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INVARIANT SUBGROUPS IN RINGS WITH INVOLUTION

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Let R be a ring with involution$*$. In this paper, we study additive subgroups A of R which are invariant under all mappings of the form $\phi_x : a \rightarrow xax^*$. That is, $xAx^* \subseteq A$, for all $x \in R$. Obvious examples of such subgroups A are ideals of R, the set of symmetric elements, and the set of skew-symmetric elements. We will prove that when R is $*$-prime, these examples are essentially the only ones.

This result gives some insight into a recent theorem of I. N. Herstein [**4**]. He proved that if A is a subring of a semi-prime ring R, such that $xAx^* \subseteq A$ for all $x \in R$, then either A contains a non-zero ideal of R or A is contained in the center of R. In the $*$-prime case, his result is a consequence of ours. Thus it would seem that the additive structure of A plays a more important role than its multiplicative structure in considerations of this type.

In work to appear, done independently, W. E. Baxter [**2**] has also generalized Herstein's result by assuming that A is merely a Jordan subring, rather than a subring. Specifically, he has proved that if R is a semi-prime ring which is 2-torsion free, and A is a Jordan subring of R such that $xAx^* \subseteq A$ for all $x \in R$, then either A contains a non-zero ideal of R, or A consists of symmetric elements and contains all the symmetric elements in some non-zero ideal of R. Baxter's result, in the $*$-prime case, is also a consequence of ours; moreover, we prove an analogous result for the case when $2R = 0$.

We note that the $xAx^* \subseteq A$ condition has appeared in several other places. P. H. Lee used the notion of a symmetric subring in [**6**] to study the subring generated by the symmetric elements: a symmetric subring A of R satisfies $xAx^* \subseteq A$, but in addition is generated by symmetric elements and contains all norms and traces. Also K. McCrimmon in [**8**] defines the kernel $K(B)$ of an ideal B of R with $B^* \subseteq B$:

$$K(B) = \left\{ b + b^* + \sum_i b_i b_i^* + \sum_j b_j s_j b_j^* | b, b_i, b_j \in B, \text{any } s_j = s_j^* \right\}.$$

$K(B)$ is a Jordan ideal of the symmetric elements of R and satisfies $xK(B)x^* \subseteq K(B)$, for every $x \in R$. Finally, our condition appears in the definition of a unitary ring (due to A. Bak). A unitary ring is a triple (R, α, A), where R is a ring with $*$, α is an element of the center Z of R, and A is an additive subgroup of R satisfying several conditions (which will be given in detail later), one of

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which being that $xAx^* \subseteq A$, for all $x \in R$. The notion of a unitary ring was used by H. Bass [**1**] in his exposition of unitary algebraic K-theory. In this paper, we will show that when R is $*$-simple, any additive subgroup A of R satisfying $xAx^* \subseteq A$, for all $x \in R$, is either an ideal of R or determines a scalar $\alpha \in Z$ such that (R, α, A) is a unitary ring.

We will say that an ideal I of R is a *$*$-ideal* if $I^* \subseteq I$. R is *$*$-simple* if it contains no proper $*$-ideals, and R is *$*$-prime* if the product of any two non-zero $*$-ideals is non-zero. We let $S = \{x \in R | x^* = x\}$ denote the symmetric elements, $K = \{x \in R | x^* = -x\}$ denote the skew symmetric elements, and $T = \{x + x^* | x \in R\}$ denote the traces in R. We let R' denote the ring obtained by adjoining a formal unit element to R.

When R is prime, the central closure and extended centroid of R were defined by Martindale in [**7**]; here we shall need the extension of these ideas to $*$-prime rings which was done by Rowen in [**9**]. The difference is that for $*$-prime rings, the ring of quotients Q is obtained from right R-module homomorphisms of $*$-ideals into R, rather than ideals as in the prime case. Let C be the center of Q. We say that C is the *extended centroid* of R. This terminology differs from that of Rowen; for, since $*$ induces an automorphism of C, one can consider the fixed ring $\hat{C}$ of $*$ on C, and it is $\hat{C}$ which Rowen calls the extended centroid. However, in this work we shall need all of C. As in the prime case, RC is called the *central closure* of R in Q. We shall need the following result of Rowen [**9**, Theorem 1]: $\hat{C}$ is a field and RC is $*$-prime.

Throughout, A will denote a non-zero additive subgroup of R satisfying $xAx^* \subseteq A$, for all $x \in R$. An immediate consequence of this, obtained by linearizing on x, is that $xay^* + yax^* \in A$, for all $a \in A$ and $x, y \in R$.

The first lemma does not require that R be $*$-prime.

LEMMA 1. *Let R be semi-prime, and assume that A does not contain a non-zero ideal of R. Then*

1) $axb^* = a^*xb$, *all* $a, b \in A$, *all* $x \in R'$;
2) *for some* $a \in A$, $a^*a \neq 0$; *and*
3) *for some* $a \in A$, $a^2 \neq 0$.

Proof. Say that $axb^* \neq a^*xb$, for some a, x, b, and let $I = R(axb^* - a^*xb)R$, a non-zero ideal of R. We claim that $I \subseteq A$. For, say $y, z \in R$. Then

$$\begin{aligned} y(axb^* - a^*xb)z &= yaxb^*z + z^*bx^*ay - z^*bx^*ay^* - ya^*xbz \\ &= ya(xb^*z) + (xb^*z)^*ay^* - ((ya^*x)bz + z^*b(ya^*x)^*) \in A \end{aligned}$$

Thus $I \subseteq A$ and 1) is proved.

For 2), assume on the contrary that $a^*a = 0$, for all $a \in A$. Then, for $x, y \in R$, we have

$$\begin{aligned} xa^2y &= xa^2y + y^*a^*ax^* \\ &= xa(ay) + (ay)^*ax^* \in A. \end{aligned}$$

Thus $I = Ra^2R \subseteq A$, a contradiction unless $a^2 = 0$, all $a \in A$. Thus 2) will follow from 3).

To prove 3), assume that $a^2 = 0$, for all $a \in A$. Then, by linearizing on a, we get $ab + ba = 0$, for $a, b \in A$. Choose $x \in R$ and let $b = xax^* \in A$. Thus $axax^* + xax^*a = 0$. Multiplying by a, $axax^*a = 0$, all $x \in R$. Linearizing on x, then multiplying on the right by xa, we obtain $axayaxa = 0$, all $x, y \in R$. Since R is semi-prime, this yields $a = 0$. That is, $A = (0)$, a contradiction. Thus 3) is proved.

LEMMA 2. *Let R be a $*$-prime ring, with $b, c \in R$ such that $c + c^* \neq 0$ and $bxc^* = b^*xc$, all $x \in R$. Then for some $\lambda \in C$, the extended centroid of R, $b = \lambda(b + b^*)$.*

Proof. Consider the *-ideal $I = R'(b + b^*)R'$ of R, and define $\lambda : I \rightarrow R$ as follows:

$$\lambda\left(\sum_i x_i(b + b^*)y_i\right) = \sum_i x_i b y_i, \quad \text{for } x_i, y_i \in R'.$$

We claim that λ is well-defined. For, say that $\sum_i x_i(b + b^*)y_i = 0$. Choose any $r \in R$, and multiply on the right by rc. Thus, using the fact that $b^*(y_i r)c = b(y_i r)c^*$,

$$\begin{aligned} 0 &= \left(\sum_i x_i b y_i\right) rc + \left(\sum_i x_i b^* y_i\right) rc \\ &= \sum_i x_i b y_i rc + \sum_i x_i b y_i rc^* \\ &= \left(\sum_i x_i b y_i\right) r(c + c^*), \quad \text{for all } r \in R. \end{aligned}$$

Since R is $*$-prime and $c + c^* \neq 0$ is symmetric, it follows that $\sum_i x_i b y_i = 0$; that is, λ is well-defined.

Now $\lambda \in C$ since λ is a bimodule homomorphism, and clearly $\lambda(b + b^*) = b$.

LEMMA 3. *Let R be $*$-prime, and assume that A does not contain a non-zero ideal of R. Then if $c + c^* \neq 0$, for some $c \in A$, there exists an invertible element $\lambda \in C$ such that $b = \lambda(b + b^*)$, for all $b \in A$.*

Proof. By Lemma 1, $axb^* = a^*xb$ for all $a, b \in A$, and thus we may apply Lemma 2 to see that for each $b \in A$, there exists $\lambda_b \in C$ such that $b = \lambda_b(b + b^*)$. We claim that λ_b is independent of b.

From Lemma 1, choose $a \in A$ such that $a^*a \neq 0$. We will show that $\lambda_b = \lambda_a$, for all $b \in A$. Now

$$\lambda_b(b + b^*)xa^* = bxa^* = b^*xa = b^*x\lambda_a(a + a^*),$$

and thus

$$\lambda_b(bxa^* + b^*xa^*) = \lambda_a(b^*xa + b^*xa^*).$$

Using $b^*xa = bxa^*$, this becomes $(\lambda_b - \lambda_a)(bxa^* + b^*xa^*) = 0$, or $(b + b^*)x(\lambda_b - \lambda_a)a^* = 0$, for all $x \in R$ (and so for all $x \in RC$). Now if $b \neq 0$, $b + b^* \neq 0$ also since $b = \lambda_b(b + b^*)$, and thus since RC is $*$-prime and $b + b^*$ is symmetric, $(\lambda_b - \lambda_a)a^* = 0$. It then follows that $(\lambda_b - \lambda_a)RCa^*a = 0$, and so since $a^*a \neq 0$ we have $\lambda_b - \lambda_a = 0$. That is, $\lambda_b = \lambda_a = \lambda$.

We now claim that λ is invertible. First, $\lambda\lambda^* \neq 0$. For if $\lambda\lambda^* = 0$, then $\lambda^*a = \lambda^*\lambda(a + a^*) = 0$. But then $a^*a = \lambda^*(a^* + a)a = (a^* + a)\lambda^*a = 0$, a contradiction. But $\lambda\lambda^* \in \hat{C}$, which is a field, and so $\lambda\lambda^*$ is invertible. Thus λ is also invertible.

Now let R be any $*$-prime ring, I a subset of R, and λ an element of C, the extended centroid of R. The following definition is inspired by [**1**, p. 75].

Definition. $S^\lambda(I) = \{x \in I | x = \lambda x^*\}$ and $S_\lambda(I) = \{x + \lambda x^* | x \in I\}$.

We can now prove our main theorem.

THEOREM. *Let R be a $*$-prime ring, and A a non-zero additive subgroup of R such that $xAx^* \subseteq A$, for all $x \in R$. Assume that A does not contain a non-zero ideal of R, and let $I = RAR$. Then $I \cap I^* \neq (0)$ and there exists an invertible $\lambda \in C$, the extended centroid of R, such that*

$$S^\lambda(R) \supseteq A \supseteq S_\lambda(I).$$

In particular, A satisfies one of the following;

i) $K \supset A \supset \{x - x^* | x \in I\}$

ii) $S \supset A \supset \{x + x^* | x \in I\} = T_I$

iii) $S \supset \beta A \supset T_{\beta I}$, *for some $\beta \in C$ with $\beta^* \neq \pm \beta$.*

Proof. If $c + c^* = 0$ for all $c \in A$, then clearly $A \subseteq S^\lambda$, where $\lambda = -1$. Thus, assume $c + c^* \neq 0$, for $c \in A$. By Lemma 3, there exists $\alpha \in C$ such that $b = \alpha(b + b^*)$, for all $b \in A$. Thus $b = \alpha(1 - \alpha)^{-1}b^* = (\alpha^{-1} - 1)^{-1}b^*$, and so $A \subseteq S^\lambda$ by letting $\lambda = (\alpha^{-1} - 1)^{-1}$.

The ideal I is spanned by elements of the form $x = yaz$, where $a \in A$. Now $x + \lambda x^* = yaz + \lambda(z^*a^*y^*) = yaz + z^*ay^* \in A$. Thus $x + \lambda x^* \in A$, for all $x \in I$, and $S_\lambda(I) \subseteq A$.

Cases i) and ii) are simply the cases when $\lambda = -1$ or $\lambda = +1$. Case iii) follows by using $\beta = \alpha^{-1}$, where $b = \alpha(b + b^*)$ for all $b \in A$ as above. $\beta^* \neq \pm\beta$, for if $\beta^* = \beta$ we have $\lambda = 1$, and if $\beta^* = -\beta$ we have $\lambda = -1$.

It remains to check that $I \cap I^* \neq (0)$. Let B be the additive subgroup generated by all xax^*, $x \in R$, $a \in A$. Then B satisfies the same hypotheses as A, and since $B \subseteq A$, B does not contain an ideal of R. Thus by Lemma 1, there exists $b \in B$ with $b^*b \neq 0$. Clearly $b \in I$; thus $I^*I \neq (0)$, and so $I \cap I^* \neq (0)$.

When R is simple, the theorem takes a particularly nice form, and so we state it separately. Recall that for a simple ring, the centroid and the extended centroid coincide.

Corollary 1. *Let R be a simple ring with $*$ of characteristic not 2, and let A be a non-zero additive subgroup of R such that $xAx^* \subseteq A$. Then $A = R$, $A = S$, $A = K$, or $A = \alpha S$, for some α in the centroid of R.*

Proof. Since $\frac{1}{2} \in C$, the centroid of R, we have $K = \{x - x^* | x \in R\}$ and $S = T$. Since R is simple, $I = R = \beta I$. Thus, from the Theorem, case i) gives $A = K$, case ii) gives $A = S$, and case iii) gives $S = \beta A$. Let $\alpha = \beta^{-1}$, and the corollary is proved.

The next corollary concerns unitary rings, which were mentioned in the introduction. Using $I = R$ and $\lambda = -\alpha$, we have that $S^{-\alpha}(R) = \{x \in R | x = -\alpha x^*\}$ and $S_{-\alpha}(R) = \{x - \alpha x^* | x \in R\}$. We can now give the definition of a unitary ring [**1**, p. 75]:

Definition. A *unitary ring* is a triple (R, α, A), where R is a ring with involution, $\alpha \in Z$, the center of R, such that $\alpha\alpha^* = 1$, and A is an additive subgroup of R satisfying $S^{-\alpha}(R) \supseteq A \supseteq S_{-\alpha}(R)$ and $xAx^* \subseteq A$, for all $x \in R$.

We note that when R is $*$-simple with a non-trivial center Z, then $Z = C$, the (extended) centroid. The next corollary shows that for $*$-simple rings, much of the definition of a unitary ring is superfluous.

Corollary 2. *Let R be a $*$-simple ring with 1, and let A be an additive subgroup of R such that $xAx^* \subseteq A$, for all $x \in R$. Then either A is an ideal of R, or there exists $\alpha \in Z$ such that (R, α, A) is a unitary ring.*

Proof. First assume that A contains an ideal of R. If R is simple, then $A = R$ and we are done. Otherwise, $R = Q \oplus Q^o$, where Q is a simple ring, Q^o the opposite ring, and $*$ interchanges the two components. In this case, Q and Q^o are the only proper ideals. Say that $A \supseteq Q$ but $A \neq Q$. Since $(1, 0) \in R$, it is easy to see that $A \cap Q^o \neq (0)$. Now, using $x = (x_1, x_2) \in R$, $xAx^* \subseteq A$ implies that $x_2(A \cap Q^o)x_1 \subseteq A \cap Q^o$. That is, $A \cap Q^o$ is an ideal of Q^o, so must equal Q^o. Then $A = Q \oplus Q^o = R$, and we are done.

We may therefore assume that A does not contain a non-zero ideal. By the theorem, there exists $\lambda \in Z$ such that $S^{\lambda}(R) \supseteq A \supseteq S_{\lambda}(I)$. Since $I \cap I^* \neq (0)$ and R is $*$-simple, $I \cap I^* = R = I$. Now let $\alpha = -\lambda$. Since $x = \lambda x^* = \lambda(\lambda^* x)$ for all $x \in A$, and since λ is invertible, $\lambda\lambda^* = 1$. It follows that $\alpha\alpha^* = 1$, and the result is proved.

We now turn to the situation when A is also a Jordan subring of R. That is, in addition to being an additive subgroup, A is closed under the two (quadratic) operations a^2 and aba, for $a, b \in A$. We actually do not need the full assumption that A is a Jordan subring, but only that A is closed under squares. For any subset $U \subseteq R$, we let $N_U = \{xx^* | x \in U\}$, the *norms* of U. As in the statement of the theorem, T_U denotes the *traces* of U. When $2R \neq (0)$, as was mentioned at the beginning of the paper, the following result is due to Baxter [**2**].

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COROLLARY 3. *Let R be $*$-prime, and let A be an additive subgroup of R such that $a^2 \in A$, for all $a \in A$, and such that $xAx^* \subseteq A$, for all $x \in R$. Then either A contains a non-zero ideal of R, or for some non-zero $*$-ideal I,*

$$S \supset A \supset N_I \cup T_I.$$

In particular, when $2R \neq (0)$, A contains the symmetric elements in some non-zero $*$-ideal.

Proof. If A does not contain a non-zero ideal of R, then by our main theorem $A \subset S^\lambda(R)$, for some invertible $\lambda \in C$. Choose any $a \in A$. Then $a = \lambda a^*$, and since $a^2 \in A$, $a^2 = \lambda(a^*)^2$. Substituting, $\lambda^2(a^*)^2 = \lambda(a^*)^2$. Now by Lemma 1, part 3), $a^2 \neq 0$ for some $a \in A$. Thus $(a^*)^2 \neq 0$, and since λ is invertible it follows that $\lambda = 1$. This gives us case ii); $S \supset A \supset T_J$, where $J = RAR$.

Now if $2R \neq (0)$, then R is 2-torsion free since R is $*$-prime. Thus $2J \neq (0)$. Clearly $2J$ is a $*$-ideal and it is trivial that $T_J \supset S \cap (2J) \supset N_{2J} \cup T_{2J}$.

We may therefore assume that $2R = (0)$. We first claim that $N_{\bar{A}} \subseteq A$, where $\bar{A}$ denotes the subring generated by A. For, say that a and b are monomials in $\bar{A}$, and write $a = a_1a_2 \cdots a_n$, $b = b_1b_2 \cdots b_m$, where $a_i, b_i \in A$. Then $aa^* = (a_1a_2 \cdots a_{n-1})a_n^2(a_1a_2 \cdots a_{n-1})^* \in A$ since $a_n^2 \in A$, and $ab^* + ba^* = (a_1a_2 \cdots a_{n-1})a_nb^* + ba_n(a_1a_2 \cdots a_{n-1})^* \in A$. Thus if $x \in \bar{A}$, write $x = \sum_i b_i$, where each b_i is a monomial in $\bar{A}$. Then $xx^* = \sum_i b_ib_i^* + \sum_{i<j} b_ib_j^* + b_jb_i^* \in A$. Thus $N_{\bar{A}} \subseteq A$.

If T_J is not commutative, then the subring generated by T_J will contain the non-zero ideal $I = R[\bar{T}_J, \bar{T}_J]R$ by [**5**, Theorem 2.1.2, p. 57]. I is clearly a $*$-ideal and $I \subset J$. Since $I \subseteq \bar{T}_J \subseteq \bar{A}$, $N_I \subseteq A$ by the previous paragraph. Also $T_I \subseteq T_J \subseteq A$; thus $A \supseteq N_I \cup T_I$ and we would be done. We may therefore assume that T_J is commutative. In this situation, we will show that $A \cap Z \neq (0)$, where Z is the center of R.

If R is not prime, then for some non-zero prime ideal P of R, $P \cap P^* = (0)$. Now $J \not\subseteq P$, since $J^* \subseteq J$, and thus in $\bar{R} = R/P$, the image $\bar{J}$ of J is non-zero. Also $\bar{P}^* = P^* + P/P \neq (0)$; thus $\bar{J} \cap \bar{P}^* \neq (0)$ since $\bar{R}$ is prime. Choose $x \in J \cap P^*$. Then $x^* \in P$, and so $x \equiv x + x^* \pmod{P}$. But $x + x^* \in T_J$, and so $\bar{J} \cap \bar{P}^* \subseteq \bar{T}_J$, which is commutative. A prime ring containing a non-zero commutative ideal is commutative; thus $\bar{R}$ is commutative. Similarly R/P^* is commutative, and it follows that R is commutative. Certainly in this case $A \cap Z \neq (0)$.

We may now assume that R is prime. Now since T_J is commutative, by Amitsur's Theorem [**5**, Theorem 5.1.2] J satisfies the standard identity of degree 4. Moreover, R must satisfy the same identity. Thus $Z \neq 0$, and the localization of R at Z is a four-dimensional simple algebra over its center F, which is the quotient field of Z [**5**, Theorems 1.3.4 and 1.4.3].

If $*$ is of the second kind on Z, choose $\alpha \in Z$ with $\alpha \neq \alpha^*$. Now for any $s \in S \cap J$, $\alpha s + (\alpha s)^* = (\alpha + \alpha^*)s \in T_J \subset A$. Thus all symmetric elements in the ideal $(\alpha + \alpha^*) J$ commute. It follows by [**5**, Theorem 2.1.5] that either

$(\alpha + \alpha^*)S \cap J \subset Z$ or that R contains a commutative ideal, in which case R is commutative. In either case, $A \cap Z \neq (0)$.

We may therefore assume that * fixes Z, and thus also F. Let B be the localization of A at Z; $B \subset Q$. If $B \cap F \neq (0)$, then $A \cap Z \neq (0)$; moreover B satisfies $xBx^* \subseteq B$. Now let $\bar{F}$ denote the algebraic closure of F, and consider $B \otimes_F \bar{F} \subseteq Q \otimes_F \bar{F}$. Using the linearizations of $a^2 \in A$ and $xAx^* \subseteq A$, we see that $B \otimes_F \bar{F} = B_1$ also satisfies $xB_1x^* \subseteq B_1$ for all $x \in Q \otimes_F \bar{F} = Q_1$ and that B_1 is closed under squares. In addition, since the center of Q_1 is $Z(Q_1) = F \otimes_F \bar{F}$, $B_1 \cap Z(Q_1) \neq (0)$ implies $B \cap F \neq (0)$, and we would be done. We may therefore work in Q_1. Since $\bar{F}$ is algebraically closed, $Q_1 \cong M_2(\bar{F})$, the 2×2 matrices over $\bar{F}$. Since $*$ is of the first kind, $*$ must either be the symplectic involution or the ordinary transpose. If $*$ is symplectic, $T_{Q_1} \subseteq Z(Q_1)$, and if $*$ is transpose and $x \in T_{Q_1}$, then $x^2 \in Z(Q_1)$. Since $B_1 \supset T_{Q_1}$, in either case $B_1 \cap Z(Q_1) \neq (0)$. We have proved that whenever T_J is commutative, $A \cap Z \neq (0)$.

The result will now follow. For, choose $a \in A \cap Z$, $a \neq 0$. Since $A \subseteq S$, $a \in S \cap Z$ and so $a^2 \neq 0$. Let $I = Ra$. Then for any $x \in I$, write $x = ra$. Then $xx^* = ra(ra)^* = ra^2r^* \in A$. Thus, $N_I \subseteq A$. To finish the theorem, use the ideal $I_1 = I \cap J$. I_1 is a $*$-ideal, as both I and J are, and $I_1 \neq (0)$ since R is $*$-prime. Then $N_{I_1} \cup T_{I_1} \subseteq A$.

We note that when $2R = (0)$, one could not hope to prove that A contained all symmetric elements in some ideal. For, if $R = M_2(F)$, where F is a field of characteristic 2 and R has the symplectic involution, let $A = F \cdot 1$, the scalars. Then $N \cup T \subseteq A$, $xAx^* \subseteq A$, but $S \not\subseteq A$.

As our final corollary, we obtain (in the $*$-prime case) the theorem of Herstein [**4**] discussed earlier.

COROLLARY 4. *Let R be a $*$-prime ring, and let A be a subring of R such that $xAx^* \subseteq A$, for all $x \in R$. Then either A contains a non-zero ideal of R, or $A \subseteq Z$, the center of R.*

Proof. By Corollary 3, if A does not contain a non-zero ideal of R, then $S \supset A \supset T_I$. Since A is a subring, A must then be commutative (for $ab = (ab)^* = b^*a^* = ba$, for all $a, b \in A$). Consequently T_I is commutative. If $2R \neq (0)$, then S_{2I} commutes, so by [**5**, Theorem 2.1.5], either $S_{2I} \subset Z$ or R contains a commutative $*$-ideal. It follows that either $S \subset Z$ or R is commutative; in either case $A \subset Z$. Also if R is not prime, the same argument as in Corollary 3 shows that R is commutative, so $A \subset Z$. We may therefore assume that R is prime and $2R = (0)$.

As in Corollary 3, since T_I is commutative, we must have $Z \neq (0)$ and the localization Q of R at Z is a simple 4-dimensional algebra. If $*$ is of the second kind, it again follows that either $S \subset Z$ or R is commutative (using Theorem 2.1.5 of [**5**]) and so again $A \subset Z$. Continuing the argument in Corollary 3, we may reduce to the case of a simple ring Q, four-dimensional over its center F, an algebraically closed field. That is, $Q \cong M_2(F)$. We now finish by using Herstein's own argument in this case: by a direct matrix computation, he proves that $*$ can not be the transpose, and that in the symplectic case the commutativity of A forces $A \subset Z$ [**5**, p. 226]. The result is proved.

We give an example to show that case iii) of the theorem can actually occur, and with a $\beta \in C$ such that $\beta R \nsubseteq R$. Let R be the set of countable by countable matrices, with entries in the complex numbers **C**, of the following form:

$$\begin{bmatrix} \boxed{M} & & & & 0 \\ & \alpha & & & \\ & & \alpha & & \\ & & & \alpha & \\ 0 & & & & \ddots \end{bmatrix},$$

where M is an $n \times n$ matrix with entries in **C**, for any n, and $\alpha \in \mathbf{R}$, the reals. Then R is a prime ring, and has an involution $*$ given by the usual Hermitian adjoint (transpose conjugate). Note the the extended centroid C of R is isomorphic to **C**. Let I be the ideal of all finite rank matrices in R. Now choose $\beta \in C$ such that $\bar{\beta} \neq \pm\beta$ (for example, $\beta = 1 + i$), and let $A = \beta^{-1}(S \cap I)$. Then $xAx^* \subseteq A$, $A \cap S = (0) = A \cap K$, and $\beta A = S \cap I$. However, $\beta R \nsubseteq R$.

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