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On Polynomial Invariants of Exceptional Simple Algebraic Groups

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Abstract. We study polynomial invariants of systems of vectors with respect to exceptional simple algebraic groups in their minimal linear representations. For each type we prove that the algebra of invariants is integral over the subalgebra of trace polynomials for a suitable algebraic system (*cf.* [27], [28], [13]).

1 Introduction

Throughout the paper the ground field F will be assumed to be algebraically closed of characteristic zero.

Let V be a finite dimensional module over an algebraic group G (we denote it by $G : V$); then G acts on the algebra $F[V]$ of polynomial functions from V to F as follows: if $\phi \in G$ and $f = f(y) \in F[V]$, then $(f \cdot \phi)(v) = f\big(\phi(v)\big)$. The subalgebra of fixed functions $F[V]^G$ is called the algebra of polynomial invariants; if V is the direct sum of several copies of a G-module W with the diagonal action of G (denote $V = \underbrace{W \oplus \cdots \oplus W}_{k} = W^k$), then $F[V]^G$ is called the *algebra of invariants of the system of k vectors of* $G : W$.

Although invariants of systems of vectors for classical linear groups are known for a long time [29], the case of exceptional simple algebraic groups has not been studied well enough: the generators (First Main Theorem) and relations between them (Second Main Theorem) are known only for the minimal (simple) module over a simple algebraic group of type G_2 [27] (for a conceptual explanation and a new proof of the result in [27] see [9]).

It is well-known that simple modules of minimal dimension for any type of exceptional simple algebraic group can be associated with a certain algebra or, more precisely, an (algebraic) *system*, that is, a vector space A endowed with a finite number of *multilinear* operations. In particular, such a group turns out to be either the automorphism group of the associated system or at least its identity component.

If A is any finite dimensional system, let x_i be the projection of A^k onto the i-th summand; in particular, $x_1, \ldots, x_k$ lie in the system (over the ring $F[A^k]$, hence over F by restriction of scalars) $\mathrm{Pol}(A^k, A) \simeq F[A^k] \otimes_F A$ of polynomial mappings from A^k to A (with pointwise operations). The system of generic elements $\mathcal{F}_k(A)$ of rank k is defined as the subsystem over F generated by $x_1, \ldots, x_k$. In particular, if A is a simple algebra generated $\leq k$ elements, then $\mathcal{F}_k(A)$ is *a free affine algebra of type A and rank k [18].*

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Now, let $g(y_1,\dots,y_r)$ be one of the multilinear operations of A and let B be a subsystem of A. If we replace $r-1$ variables in g with some elements in B, then we get a linear operator in $\operatorname{End}_F A$ called a *multiplication operator*. For example, if A is an algebra and g is the multiplication, then in this way we get two operators $L_b\colon y \to by$ and $R_b\colon y \to yb$. Denote by $M^A(B)$ the subalgebra of $\operatorname{End}_F A$ generated by all such operators. In particular, $M(A) = M^A(A)$ is called *the multiplication algebra*. A subsystem $B \subseteq A$ is said to be *ad-nilpotent*, if $M^A(B)$ is nilpotent.

Also, for $g(y_1,\dots,y_r)$ as above and any $i \in \{1,\dots,r\}$ we get a multilinear mapping from A^{r-1} to $\operatorname{End}_F A$:

$$E = E(y_1,\dots,y_{i-1},y_{i+1},\dots,y_r)\colon y_i \to g(y_1,\dots,y_r).$$

The composition of E with $u_1,\dots,u_{i-1},u_{i+1},\dots,u_r \in \mathcal{F}_k(A)$ gives a polynomial mapping

$$E_1(x_1,\dots,x_k) = E(u_1,\dots,u_{i-1},u_{i+1},\dots,u_r)$$

from A^k to $\operatorname{End}_F A$. Denote by $\operatorname{Mult}_k(A)$ the subalgebra of $\operatorname{Pol}(A^k,\operatorname{End}_F A) \simeq F[A^k] \otimes_F \operatorname{End}_F A$ generated over F by all such mappings. Obviously, any values of $y_1,\dots,y_k$ in A define uniquely a homomorphism from $\operatorname{Mult}_k(A)$ to $M(A)$. Moreover, if the values belong to a subsystem B, then the image of $\operatorname{Mult}_k(A)$ is contained in $M^A(B)$.

Next, for every $E \in \operatorname{Mult}_k(A)$ the mapping $\operatorname{Tr}_A(E)$, where $\operatorname{Tr}_A(x)$ is the trace of a linear operator $x \in \operatorname{End}_F A$, belongs to $F[A^k]$. Moreover, it is easy to check that $\operatorname{Tr}_A(E)$ is an invariant of the automorphism group G of A; the subalgebra $\operatorname{Tr}[A^k]$ generated by 1 and all such elements is called *the algebra of trace polynomials of A of rank k* (*cf.* [13], [20]).

Proposition 1 *Let A be a finite dimensional system, then* $\operatorname{Tr}[A^k]$ *is a finitely generated algebra over F.*

Proof Observe that $P = \operatorname{Tr}[A^k]$, $B = \mathcal{F}_k(A)$ and $\operatorname{Mult}_k(A)$ ($\subseteq \operatorname{Pol}(A^k,\operatorname{End}_F A)$) are homogeneous. Denote by P_+ the sum of the homogeneous components of P of positive degree, then P_+ is an ideal of P spanned by all $\operatorname{Tr}_A(E)$, where E ranges over the sum C of homogeneous components of $\operatorname{Mult}_k(A)$ of positive degree.

Next, in $\operatorname{Pol}(A^k, A) \simeq F[A^k] \otimes_F A$ we consider the F-subsystem $\tilde{B} = PB = B + P_+B$ (PB is the P-submodule of $\operatorname{Pol}(A^k,A)$ generated by B) and its ideal $\tilde{B}_+ = P_+B$. Similarly, in $\operatorname{Pol}(A^k,\operatorname{End}_F A)$ we consider the subalgebra $\tilde{C} = C + P_+C$ and its ideal $\tilde{C}_+ = P_+C$.

Since the number of operations of the system A is finite, there is $r \in \mathbf{N}$ such that every operation on A depends on $\leq r$ variables. By induction on $m \in \mathbf{N}$ it is easy to prove that every homogeneous component B_s of B of degree $s \geq r^m$ is contained in the subspace spanned by elements $E \cdot b$ where $b \in B$ and E ranges over the m-th power of C:

$$\sum_{s \geq r^m} B_s \subseteq C^m \cdot B.$$

Indeed, if $m = 1$, then it is evident; next, B_s for $s \geq r^{m+1}$ is spanned by $b = g(b_1,\dots,b_l)$, where $g = g(y_1,\dots,y_l)$ is an operation of A and $b_1,\dots,b_l$ are homogeneous elements in B. We need to show that $b \in C^{m+1} \cdot B$; we may assume that $l \geq 2$. By definition, $l \leq r$,

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hence, for some $i \in \{1, \dots, l\}$ there is b_i with degree being not less than r^m; by induction, it belongs to $C^m \cdot B$, hence, $b \in C^{m+1} \cdot B$.

Denote $n = \dim_F A$; by Cayley-Hamilton's theorem (*e.g.*, [21, p. 18]), every element in C is integral over P_+ of degree $\leq n$; obviously, $\tilde{C}$ has the same property, hence $\tilde{C}/\tilde{C}_+$ is a nil algebra of index n, and by Nagata-Higman's theorem [21, p. 149], it is nilpotent. Let m be the nilpotency index, then $\tilde{C}^m \subseteq \tilde{C}_+$ and, therefore, every homogeneous component B_s of degree $s \geq r^m$ is contained in the P-module Q generated by B_t, $t < r^m$:

$$\tilde{B} = P\Big(\sum_{t<r^m} B_t\Big), \tag{1}$$

so that

$$\tilde{C} = P\Big(\sum_{t<(r-1)r^m} C_t\Big) = \sum_{t<(r-1)r^m} C_t + P_+\tilde{C}.$$

Taking traces, we prove that P is generated by the homogeneous elements in P of degree $\leq (r-1)r^m$. Since these form a finite dimensional F-subspace of P, we conclude that P is finitely generated. ■

Therefore, if G is the group of automorphisms of a system A, then $\mathrm{Tr}[A^k]$ is a homogeneous affine subalgebra of $F[A^k]^G$.

Coming back to the simple algebraic group of type G_2, this group is represented as the automorphism group $G = \mathrm{Aut}(\mathbf{O})$ of the Cayley-Dickson algebra $\mathbf{O}$ (*e.g.*, [23]). Every element $x \in \mathbf{O}$ satisfies the equation

$$x^2 - \mathrm{tr}(x)x + n(x) = 0,$$

where $\mathrm{tr}(x)$ is the trace of a (which is one fourth of the trace of the operator of left or right multiplication by a). As a G-module, $\mathbf{O}$ is the direct sum of the trivial one $F \cdot 1$ and the simple module $\mathbf{O}' = \{a \in \mathbf{O} \mid \mathrm{tr}(a) = 0\}$ ($\dim_F \mathbf{O}' = 7$). The generators of the algebra of invariants $F[\mathbf{O}^k]^G$ ($F[\mathbf{O}'^k]^G$) are trace polynomials (*cf.* [27]).

On the other hand, let $\mathbf{O}_3 = \mathrm{Mat}_3(\mathbf{O}) = \mathbf{O} \otimes_F \mathrm{Mat}_3(F)$ be the algebra of 3×3-matrices over $\mathbf{O}$. Define an involution $a \otimes b \mapsto \bar{a} \otimes b^\top$, where $b \mapsto b^\top$ is the matrix transposition. The subspace $H(\mathbf{O}_3)$ of symmetric elements with the symmetrized multiplication $a \circ b = 1/2(ab+ba)$ is a central simple Jordan algebra $\mathbf{A}$ of dimension 27, which is called the *Albert algebra* [15].

Every $x \in \mathbf{A}$ satisfies a cubic equation:

$$x^3 - \mathrm{tr}(x)x^2 + S(x)a - N(x) = 0, \tag{2}$$

where $\mathrm{tr}(x), S(x), N(x) \in F$ and $\mathrm{tr}(x)$ is one ninth of the trace of the multiplication by the element x. The group $G = \mathrm{Aut}(\mathbf{A})$ is simple of type F_4 (*e.g.* [15]) and $\mathbf{A}$ is the direct sum of the trivial G-submodule $F \cdot 1$ and the simple one, $\mathbf{A}' = \{a \in \mathbf{A} \mid \mathrm{tr}(a) = 0\}$, $\dim_F \mathbf{A}' = 26$.

The group $\mathcal{M}$ of linear automorphisms of $\mathbf{A}$ which preserve the norm $N(x)$ is a simple group of type E_6 and $\mathbf{A}$ is a simple $\mathcal{M}$-module of minimal dimension [14]. Also, $\mathcal{M}$ has an outer automorphism μ of second order that defines the contragredient representation of $\mathcal{M}$ in $\mathrm{End}_F \mathbf{A}$; denote this module by $\mathbf{A}^*$.

In the next section we will consider an algebra $\mathcal{A}$ with involution where the identity component of $\mathrm{Aut}(\mathcal{A})$ is $\mathcal{M}$; as an $\mathcal{M}$-module, $\mathcal{A}$ is the direct sum of $\mathbf{A}$, $\mathbf{A}^*$ and two copies of the one-dimensional trivial $\mathcal{M}$-module; $\mathcal{A}$ is called the *Chevalley algebra* (*cf.* [26]).

In the final section we will consider the Freudenthal triple system $\mathcal{T}$; as a vector space it coincides with $\mathcal{A}$ and it is endowed with two 3-linear operations; the automorphism group of $\mathcal{T}$ is a simple algebraic group of type E_7 which acts irreducibly on $\mathcal{T}$ [7], [5].

If $\mathcal{L}$ is a simple Lie algebra of type E_8, then $G = \mathrm{Aut}(\mathcal{L})$ is a simple group of type E_8 [16, p. 281] and $\mathcal{L}$ is a G-module of minimal dimension.

One of the main results of this work is about the null-cone of $G : V$ for a finite dimensional module V over an algebraic group G; let us recall the definition. Let P be a subalgebra of $F[V]^G$; for a point $y \in V$ denote by $\pi_P(y)$ the *fibre* of P at point y, this is the level set of the subalgebra P at the point y:

$$\pi_P(y) := \{y' \in V \mid f(y) = f(y')\, \forall f \in P\}.$$

If $P = F[V]^G$, then $\pi_P(0)$ is called the *null-cone* of $G : V$ [19, p. 196].

Theorem 1 *Let A be the system associated with a minimal irreducible representation of an exceptional simple algebraic group, that is, $A \in \{\mathbf{O}, \mathbf{A}, \mathcal{A}, \mathcal{T}, \mathcal{L}\}$, and let $G = \mathrm{Aut}(A)$.*

A point $x = (a_1, \dots, a_k) \in A^k$ belongs to the null-cone $G : A^k$ iff the subsystem A_x generated by $a_1, \dots, a_k$ in A is ad*-nilpotent.*

The next proposition, which goes back to Hilbert [8] enables us to rewrite Theorem 1 in terms of trace polynomials.

Proposition 2 *Let V be a finite dimensional module over a linearly reductive algebraic group G and let P be a homogeneous affine subalgebra of $F[V]^G$. Then $F[V]^G$ is a finitely generated P-module if and only if*

$$\pi_P(0) = \pi_{F[V]^G}(0). \tag{3}$$

Now, pick $x = (a_1, \dots, a_k) \in \pi_{\mathrm{Tr}[A^k]}(0)$, then the subsystem A_x generated by $a_1, \dots, a_k$ is ad-nilpotent. Indeed, the condition $x = (a_1, \dots, a_k) \in \pi_{\mathrm{Tr}[A^k]}(0)$ implies that the trace of any element $E(a_1, \dots, a_k)$, where $E(x_1, \dots, x_k) \in \mathrm{Mult}_k(A)$, is zero, and this forces that any element in $M^A(A_x)$ is nilpotent, hence, this finite dimensional associative algebra is nilpotent, therefore, A_x is ad-nilpotent, and conversely. So, we get the following statement which is equivalent to Theorem 1.

Corollary 1 *Let A be the system associated with the minimal irreducible representation of an exceptional simple algebraic group, let $G = \mathrm{Aut}(A)$. Then $F[A^k]^G$ is a finitely generated module over* $\mathrm{Tr}[A^k]$.

In fact, this is a new result only for the Chevalley algebra and the Freudenthal triple system; indeed, for $A = \mathbf{O}$ it follows from [27], [28], the case $A = \mathbf{A}$ was considered in [13] and for the adjoint representation of a linearly reductive algebraic group it was done in [22]. We will prove the similar statement for a wide class of simple algebras which includes $\mathcal{A}$ (Theorem 2) and for $\mathcal{T}$ (Theorem 3).

It was conjectured in [20, Conjecture 1] that $F[A^k]^{\mathrm{Aut}\,A}$ is integral over $\mathrm{Tr}[A^k]$ for any finite dimensional simple algebra (that is, a system with just one bilinear multiplication). Thus, Corollary 1 (as well as Corollary 2 in the next section) gives a proof of (an obvious generalization of) this conjecture for some systems. The idea of considering the null-cone and applying Hilbert's theorem to prove this conjecture was formulated too in [20, Conjecture 1*].

With the same notations as in the previous corollary, let G° be the identity component of G, then G° is a simple algebraic group; since G/G° is finite, $F[A^k]^{G^\circ}$ is integral over $F[A^k]^G$, hence, since $F[A^k]^{G^\circ}$ is an affine algebra (*e.g.* [19, Section 3.4]), it is a finitely generated module over $\mathrm{Tr}[A^k]$.

Now, if we replace the generators $x_1, \dots, x_k$ in trace polynomials with the projectors $x_1', \dots, x_k'$ onto the corresponding simple G°-submodule $V \subseteq A$, we will get a subalgebra $\mathrm{Tr}[V^k]$ of $F[V^k]^{G^\circ}$ and the latter one is a finitely generated module over it. Thus, Corollary 1 provides an effective way to construct a "significant" part of polynomial invariants of exceptional simple algebraic groups. The problem whether there are invariants of a different type is unsolved so far (except for the cases mentioned above).

Next, if $f(x_1, \dots, x_k) \in \mathcal{F}_k(A)$, then for all $u_1, \dots, u_k \in \mathcal{F}_k(A)$ we have $f(u_1, \dots, u_k) \in \mathcal{F}_k(A)$. In other words, such a replacement of variables defines an algebra endomorphism of $\mathcal{F}_k(A)$ and, conversely, every endomorphism has such a form (the $\mathcal{F}_k(A)$ is a free algebra of rank k in the variety of algebras generated by A).

Notice that an endomorphism of $\mathcal{F}_k(A)$ induces an endomorphism ϕ of $F[A^k]$ (the replacement of the variables $x_1, \dots, x_k$ with elements in $\mathcal{F}_k(A)$). Moreover, the subalgebra of invariants $F[A^k]^G$ is stable under ϕ, that is, $F[A^k]^G$ is a *fully characteristic* subalgebra of $F[A^k]$. Therefore, it makes sense to describe generators of $F[A^k]^G$ as a fully characteristic algebra (in classical terms we are looking for *typical* generators [29]).

For example, by [27], [28], the algebra $F[\mathbf{O}^k]^G$ is generated $\mathrm{tr}(x_1)$. In the case of the Albert algebra, the fully characteristic subalgebra P generated by $\mathrm{tr}(x_1)$ satisfies (3), therefore, the algebra $F[\mathbf{A}^k]^G$ is a finitely generated module over P. If A is a simple Lie algebra, then the same holds for the fully characteristic subalgebra of $F[A^k]^G$ generated by $\mathrm{tr}(R_{x_1}^j)$, $j \in \{2, \dots, \dim_F(A)\}$.

2 Structurable Algebras and Invariants of E_6

First of all, let us recall the definition of structurable algebra [1].

Let A be an algebra over F with an involution $x \mapsto \bar{x}$. Then it is decomposed into the direct sum of the subspaces of symmetric and skew-symmetric elements, that is, $A = \mathcal{H}(A) \dotplus \mathcal{S}(A)$. Then A is said to be a *structurable algebra* if it satisfies the following identities ($\forall x, y \in A, \forall a, b, c \in \mathcal{H}(A), \forall s \in \mathcal{S}(A)$):

$$(s, x, y) = -(x, s, y); \tag{4}$$

$$(a, b, c) - (b, a, c) = (c, a, b) - (c, b, a); \tag{5}$$

$$2/3\big[[a^2, a], b\big] = (b, a^2, a) - (b, a, a^2), \tag{6}$$

where $[a, b] = ab - ba$ and $(a, b, c) = (ab)c - a(bc)$ are the commutator and associator respectively.

If A is unital (one can join 1 in the external way, putting $\bar{1} = 1$), then (4)–(6) are equivalent to the following identity

$$[T_z, V_{x,y}] = V_{T_z x,y} - V_{x,T_{\bar{z}} y},$$

where $V_{x,y} \in \operatorname{End}_F A$, $V_{x,y}z = (x\bar{y})z + (z\bar{y})x - (z\bar{x})y$ and $T_z = V_{z,1}$.

Note that the class of structurable algebras includes Jordan algebras where the involution is the identity mapping (in particular, the Albert algebra) and alternative algebras with an involution (for example, **O**).

Also, since the involution is regarded as an additional operation on A, subalgebras and ideals are (by definition) closed by the involution, automorphisms and derivations commute with the involution and so on. For example, the algebra of generic elements $\mathcal{F}_k(A)$ is generated as an algebra over F by $x_1, \ldots, x_k, \bar{x}_1, \ldots, \bar{x}_k$.

Now we will give an important example of structurable algebra. Let $N(x, y, z)$ be the complete linearization of the norm $N(x)$ in the Albert algebra $\mathbf{A}$ (see (2)), where $N(x, x, x) = 6N(x)$; since $\operatorname{tr}(xy)$ is a Killing form of $\mathbf{A}$, we can define a quadratic mapping from $\mathbf{A}$ to $\mathbf{A}^*$, $x \mapsto x^\#$, where $\operatorname{tr}(x^\# y) = 1/2N(x, x, y)$ (here $x^\# y$ is the product of elements in the Albert algebra); also, put $a \times b = (a + b)^\# - a^\# - b^\#$. Consider the direct sum of $\mathbf{A}$, $\mathbf{A}^*$ and two copies of F written in the matrix form:

$$\mathcal{A} = \begin{pmatrix} F & \mathbf{A} \\ \mathbf{A}^* & F \end{pmatrix}.$$

Next, define the multiplication on $\mathcal{A}$:

$$\begin{pmatrix} \alpha_1 & a_1 \\ b_1 & \beta_1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_2 & a_2 \\ b_2 & \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\alpha_2 + (a_1, b_2) & \alpha_1 a_2 + \beta_2 a_1 + b_1 \times b_2 \\ \alpha_2 b_1 + \beta_1 b_2 + a_1 \times a_2 & (b_1, a_2) + \beta_1\beta_2 \end{pmatrix}$$

and an involution

$$\overline{\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}} = \begin{pmatrix} \beta & a \\ b & \alpha \end{pmatrix}.$$

This algebra $\mathcal{A}$ is a structurable one (the algebra of an admissible triple [1]), it is called the *Chevalley algebra* (*cf.* [26]). From the definition of $\mathcal{A}$ it follows easily that the automorphism group of $\mathcal{A}$ consists of two components: $\mathcal{M}$ and $\tau\mathcal{M}$, where τ is an automorphism of period 2 which swops the lines and columns and for all $a \in \mathbf{A}$ and $g \in \mathcal{M}$ we have $\tau\big(g(a)\big) = \mu(g)\big(\tau(a)\big)$, where μ is an outer automorphism of $\mathcal{M}$ (see the Introduction).

Let A be a unital structurable algebra; denote by $\operatorname{Inder}(A)$ the algebra of inner derivations of A (spanned by derivations $D_{a,b}$, see [1]), then $\operatorname{Instrl}(A) = T_A + \operatorname{Inder}(A)$, where $T_A = \{T_a \mid a \in A\}$, is a subalgebra of the Lie algebra $\operatorname{End}_F(A)^{(-)}$. Also, on the vector space

$$\mathcal{K}(A) = \mathcal{S}(A) \dotplus A \dotplus \operatorname{Instrl}(A) \dotplus A^\sim \dotplus \mathcal{S}(A)^\sim,$$

where $A^\sim \simeq A$, $\mathcal{S}(A)^\sim \simeq \mathcal{S}(A)$, one can define the structure of a $\mathbf{Z}$-graded Lie algebra of length 5 [1] (the Kantor-Koecher-Tits construction for structurable algebras). In particular, the component $\mathcal{K}_1(A)$ ($\mathcal{K}_{-1}(A)$) is equal to A ($A^\sim$); we will also make use of the following properties of the multiplication $[a, b]$ on $\mathcal{K}(A)$:

(a) $\mathcal{K}(A)$ is generated by $A \cup A^{\sim}$;
(b) for all $a \in A$ and $b \in A^{\sim}$

$$[a, b] = V_{a,b} \in \operatorname{Instrl}(A) = \mathcal{K}_0(A).$$

Let ϕ be an automorphism of the graded Lie algebra $\mathcal{K}(A)$, then its restriction to $\mathcal{K}_1(A) = A$ is an autotopy of the structurable algebra A [2]. It means that there exists $\hat{\phi} \in \operatorname{End}_F(A)$ such that for all $x, y, z \in A$ we have

$$\phi(V_{x,y}z) = V_{\phi(x),\hat{\phi}(y)}\phi(z). \tag{7}$$

The set of all autotopies of A is a group $\Gamma(A)$ called *the structure group* of A (a closed subgroup of $\mathrm{GL}(A)$) [2, p. 135]; $\phi \mapsto \hat{\phi}$ is an automorphism of $\Gamma(A)$ of period 2; besides, the automorphism group of A is the subgroup of all elements in $\Gamma(A)$ which fix the identity element of A [2, p. 134]:

$$\operatorname{Aut}(A) = \{\phi \in \Gamma(A) \mid \phi(1) = 1\},$$

Conversely, by [2, Proposition 12.3, Corollary 8.6], every autotopy ϕ of A can be extended uniquely to an automorphism $\tilde{\phi}$ of the graded Lie algebra $\mathcal{K}(A)$ and this defines an isomorphism of $\Gamma(A)$ onto the automorphism group of the $\mathbf{Z}$-graded Lie algebra $\mathcal{K}(A)$.

Observe that the elements $1 \in L_1 = A$, $2 \in L_{-1} = A^{\sim}$ and $2T_1 \in L_0 = \operatorname{Instrl}(A)$ form a standard basis of a simple 3-dimensional Lie subalgebra H of $\mathcal{K}(A)$ (in particular, the $\mathbf{Z}$-grading is defined by the action of the semisimple element $2T_1$). Summarizing, we get the following statement.

Lemma 1 *Let $\operatorname{Aut}_H\big(\mathcal{K}(A)\big)$ be the subgroup of all automorphisms of the Lie algebra $\mathcal{K}(A)$ which fix the elements in H (the group of automorphisms over H), then for every $\psi \in \operatorname{Aut}_H \mathcal{K}(A)$ the restriction $\psi|_A$ to the graded component $\mathcal{K}_1(A) = A$ is an automorphism of the structurable algebra A and every automorphism of A has such a form.*

Now we are going to prove an analog of Theorem 1 for structurable algebras.

Theorem 2 *Let A be a finite dimensional simple structurable algebra and let $G = \operatorname{Aut}(A)$. Then a point $x = (a_1, \ldots, a_k) \in A^k$ is in the null-cone $G : A^k$ iff the subalgebra $A_x = \operatorname{alg}_F\{a_1, \ldots, a_k\} \subseteq A$ generated by $a_1, \ldots, a_k$ is nilpotent.*

Proof If x is an element in the null-cone, then for every $f \in \operatorname{Tr}[A^k]$ we have $f(a_1, \ldots, a_k) = 0$, hence, A_x is a nilpotent subalgebra of A.

Conversely, suppose that A_x is nilpotent.

Lemma 2 *Let B be a nilpotent subalgebra of a structurable algebra A, then*

$$M^B(A) = \operatorname{alg}_F\{R_b, L_b \mid b \in B\} \subseteq \operatorname{End}_F(A)$$

is nilpotent.

Proof Consider the algebra A as a structurable bimodule N over B, then the split null extension $C = B \dotplus N$ [24, p. 485] is a solvable structurable algebra; by [25] it is nilpotent, hence, $M^B(C)$ is nilpotent. Notice that $M^B(A) \simeq M^B(N)$ is a homomorphic image of $M^B(C)$, therefore, $M^B(A)$ is also nilpotent. ∎

We continue the proof of Theorem 2. Denote $L = \mathcal{K}(A)$; by [1, Corollary 6], the algebra L is simple. Next, let B_x be the subalgebra of A generated by A_x and the identity element 1, consider the subalgebra of L

$$\tilde{\mathcal{K}}(B_x) = \mathcal{S}(B_x) \dotplus B_x \dotplus (T_{B_x} \dotplus \operatorname{alg}\{D_{a,b} \mid a, b \in B_x\}) \dotplus B_x^{\sim} + \mathcal{S}(B_x)^{\sim};$$

it is equal to the sum of H and the ideal I of $\tilde{\mathcal{K}}(B_x)$ generated by $A_x \cup A_x^{\sim}$. By the previous lemma every element in I is ad-nilpotent (that is, for all $a \in I$ the operator of adjoint multiplication $\operatorname{ad}(a)\colon b \mapsto [b, a]$ in $\operatorname{End}_F L$ is nilpotent), since so are its homogeneous components (we use here Engel-Jacobson's theorem [16, Theorem 2.1]).

Let K be the automorphism group $\operatorname{Aut}(L)$ of the algebra L without grading; note that K is a closed subgroup of $\operatorname{GL}(L)$ and its Lie algebra $\mathcal{L}(K)$ is the derivation algebra $\operatorname{Der}(L)$ [11, 13.2], where elements in K act by conjugation (the adjoint representation of K). Since L is simple, all derivations of L are inner, *i.e.*,

$$\operatorname{Der}(L) = \operatorname{ad}(L) = \{\operatorname{ad}(a) \mid a \in L\} \simeq L.$$

Denote by N the normalizer $N_K(I) = \{\phi \in K : \phi(I) \subseteq I\}$ of I in K; notice that the normalizer in $\operatorname{GL}(L)$ of any subspace is a closed subgroup in $\operatorname{GL}(L)$, hence $N_K(I)$ is closed in K:

$$N_K(I) = K \cap N_{\operatorname{GL}(L)}(I).$$

Next, let S be the subgroup generated by $\exp\bigl(\operatorname{ad}(a)\bigr)$, for all $a \in I$; observe that S is a closed connected subgroup of $\operatorname{Aut}(L)$ [11, 7.5]; also, $S \subseteq N$, hence S lies in the identity component N° of N. Moreover, S is a normal subgroup of N consisting of unipotent linear transformations of L; hence, by [11, 19.5], S lies in the unipotent radical $R = \mathcal{R}_u(N) = \mathcal{R}_u(N^\circ)$. Thanks to [11, 13.1] the Lie algebra $\mathcal{L}(S)$ contains $\operatorname{ad}(I)$, hence $\operatorname{ad}(I) \subseteq \mathcal{L}\bigl(\mathcal{R}_u(N^\circ)\bigr)$.

Next, consider the subgroup P generated by $\exp\bigl(\operatorname{ad}(a)\bigr)$ where a ranges over the strongly nilpotent elements in H; by [10, p. 55], P is a closed connected subgroup of K and its Lie algebra is equal to H [6, p. 175], hence P is a simple algebraic subgroup of K. Moreover, its generators keep I being invariant, hence $P \subseteq N^\circ$ and, therefore, P lies in a Levi factor Q of N° [11, 30.2].

By [22, 2.6 and 2.3], in K° there is a one-parametric subgroup λ_t, $t \in F^* = F \setminus \{0\}$, such that:

(a) $\lim_{t\to 0} \lambda_t(a)$ exists for all $a \in \mathcal{L}(N^\circ)$;
(b) $\lim_{t\to 0} \lambda_t(a) = 0$ for all $a \in \mathcal{L}(R)$;
(c) $\lambda_t(a) = a$ for all $a \in \mathcal{L}(Q)$.

Recall that $\operatorname{ad}(H) \subseteq \mathcal{L}(Q)$, hence, by (c), $\forall t \in F^*$ λ_t fixes all elements in $\operatorname{ad}(H)$ and, therefore, it fixes all elements in H. It means that λ_t is an automorphism of the graded Lie algebra L and, by Lemma 1, the restriction of λ_t onto $A = L_1$ is an automorphism of A for all $t \in F^*$. From (b) and since $\operatorname{ad}(I) \subseteq \mathcal{L}(R)$, it follows that 0 lies in the closure of the orbit of x and hence x lies in the null-cone of $G : A^k$. The theorem is proven. ■

Proposition 2 yields immediately the following statement.

Corollary 2 *Let A be a finite dimensional simple structurable algebra and let $G = \operatorname{Aut}(A)$. Then $F[A^k]^G$ is a finitely generated module over $\operatorname{Tr}[A^k]$.*

It is known [4] that a finite-dimensional structurable algebra A has a Killing form $\langle x, y\rangle = \operatorname{Tr}_A(L_{\bar{x}y+\bar{y}x})$ and the radical (the maximal solvable ideal) of A coincides with the radical of the form. Also, by [25] the radical of A is nilpotent. Observe that for $A = \mathcal{A}$ we have $\operatorname{Tr}_A(L_x) = 2\operatorname{tr}_{\mathcal{A}}(x)$, where $\operatorname{tr}_{\mathcal{A}}(x)$ is the sum of the diagonal entries of x. For every subalgebra $B \subseteq \mathcal{A}$ if $\operatorname{tr}_{\mathcal{A}}(b) = 0$ for all $b \in B$, then B is nilpotent: indeed, suppose that B is not nilpotent, then B is an ideal of the structurable algebra $F \cdot 1 \dotplus B$, so by [4, Theorem 7] there is a nonzero idempotent $e = \bar{e}$ in B. Now by [4, Theorem 11] we have $\operatorname{Tr}_{\mathcal{A}}(L_e) \neq 0$, hence, $\operatorname{tr}_{\mathcal{A}}(e) \neq 0$, a contradiction.

Let $\operatorname{tr}[\mathcal{A}^k]$ be the subalgebra of $F[\mathcal{A}^k]^G$ generated by 1 and $\operatorname{tr}_{\mathcal{A}}(u)$, where $u \in \mathcal{F}_k(\mathcal{A})$. Corollary 2 can then be transformed into the following statement.

Corollary 3 *Let $G = \operatorname{Aut}(\mathcal{A})$, then $F[\mathcal{A}^k]^G$ is a finitely generated module over $\operatorname{tr}[\mathcal{A}^k]$.*

Proof From (1) it follows that given an homogeneous basis Ω of $\sum_{i<2^m} \mathcal{F}_k(\mathcal{A})_t$, and any $x = (a_1, \dots, a_k) \in \mathcal{A}^k$, the subalgebra A_x is spanned by the elements $f(a_1, \dots, a_k)$, where f ranges over Ω. Let D denote the subalgebra of $\operatorname{tr}[\mathcal{A}^k]$ generated by 1 and $\operatorname{tr}_{\mathcal{A}}(u)$, where $u \in \Omega$. Then

$$\pi_{F[\mathcal{A}^k]^G}(0) = \pi_{\operatorname{Tr}[\mathcal{A}^k]}(0) \subseteq \pi_{\operatorname{tr}[\mathcal{A}^k]}(0) \subseteq \pi_D(0),$$

since $D \subseteq \operatorname{tr}[\mathcal{A}^k] \subseteq \operatorname{Tr}[\mathcal{A}^k]$. But given any $x = (a_1, \dots, a_k) \in \pi_D(0)$ and any $f \in \mathcal{F}_k(\mathcal{A})$, $f(a_1, \dots, a_k)$ is a linear combination of the elements $u(a_1, \dots, a_k)$, where u ranges over Ω, and hence $\operatorname{tr}_{\mathcal{A}}\big(f(a_1, \dots, a_k)\big) = 0$. From the comments above and Lemma 2 it follows that A_x is ad-nilpotent, so $x \in \pi_{\operatorname{Tr}[\mathcal{A}^k]}(0) = \pi_{F[\mathcal{A}^k]^G}(0)$. Now, by Proposition 2, $F[\mathcal{A}^k]^G$ is a finitely generated D-module, so a finitely generated module over $\operatorname{tr}[\mathcal{A}^k]$ too. ■

Now, let us consider the $\mathcal{M}$-module

$$\mathbf{A}^k \oplus (\mathbf{A}^*)^k \simeq (\mathbf{A} \oplus \mathbf{A}^*)^k \subseteq \mathcal{A}^k.$$

Recall that the connected component of $\operatorname{Aut}(\mathcal{A})$ is $\mathcal{M}$, hence, $F[\mathcal{A}^k]^{\mathcal{M}}$ is also a finitely generated module over $\operatorname{tr}[\mathcal{A}^k]$. Let $y_1, \dots, y_k$ ($z_1, \dots, z_k$) be the projectors onto the direct summands of $\mathbf{A}^k$ (respectively, $(\mathbf{A}^*)^k$); if we replace the generic elements x_i of $\mathcal{A}$ in the trace polynomials in $\operatorname{tr}[\mathcal{A}^k]$ with

$$\begin{pmatrix} 0 & y_i \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ z_i & 0 \end{pmatrix}$$

for all $i \in \{1, \dots, k\}$, we will get invariants in $F[\mathbf{A}^k \oplus (\mathbf{A}^*)^k]^{\mathcal{M}}$. Denote by P the subalgebra generated by them; obviously, $F[\mathbf{A}^k \oplus (\mathbf{A}^*)^k]^{\mathcal{M}}$ is a finitely generated module over P.

Also, from the definition of the multiplication of $\mathcal{A}$ it easily follows that the generators of P are obtained in the following way.

Let S^+, S^- be the minimal sets of polynomials satisfying

1. $y_1, \dots, y_k \in S^+, z_1, \dots, z_k \in S^-$;
2. if $a_\sigma, b_\sigma \in S^\sigma$, then $a_\sigma \times b_\sigma \in S^{-\sigma}$ for all $\sigma \in \{+, -\}$.

Then the set of polynomials $\mathrm{tr}_{\mathbf{A}}(ab)$, where $a \in S^+$, $b \in S^-$, generates P.

3 Invariants of E_7

Let us recall first the definition of Freudenthal triple system (*cf.* [5]). On the vector space $\mathcal{A}$ define two forms (*cf.* [3, p. 192], [5, p. 87]):

$$\nu(x) = 4\alpha N(a) + 4\beta N(b) - 4\,\mathrm{tr}_{\mathbf{A}}(a^\# b^\#) + \big(\alpha\beta - \mathrm{tr}_{\mathbf{A}}(ab)\big)^2,$$
$$\langle x, y\rangle = \mathrm{tr}_{\mathcal{A}}\big((sx)\bar{y}\big) = \alpha\delta - \beta\gamma + \mathrm{tr}_{\mathbf{A}}(ac) - \mathrm{tr}_{\mathbf{A}}(bd)$$

where

$$x = \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}, \quad y = \begin{pmatrix} \gamma & c \\ d & \delta \end{pmatrix}$$

and s is the generator of the subspace $\mathcal{S}(\mathcal{A})$:

$$s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is known [7], [5] that the group of invertible linear transformations $\mathcal{E} \subseteq \mathrm{End}_F \mathcal{A}$ preserving ν and $\langle *, *\rangle$ is a simple algebraic group of type E_7.

Denote by $\nu(x, y, z, t)$ the complete linearization of $\nu(x)$, where $\nu(x, x, x, x) = 24\nu(x)$. Since $\langle x, y\rangle$ is nondegenerate, we may define a 3-linear operation $(x, y, z)_1$ on $\mathcal{A}$:

$$\langle x, (y, z, t)_1\rangle = \frac{1}{2}\nu(x, y, z, t).$$

The vector space $\mathcal{A}$ with the operation $(x, y, z)_1$ and the form $\langle x, y\rangle$ is a Freudenthal triple system. For us it is more convenient to replace the form $\langle x, y\rangle$ with a 3-linear operation

$$(x, y, z)_2 = \langle x, y\rangle z.$$

Denote by $\mathcal{T}$ the vector space $\mathcal{A}$ with the operations $(x, y, z)_1$, $(x, y, z)_2$.

Lemma 3 *The automorphism group* $\mathrm{Aut}(\mathcal{T})$ *is equal to* $\mathcal{E}$.

Proof Let $\phi \in \mathrm{Aut}(\mathcal{T})$, then $\phi\big((x, y, z)_2\big) = \big(\phi(x), \phi(y), \phi(z)\big)_2 = \langle\phi(x), \phi(y)\rangle \cdot \phi(z)$, on the other hand, since ϕ is linear, $\phi\big((x, y, z)_2\big) = \langle x, y\rangle\phi(z)$ for all $x, y, z \in \mathcal{T}$. Hence, $\langle\phi(x), \phi(y)\rangle = \langle x, y\rangle$.

Next,

$$\nu\big(\phi(x),\phi(y),\phi(z),\phi(t)\big) = 2\langle\phi(x),\big(\phi(y),\phi(z),\phi(t)\big)_1\rangle = 2\langle\phi(x),\phi\big((y,z,t)_1\big)\rangle$$
$$= 2\langle x,(y,z,t)_1\rangle = \nu(x,y,z.t),$$

hence, $\phi \in \mathcal{E}$.

Conversely, since $\langle x, y\rangle$ is nondegenerate, we get $\mathcal{E} \subseteq \mathrm{Aut}(\mathcal{T})$, therefore, $\mathcal{E} = \mathrm{Aut}(\mathcal{T})$. ■

Lemma 4 *The group $\mathcal{E}$ lies in $\Gamma(\mathcal{A})$. Also, let F^* be the multiplicative group of the ground field $F \subseteq \mathrm{End}_F(\mathcal{A})$, then $\Gamma(\mathcal{A}) = F^* \cdot \mathcal{E}$.*

Proof By (2.16) in [3], for all $x, y, z \in \mathcal{A}$

$$(x,y,z)_1 = 2V_{x,sy}(z) - \langle y,z\rangle x - \langle y,x\rangle z - \langle x,z\rangle y. \tag{8}$$

Also, $L_s \in \Gamma$, there $\hat{L}_s = -L_s$ [2, Section 11]. Hence, if $\phi \in \mathcal{E}$, then we put $\hat{\phi} = L_s\phi L_s$ and equality (7) follows from (8), therefore, $\phi \in \Gamma$.

Next, pick $\phi \in \Gamma(\mathcal{A})$ and put $a = \phi(1)$; then, by [5, Theorem 3], there is $\psi \in \mathcal{E}$ such that $\psi(a) = \beta 1$, where $\beta^4 = \nu(a)$. Hence, the composition of autotopies $1/\beta \in F^*$, ψ and ϕ fixes 1, hence, it is an automorphism of $\mathcal{A}$ [2, Corollary 8.6] and, therefore, belongs to $\mathcal{E}$. ■

The multiplication operators of $\mathcal{T}$ have the following form:

$$(*,y,z)_i\colon x \mapsto (x,y,z)_i;$$
$$(x,*,z)_i\colon y \mapsto (x,y,z)_i;$$
$$(x,y,*)_i\colon z \mapsto (x,y,z)_i,$$

where $i \in \{1,2\}$. So, the subalgebra $\mathrm{Mult}_k(\mathcal{T}) \subseteq \mathrm{Pol}(\mathcal{T}^k, \mathrm{End}_F\mathcal{T})$ is generated by $\{(*,a,b)_i, (a,*,b)_i, (a,b,*)_i \mid i = 1,2; a,b \in \mathcal{F}_k(\mathcal{T})\}$ and the algebra of trace polynomials $\mathrm{Tr}[\mathcal{T}^k]$ is defined as

$$\mathrm{alg}_F\{1, \mathrm{Tr}_{\mathcal{T}}(E) \mid E \in \mathrm{Mult}_k(\mathcal{T})\} \subseteq F[\mathcal{T}^k]^{\mathcal{E}}.$$

We will prove now Corollary 1 for $A = \mathcal{T}$; by Proposition 2 this will complete the proof of Theorem 1.

Theorem 3 *The algebra $F[\mathcal{T}^k]^{\mathcal{E}}$ is a finitely generated module over the algebra of trace polynomials* $\mathrm{Tr}[\mathcal{T}^k]$.

We divide the proof into several steps.

Let A be a simple structurable algebra and let $\Gamma(A)$ be the structure group of A; also, denote by $A^\sim$ the $\Gamma(A)$-module, where every $\phi \in \Gamma(A)$ acts as $\hat{\phi}$. First of all, we need to describe generators of the algebra $F[(A \oplus A^\sim)^k]^{\Gamma(A)}$; it is convenient to do it in terms of pairs .

Definition 1 **(*cf.* [17])** The direct sum of two vector spaces $P = P^+ \oplus P^-$ equipped with two 3-linear operations: $\sigma \in \{+, -\}$

$$P^\sigma \times P^{-\sigma} \times P^\sigma \longrightarrow P^\sigma,$$

$(a^\sigma, b^{-\sigma}, c^\sigma) \mapsto \{a^\sigma\, b^{-\sigma}\, c^\sigma\}$ is called a *pair* ($-\sigma$ means +, if $\sigma = -$ and $-\sigma = -$, *otherwise*).

The subpair $\mathcal{F}_k(P)$ of generic elements of a pair P of rank k is generated in the pair

$$\mathrm{Pol}(P^k, P) = (F[P^k] \otimes_F P^+) \oplus (F[P^k] \otimes_F P^-)$$

by $2k$ projectors x_i^σ ($i \in \{1, \dots, k\}$, $\sigma \in \{+, -\}$). Next, the algebra $\mathrm{Mult}_k(P)$ ($\leq \mathrm{Pol}(P^k, \mathrm{End}_F P)$) is generated by multiplication operators ($\sigma \in \{+, -\}$):

$$\{* \, b^{-\sigma}\, c^\sigma\}, \quad \{a^\sigma \, * \, c^\sigma\}, \quad \{a^\sigma\, b^{-\sigma} \, *\},$$

where $a^\sigma, c^\sigma \in \mathcal{F}_k(P)^\sigma$, $b^{-\sigma} \in \mathcal{F}_k(P)^{-\sigma}$.

Finally, the algebra of trace polynomials $\mathrm{Tr}[P^k]$ is generated by 1 and $\mathrm{Tr}_P(E)$, where $E \in \mathrm{Mult}_k(P)$. Notice that for any $\phi \in \mathrm{End}_F P$ we have a decomposition

$$\phi(a^+, a^-) = \big(\phi_+(a^+) + \phi_{-,+}(a^-), \phi_{+,-}(a^+) + \phi_-(a^-)\big),$$

where $\phi_\sigma \in \mathrm{End}_F P^\sigma$ and $\phi_{\sigma,-\sigma} \in \mathrm{Hom}_F(P^\sigma, P^{-\sigma})$. Therefore, by taking traces,

$$\mathrm{Tr}_P(\phi) = \mathrm{Tr}_{P^+}(\phi_+) + \mathrm{Tr}_{P^-}(\phi_-). \tag{9}$$

Also, for any $E \in \mathrm{Mult}_k(P)$, both E_+ and E_- belong to $\mathrm{Mult}_k(P)$ (we identify $\mathrm{End}_F P^\sigma$ with a subalgebra of $\mathrm{End}_F P$ in the usual way).

Like in Proposition 1 one can show that $\mathrm{Tr}[P^k]$ is affine.

For a structurable algebra A define a pair $P = P^+ \oplus P^-$ letting $P^+ = A$, $P^- = A^\sim$ and

$$\{a^+\, b^-\, c^+\} = V_{a^+, b^-} c^+, \quad \{a^-\, b^+\, c^-\} = V_{a^-, b^+} c^-$$

for $a^\sigma, b^\sigma, c^\sigma \in P^\sigma$, $\sigma \in \{+, -\}$. By definition, the automorphism group of P consists of all pairs $(\phi, \hat{\phi})$, where $\phi \in \Gamma(A)$, and, therefore, it is isomorphic to $\Gamma(A)$.

Proposition 3 *Let A be a simple structurable algebra, then $F[(A \oplus A^\sim)^k]^{\Gamma(A)} = F[P^k]^{\Gamma(A)}$ is a finitely generated module over* $\mathrm{Tr}[P^k]$.

Proof Let $x = (a_1, b_1, \dots, a_k, b_k) \in P^k$ and let $M^+ \oplus M^-$ be the subpair of P generated by the elements $a_1, \dots, a_k \in P^+$, $b_1, \dots, b_k \in P^-$. Since $P = A \oplus A^\sim$ is contained in the $\mathbf{Z}$-graded Lie algebra $L = \mathcal{K}(A)$, we may consider the subalgebra M generated by $a_1, \dots, a_k \in A = L_1$ and $b_1, \dots, b_k \in A^\sim = L_{-1}$. From the definition of multiplication $[a, b]$ in L [1] it follows that $M = \sum_{i=-2}^{2} M_i$, where $M_1 = M^+$, $M_{-1} = M^-$, $M_0 = [M_1, M_{-1}]$ and $M_{\pm 2} = [M_{\pm 1}, M_{\pm 1}]$.

Now, suppose that $x \in \pi_{\mathrm{Tr}[P^k]}(0)$, let us prove that M is ad-nilpotent; because of Engel-Jacobson's theorem [16, Theorem 2.1] it suffices to show that $\mathrm{ad}(a)$ is nilpotent for all $a \in M_0$, since the homogeneous elements in nonzero components are clearly ad-nilpotent. But

$$[M_1, M_{-1}] \subseteq \mathrm{vect}_F\{V_{a,b} \mid a \in M^+, b \in M^-\},$$

therefore, by our hypothesis on x, for all $u \in M_0$, $\mathrm{Tr}_A(u^n) = \mathrm{Tr}_{A^\sim}(u^n) = 0$. Hence, $\mathrm{ad}(u)$ acts nilpotently on $L_{\pm 1}$ and, since L is generated by $L_1 \cup L_{-1}$, by the Jacobi identity, it acts nilpotently on the whole L.

So M is a nilpotent subalgebra and it is homogeneous (invariant under T_1). Also, A is simple, therefore, by [1], L is also simple; like in the proof of Theorem 2 we can show that x belongs to the null-cone of $G_0 : L$, where G_0 is the subgroup of $\mathrm{Aut}(L)$ which keeps T_1 being fixed, that is, the automorphism group of the $\mathbf{Z}$-graded Lie algebra L. Recall that G_0 is precisely the structure group $\Gamma(A)$ [2], so Proposition 2 completes the proof. ∎

The linear mapping $L_s \in \mathrm{End}_F \mathcal{A}$ is invertible, where $(L_s)^{-1} = L_s$; taking $a \in \mathcal{A}^\sim$ to $sa \in \mathcal{T}$, we identify the vector spaces $\mathcal{A}^\sim$ and $\mathcal{T}$. It defines an isomorphism $\phi \mapsto L_s \phi L_s$ from $\mathrm{End}_F \mathcal{T}$ onto $\mathrm{End}_F \mathcal{A}^\sim$.

Now let $P = \mathcal{A} \oplus \mathcal{A}^\sim$ and consider the linear bijection ψ from $\mathcal{T}^{2k}$ to $P^k \simeq \mathcal{A}^k \oplus (\mathcal{A}^\sim)^k$ given by

$$\psi\colon (a_1, \dots, a_{2k}) \mapsto (a_1, \dots, a_k, sa_{k+1}, \dots, sa_{2k});$$

it induces corresponding identifications:

$$\begin{gathered}
\hat{\psi}\colon F[P^k] \longrightarrow F[\mathcal{T}^{2k}] \\
f \mapsto f \circ \psi \\
\tilde{\psi}^+\colon \ \mathrm{Pol}(P^k, \mathcal{A}) \longrightarrow \mathrm{Pol}(\mathcal{T}^{2k}, \mathcal{T}) \\
f \mapsto f \circ \psi \\
\tilde{\psi}^-\colon \ \mathrm{Pol}(P^k, \mathcal{A}^\sim) \longrightarrow \mathrm{Pol}(\mathcal{T}^{2k}, \mathcal{T}) \\
f \mapsto L_s \circ f \circ \psi \\
\bar{\psi}^+\colon \ \mathrm{Pol}\big(P^k, \mathrm{End}_F(\mathcal{A})\big) \longrightarrow \mathrm{Pol}\big(\mathcal{T}^{2k}, \mathrm{End}_F(\mathcal{T})\big) \\
f \mapsto f \circ \psi \\
\bar{\psi}^-\colon \ \mathrm{Pol}\big(P^k, \mathrm{End}_F(\mathcal{A}^\sim)\big) \longrightarrow \mathrm{Pol}\big(\mathcal{T}^{2k}, \mathrm{End}_F(\mathcal{T})\big) \\
f \mapsto L_s \circ (f \circ \psi) \circ L_s
\end{gathered}$$

where the latter line should be understood as follows:

$$\bar{\psi}^-(f)(a_1, \dots, a_{2k}) = L_s \circ f(a_1, \dots, a_k, sa_{k+1}, \dots, sa_{2k}) \circ L_s.$$

Notice that for any $f \in \mathrm{Pol}(P^k, \mathrm{End}_F \mathcal{A})$ and $g \in \mathrm{Pol}(P^k, \mathrm{End}_F \mathcal{A}^\sim)$,

$$\hat{\psi}\big(\mathrm{Tr}_{\mathcal{A}}(f)\big) = \mathrm{Tr}_{\mathcal{T}}\big(\bar{\psi}^+(f)\big) \quad \text{and} \quad \hat{\psi}\big(\mathrm{Tr}_{\mathcal{A}^\sim}(g)\big) = \mathrm{Tr}_{\mathcal{T}}\big(\bar{\psi}^-(g)\big). \tag{10}$$

The isomorphism $P^k \simeq \mathcal{A}^k \oplus (\mathcal{A}^\sim)^k$ defines an isomorphism $F[P^k] \simeq F[\mathcal{A}^k] \otimes_F F[(\mathcal{A}^\sim)^k]$, which in turn provides $F[P^k]$ the structure of a bigraded algebra, so that we can speak of the degree in $\mathcal{A}^k$ and the degree in $(\mathcal{A}^\sim)^k$. Let Δ_k be the subspace of $F[P^k]$ spanned by the elements $f(x)g(y)$ ($\simeq f(x) \otimes g(y)$), where x (respectively, y) is the projector of P^k onto $\mathcal{A}^k$ ($(\mathcal{A}^\sim)^k$), for homogeneous f and g of the same degree. That is, Δ_k is the subspace of $F[P^k]$ spanned by all homogeneous polynomials $f = f(x, y)$ such that $\deg_x f = \deg_y f$. Obviously, it is a subalgebra of $F[P^k]$; moreover, $F[P^k]^{\Gamma(\mathcal{A})} \subseteq \Delta_k$ because $F[P^k]^{\Gamma(\mathcal{A})}$ is homogeneous, the scalars F^* lie in $\Gamma(\mathcal{A})$ and for all $\alpha \in F^*$, $a \in \mathcal{A}$ and $b \in \mathcal{A}^\sim$ the mapping α takes a to αa and b to $\alpha^{-1} b$.

Let $\hat{\Delta}_k$ be the image of Δ_k under $\hat{\psi}$, then $\hat{\Delta}_k$ is spanned by all homogeneous polynomials with the degree in the first k projectors being equal to that of the last k ones. Put $\Lambda_k = \hat{\Delta}_k \cap F[\mathcal{T}^{2k}]^{\mathcal{E}}$.

Proposition 4 *The mapping $\hat{\psi}$ restricts to an algebra isomorphism from $F[P^k]^{\Gamma(\mathcal{A})}$ onto Λ_k.*

Proof For all $f \in F[P^k]^{\Gamma(\mathcal{A})}$ and $\phi \in \mathcal{E}$ ($\subseteq \Gamma(\mathcal{A})$)

$$\begin{aligned}
\big(\hat{\psi}(f)\cdot\phi\big)(a_1,\dots,a_{2k}) &= \hat{\psi}(f)\big(\phi(a_1),\dots,\phi(a_{2k})\big)\\
&= f\big(\phi(a_1),\dots,\phi(a_k), s\phi(a_{k+1}),\dots,s\phi(a_{2k})\big)\\
&= f\big(\phi(a_1),\dots,\phi(a_k), \hat{\phi}(sa_{k+1}),\dots,\hat{\phi}(sa_{2k})\big)\\
&= (f\cdot\phi)(a_1,\dots,a_k, sa_{k+1},\dots,sa_{2k})\\
&= f(a_1,\dots,a_k, sa_{k+1},\dots,sa_{2k})\\
&= \hat{\psi}(f)(a_1,\dots,a_{2k})
\end{aligned}$$

so $\hat{\psi}(f) \in \Lambda_k$. In the other way, if $f \in \Lambda_k$ and $g \in \Delta_k$ with $\hat{\psi}(g) = f$, since $g \in \Delta_k$, it is invariant under the action of $F^* \subseteq \Gamma$. Also, for all $\phi \in \mathcal{E}$:

$$\begin{aligned}
(g\cdot\phi)(a_1,\dots,a_{2k}) &= g\big(\phi(a_1),\dots,\phi(a_k), \hat{\phi}(a_{k+1}),\dots,\hat{\phi}(a_{2k})\big)\\
&= g\big(\phi(a_1),\dots,\phi(a_k), s\phi(sa_{k+1}),\dots,s\phi(sa_{2k})\big)\\
&= f\big(\phi(a_1),\dots,\phi(a_k), \phi(sa_{k+1}),\dots,\phi(sa_{2k})\big)\\
&= (f\cdot\phi)(a_1,\dots,a_k, sa_{k+1},\dots,sa_{2k})\\
&= f(a_1,\dots,a_k, sa_{k+1},\dots,sa_{2k})\\
&= \hat{\psi}(g)(a_1,\dots,a_k, sa_{k+1},\dots,sa_{2k})\\
&= g(a_1,\dots,a_k, a_{k+1},\dots,a_{2k}) \quad \text{since } L_s^2 = \mathrm{id},
\end{aligned}$$

as required. ∎

Now, pick a homogeneous $h \in F[\mathcal{T}^k]^{\mathcal{E}}$ and define f in $F[\mathcal{T}^{2k}]^{\mathcal{E}}$ letting

$$f = h(x_1,\dots,x_k)\, h(x_{k+1},\dots,x_{2k}).$$

Obviously $f \in \Lambda_k$, hence there is a $g \in F[P^k]^{\Gamma(A)}$ such that $\hat{\psi}(g) = f$. By Proposition 3, g is integral over $\mathrm{Tr}[P^k]$, so if we show that $\hat{\psi}(\mathrm{Tr}[P^k]) \subseteq \mathrm{Tr}[\mathcal{T}^{2k}]$, then we will have f to be integral over $\mathrm{Tr}[\mathcal{T}^{2k}]$, which yields (replacing x_{k+i} with x_i for all $i \in \{1, \ldots, k\}$) h^2, and hence h, to be integral over $\mathrm{Tr}[\mathcal{T}^k]$. Therefore, this will complete the proof of Theorems 3 and 1.

The isomorphisms above obtained from ψ allow us to define a linear mapping

$$\tilde{\psi}\colon \mathrm{Pol}(P^k, P) \longrightarrow \mathrm{Pol}(\mathcal{T}^{2k}, \mathcal{T})$$

by means of

$$\tilde{\psi}(f, g) = \tilde{\psi}^+(f) + \tilde{\psi}^-(g)$$

for all $f \in \mathrm{Pol}(P^k, \mathcal{A})$ and all $g \in \mathrm{Pol}(P^k, \mathcal{A}^\sim)$, and in exactly the same way we define a linear mapping

$$\bar{\psi}\colon \mathrm{Pol}\big(P^k, \mathrm{End}_F(\mathcal{A}) \oplus \mathrm{End}_F(\mathcal{A}^\sim)\big) \longrightarrow \mathrm{Pol}\big(\mathcal{T}^{2k}, \mathrm{End}_F(\mathcal{T})\big).$$

Lemma 5 $\tilde{\psi}\big(\mathcal{F}_k(P)\big) \subseteq \mathcal{F}_{2k}(\mathcal{T})$ *and* $\bar{\psi}\big(\mathrm{Mult}_k(P)\big) \subseteq \mathrm{Mult}_{2k}(\mathcal{T})$.

Proof Let $z_1, \ldots, z_{2k}$ be the projections (generic elements) in $\mathcal{T}^{2k}$. Then since L_s^2 is the identity, it follows that $\tilde{\psi}(x_i) = z_i$ and $\tilde{\psi}(y_i) = z_{k+i}$ for all $i \in \{1, \ldots, k\}$.

Hence, by (8), if $f, g, h \in \mathcal{F}_{2k}(\mathcal{T})$, then $V_{f, L_s \circ g} h \in \mathcal{F}_{2k}(\mathcal{T})$. Therefore, for all $f_1, f_2 \in \mathcal{F}_k^+$ and all $g \in \mathcal{F}_k^-$ (where $\mathcal{F}_k(\mathcal{A} \oplus \mathcal{A}^\sim) = \mathcal{F}_k^+ \oplus \mathcal{F}_k^-$), if $\tilde{\psi}(f_1)$, $\tilde{\psi}(f_2)$ and $\tilde{\psi}(g)$ belong to $\mathcal{F}_{2k}(\mathcal{T})$, then

$$\begin{aligned}
\tilde{\psi}(V_{f_1,g} f_2) &= \tilde{\psi}^+(V_{f_1,g} f_2)\\
&= (V_{f_1,g} f_2) \circ \psi = V_{f_1 \circ \psi, g \circ \psi} f_2 \circ \psi\\
&= V_{\tilde{\psi}(f_1), L_s \circ \tilde{\psi}(g)} \tilde{\psi}(f_2) \in \mathcal{F}_{2k}(\mathcal{T}).
\end{aligned}$$

In the same way, for all $f \in \mathcal{F}_k^+$, $g_1, g_2 \in \mathcal{F}_k^-$, if $\tilde{\psi}(f)$, $\tilde{\psi}(g_1)$ and $\tilde{\psi}(g_2)$ belong to $\mathcal{F}_{2k}(\mathcal{T})$ then

$$\begin{aligned}
\tilde{\psi}(V_{g_1,f} g_2) &= \tilde{\psi}^-(V_{g_1,f} g_2)\\
&= L_s \circ (V_{g_1,f} g_2) \circ \psi = L_s \circ V_{g_1 \circ \psi, f \circ \psi} g_2 \circ \psi\\
&= -V_{L_s \circ g_1 \circ \psi, L_s \circ f \circ \psi} L_s \circ g_2 \circ \psi \quad \text{since } L_s \in \Gamma \text{ and } \hat{L}_s = -L_s\\
&= -V_{\tilde{\psi}(g_1), L_s \circ \tilde{\psi}(f)} \tilde{\psi}(g_2) \in \mathcal{F}_{2k}(\mathcal{T}) \quad \text{by (8).}
\end{aligned}$$

So, the first part is proven. Next, from the above it follows that for all $a^\sigma, c^\sigma \in \mathcal{F}_k^\sigma$ and $b^{-\sigma} \in \mathcal{F}_k^{-\sigma}$ ($\sigma = \pm$):

$$\tilde{\psi}(\{a^\sigma, b^{-\sigma}, c^\sigma\}) = \sigma V_{\tilde{\psi}(a^\sigma), L_s \circ \tilde{\psi}(b^{-\sigma})} \tilde{\psi}(c^\sigma),$$

so by (8), the operators $\bar{\psi}(\{a^\sigma, b^{-\sigma}, *\})$, $\bar{\psi}(\{a^\sigma, *, c^\sigma\})$, and $\bar{\psi}(\{*, b^{-\sigma}, c^\sigma\})$ belong to $\mathrm{Mult}_{2k}(\mathcal{T})$, therefore, $\bar{\psi}\big(\mathrm{Mult}_k(P)\big) \subseteq \mathrm{Mult}_{2k}(\mathcal{T})$.

Recall that $P^+ = \mathcal{A}$ and $P^- = \mathcal{A}^\sim$; by (9) for any $x \in \operatorname{End}_F P$

$$\operatorname{Tr}_P(x) = \operatorname{Tr}_{\mathcal{A}}(x_+) + \operatorname{Tr}_{\mathcal{A}^\sim}(x_-) = \operatorname{Tr}_{\mathcal{J}}(x_+) + \operatorname{Tr}_{\mathcal{J}}(x_-),$$

hence, for all $E \in \operatorname{Mult}_k(P)$

$$\begin{aligned}\hat{\psi}\big(\operatorname{Tr}_P(E)\big) &= \hat{\psi}\big(\operatorname{Tr}_{\mathcal{A}}(E_+)\big) + \hat{\psi}\big(\operatorname{Tr}_{\mathcal{A}^\sim}(E_-)\big)\\ &= \operatorname{Tr}_{\mathcal{J}}\big(\bar{\psi}^+(E_+)\big) + \operatorname{Tr}_{\mathcal{J}}\big(\bar{\psi}^-(E_-)\big) \in \operatorname{Tr}[\mathcal{J}^{2k}]\end{aligned}$$

where we have used (10). Hence, $\hat{\phi}(\operatorname{Tr}[P^k]) \subseteq \operatorname{Tr}[\mathcal{J}^{2k}]$, as required, and this concludes the proof of Lemma 5 and, therefore, of Theorems 3 and 1.

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