THE REFLECTION PRINCIPLE FOR BANACH SPACE-VALUED ANALYTIC FUNCTIONS

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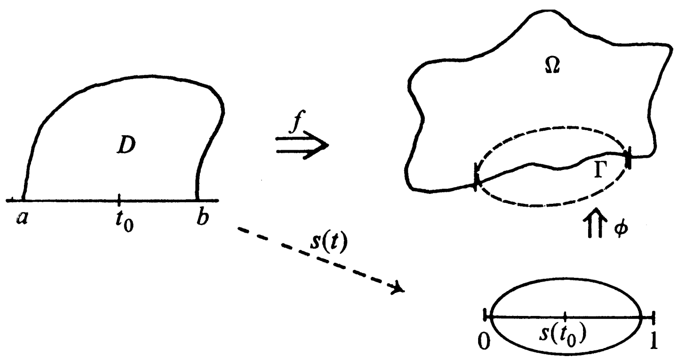
We give sufficient conditions for the continuation of an analytic function with values in a Branch space. For analytic functions taking complex numbers as values, the principle is known as the Schwarz Reflection Principle.

A function defined on a domain of the complex plane with values in a Banach space X is *analytic* if it possesses at each point z_0 of the domain a convergent power series in z, with coefficients in X.

THEOREM. *Let D be a domain in the upper half-plane, and E a regular subset of the boundary of D. Suppose that E is an interval of the real axis (a, b). Let f be an analytic function defined on D, continuous up to E, taking values in a Banach space X. Let the image of D under f be Ω, and let Γ be the part of the boundary of Ω which is the image of E under f. Suppose that Γ is an analytic arc in X. Then f can be continued analytically across E, to a domain containing $D + E$.*



We call E a *regular* subset of the boundary of D if the following is true: on the side of E on which D lies, all points sufficiently close to E belong to D. This hypothesis, which is necessary even in the one-dimensional case, is usually omitted from the statement of the theorem. We require that the interval (a, b) be a regular boundary arc so that after reflection, each $t \in (a, b)$ will be an interior point of the extended domain. This will not be the case if we allow the possibility of slit domains, say

$$\{|z| < 1\} \cap \{\text{Im } z > 0\} \backslash \{\lambda i | \ 0 < \lambda < \tfrac{1}{2}\}.$$

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An arc Γ in a Banach space is analytic if Γ is non-self intersecting, and there exists an analytic function $\phi(z)$ defined in a neighbourhood of $(0, 1)$, and Γ is the image under ϕ of $(0, 1)$. Further, it is required that $\phi'(s) \neq 0$, for all $0 < s < 1$. We shall show later that this last assumption is essential.

At first glance, this theorem may seem to be a straightforward generalization of the case when X is the complex plane. This is not the case, however, for in the one-dimensional case, the hypothesis of analyticity of the arc Γ (under the map ϕ) ensures that f and ϕ have a part of their ranges in common. In the general case, we have no guarantee *a priori* of this. Further, the fact that f and ϕ each carry an interval onto the same *point set* does not on the face of it imply that f and ϕ are analytically related. The difficulty lies in showing that a re-parametrization of the interval, $s(t)$, chosen so that $f(t) = \phi(s(t))$, is actually analytic. In the one-dimensional case, this follows easily by considering ϕ^{-1}, which we know to be analytic, and this is not an appeal we can make in the general case.

Proof. With f, Γ, and ϕ as in the hypothesis, we have that for every $t \in (a, b)$ there exists a unique $s \in (0, 1)$ such that $f(t) = \phi(s)$. Thus s is a function of t, and we write

$$f(t) = \phi(s(t)). \tag{1}$$

We now show that s is an analytic function in a neighbourhood of (a, b). Let $t_0 \in (a, b)$. By hypothesis, $\phi'(s(t_0)) \neq 0$ so there exists a linear functional T on X with $T[\phi'(s(t_0))] \neq 0$. Define $\phi_T = T \circ \phi; f_T = T \circ f$. Then ϕ_T and f_T are analytic functions in the "usual" sense (**1**, p. 224).

Note that $\phi_T'(s(t_0)) = T[\phi'(s(t_0))] \neq 0$, and

$$f_T(t) = \phi_T(s(t)).$$

Since $\phi_T'(s(t_0)) \neq 0$, we see that ϕ_T is a one-to-one analytic function in a neighbourhood of $s(t_0)$, and hence ϕ_T^{-1} is a well-defined analytic function in a neighbourhood of $f_T(t_0)$. Then we have

$$s(t) = \phi_T^{-1}(f_T(t))$$

on an interval of the real axis about t_0. However, in a "half-disc" above t_0 (i.e. a complex neighbourhood of t_0 intersected with the upper half-plane), $\phi_T^{-1}(f_T(z))$ is an analytic function. Thus $s(z) \equiv \phi_T^{-1}(f_T(z))$ is analytic in a "half disc" above t_0, with real boundary values $s(t)$, satisfying (1). However, t_0 was an arbitrary point, and since the definition of $s(t)$ is independent of the choice of T, we have that $s(z)$ is an analytic function in a domain above the real axis, with real boundary values on (a, b). By the usual Reflection Principle, $s(z)$ is analytic in a full neighbourhood of (a, b).

Let $\psi(z) = \phi(s(z))$. ψ is analytic in a neighbourhood of the interval (a, b), and by (1), $\psi(t)$ agrees with $f(t)$ on (a, b). We could now conclude that ψ is the analytic continuation of f across (a, b) by an application of Morera's Theorem for Banach space-valued analytic functions. Instead, let us prove

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this directly. For any linear functional T on X, consider f_T and ψ_T as defined earlier. The function $f_T - \psi_T$ is analytic in a domain above (a, b), and has 0 boundary values on (a, b). We conclude that $(f - \psi)_T = f_T - \psi_T \equiv 0$. However, this is true for any linear functional T, and hence $f \equiv \psi$. As ψ is analytic in a neighbourhood of (a, b), it represents the analytic continuation of f across (a, b).

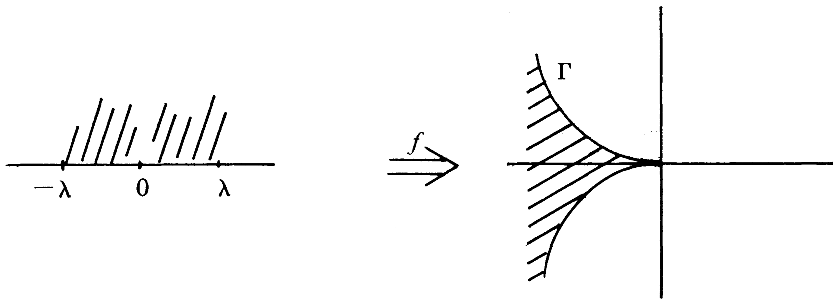
As in the usual Reflection Principle, we can relax the assumptions on D and E to read: D is a domain in the complex plane, E is a regular subset of the boundary of D, and E is an analytic arc (in the usual sense).

We cannot, in general, remove the hypothesis that $\phi'(t) \neq 0$ for all $t \in (0, 1)$, as the following example will show.

Consider

$$\phi(z) = \left(\frac{z}{i + z}\right)^2,$$

which maps an interval $(-\lambda, \lambda)$ onto a cusp, Γ:



Suppose that D is a domain bounded in part by $(-\lambda, \lambda)$, and F is a domain in the left half-plane, bounded by Γ. If $f\colon D \to F$, and $f(0) = 0$, then it is easily shown that f cannot possess a power series at 0 (and hence cannot be continued across $(-\lambda, \lambda)$).

Reference

1. N. Dunford and J. T. Schwartz, *Linear operators.* Part I: *General theory* (Interscience, New York, 1958).

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