^{(n)}$ is also locally finite. In the present case it follows that $\mathfrak{W}^{(n)}$ is locally finite for some n if, for each i, $\mathfrak{W}_i^{(n(i))}$ is locally finite for some $n(i)$. Thus to establish the theorem we prove, for example, that $\mathfrak{W}_1^{(n)}$ is locally finite for some n.

The variety $\mathfrak{S}$ is finitely based, by 34.24 of [5]. Suppose that it is defined by the law $s = s(x_1, \cdots, x_l)$. Let $e = ab$ where a is a product of the primes dividing $m = |G_1|$ and $(a, b) = 1$. Suppose that $G \in W_1$ and $H \leqq G$ is as before. If $g \in G$ has order dividing a, then, since H is normal in G, $\operatorname{gp}\{H, g\}$ has exponent dividing a. It follows that the word

$$(sx_{l+1}^b)^a \tag{2}$$

is a law of every element of W_1, and hence of $\mathfrak{W}_1$. Let

$$s_i = s(x_{(i-1)l+1}, \cdots, x_{il}), \qquad i = 1, \cdots, m+1,$$

$$w_{i,j} = y_{i,j}^{-b} \; s_i s_j^{-1} \; y_{i,j}^b, \qquad 1 \leqq i < j \leqq m+1, \qquad \text{and}$$

$$w = [x^b, w_{1,2}, w_{1,3}, \cdots, w_{m,m+1}], \tag{3}$$

where x and the $y_{i,j}$ are distinct variables not occurring in the words $s_1, \cdots, s_{m+1}$. The word w is a modification of the chief centralizer law, 52.31 of [5].

LEMMA 1. *The word w is a law of* $\mathfrak{W}_1$.

PROOF. Suppose that $G \in W_1$. Then G has a normal subgroup H such that $G/H \in \mathfrak{S}$ and

$$H = H_1 \times \cdots \times H_r$$

with $H_t \cong G_1$, $t = 1, \cdots, r$. First we prove that each H_t is normalized by every element g of G of order dividing a; and to show this we may take g of p-power order, where p is a prime dividing a. Let S be a Sylow p-subgroup of $\mathrm{gp}\{H, g\}$ which contains a Sylow p-subgroup of H_t, and let $g' \in S$. Then, since $G \in \mathfrak{V}$, S is abelian and

$$H_t \cap H_t^{g'} \neq \{1\}.$$

Since $H_t^{g'} \lhd H$ it follows that g' normalizes H_t. Thus S normalizes H_t, and so does $g \in HS$.

For $t = 1, \cdots, r$, let π_t be the projection of H onto H_t. Also let F be the free group on the variables of w and $\phi: F \to G$ a homomorphism. We have to show that $w\phi = 1$. The $s_i\phi$ lie in H and the $s_i\phi\pi_t$ in H_t. Since $|H_t| = m$, for each t there exist i and j $(1 \leqq i < j \leqq m+1)$ such that $(s_i s_j^{-1})\phi\pi_t = 1$. However the elements $x^b\phi$, $y_{1,2}^b\phi, \cdots, y_{m,m+1}^b\phi$ have order dividing a, and so normalize each H_t. It follows easily that $w\phi\pi_t = 1$ for all t. Thus $w\phi = 1$ and the lemma is proved.

Since the words of (2) and (3) are laws of $\mathfrak{W}_1$, we can choose an integer n so that they are laws of $\mathfrak{W}_1^{(n)}$. The theorem now follows from

LEMMA 2. $\mathfrak{W}_1^{(n)}$ *is locally finite.*

PROOF. Let G be a finitely generated group in $\mathfrak{W}_1^{(n)}$. We have to show that G is finite. Let H be the verbal subgroup of G corresponding to $\mathfrak{S}$. Then, since $\mathfrak{S}$ is locally finite, H has finite index in G and is finitely generated. It remains to show that H is finite.

H is generated by the values of s in G. These have order dividing a and so are bth powers. Since G has the law (2), a well-known argument—see the last proof on p. 92 of [5]—shows that H has exponent dividing a. Also, any nilpotent factor of H is abelian, since $H \in \mathfrak{W}_1^{(n)}$ and $n \geqq 2$. Thus by 52.22 of [5] it is sufficient to show that there is a bound on the orders of the chief factors of H.

Let K/L be a chief factor of H, and C the centralizer of K/L in H. By 52.54 of [5], it is sufficient to bound the order of H/C. We show that H/C is an m-generator group: then, since $n \geqq m$, $H/C \in \mathfrak{W}_1$, and our result follows since $\mathfrak{W}_1$ is locally finite.

Let F be the free group on the variables of w. Then H is generated by the elements of the form $s\phi$, where $\phi: F \to G$ is a homomorphism. Thus H/C is generated by the elements $s\phi C$ and it will suffice to show that there are at most m

distinct elements of this form. Thus it suffices to show that, for any homomorphism $\phi: F \to G$, at least one of $(s_1 s_2^{-1})\phi, \cdots, (s_m s_{m+1}^{-1})\phi$ lies in C. Assume this is false for a given $\phi: F \to G$. Then, by the argument of the proof of the second part of 52.32 of [5], there exist elements $h \in K \setminus L$ and $h_{1,2}, \cdots, h_{m,m+1}$ of H such that

$$[h, (s_1 s_2^{-1})\phi^{h_{1,2}}, \cdots, (s_m s_{m+1}^{-1})\phi^{h_{m,m+1}}] \neq 1.$$

But since H has exponent prime to b, this is a value in G of the word w, contradicting the fact that w is a law of G.

This completes the proof of Lemma 2, and so of the theorem.

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