BULL. AUSTRAL. MATH. SOC. 22D25
VOL. 47 (1993) [169–174]

THE STRUCTURE OF CERTAIN GROUP C^*-ALGEBRAS

MILAN PAHOR

Let G be a separable locally compact group which admits a non-trivial compact normal subgroup. It is shown that the group C^*-algebra $C^*(G)$ of G may be decomposed as a direct sum of ideals whose structure is determined up to $*$-isomorphism. Applications are given to Type 1, $[FD]^-$ groups; in particular it is shown that the group C^*-algebra of such a group is a direct sum of homogeneous C^*-algebras.

1. INTRODUCTION

One form of the classical Peter-Weyl theorem states that if K is a separable compact group then $C^*(K) \cong \bigoplus_{\pi \in \widehat{K}} M_{\mathrm{Dim}(\pi)}(C)$ (where $M_{\mathrm{Dim}(\pi)}(C)$ is the matrix algebra whose dimension is that of π) and hence that $\mathrm{Prim}(C^*(K))$ is discrete. Incorporating this result we show that if G is a separable locally compact group which admits a non-trivial compact normal subgroup then $\mathrm{Prim}(C^*(G))$ may be expressed as a disjoint union of clopen subsets; implying that $C^*(G)$ may be decomposed as a direct sum of ideals. The structure of these ideals is then determined up to $*$-isomorphism. A corollary of these results is that the group C^*-algebra of a Type 1 $[FD]^-$ group is a direct sum of homogeneous C^*-algebras, this being a substantial improvement on [11, Theorem 4]. Our method of proof is to express $C^*(G)$ as a twisted crossed product and then implement Green's algebraic generalisation [4] of the Mackey analysis [8] of the representation theory of group extensions.

Notation will be as follows: Given a separable locally compact group G and a 2-cocycle σ on G, $C^*(G)$ is the usual group C^*-algebra of G and $C^*(G,\sigma)$ the twisted group C^*-algebra determined by σ. $\mathrm{Prim}(C^*(G))$ is the space of all primitive ideals of $C^*(G)$ with the Jacobson hull-kernel topology and $\widehat{G}$ the space of (unitary equivalence classes of) irreducible representations of G, endowed with the Fell topology (which is the inverse image of the Jacobson topology under the canonical map $\widehat{G}(=(C^*(G))^\wedge) \to \mathrm{Prim}(C^*(G))$. Given a representation or an automorphism θ of a C^*-algebra A, $\widetilde{\theta}$ will denote its natural extension to the multiplier algebra $M(A)$ of A. All ideals are assumed to be closed and two-sided, all $*$-representations non-degenerate, and all groups and C^*-algebras separable.

Received 18 February 1992

https://doi.org/10.1017/S0004972700012375 Published online by Cambridge University Press

2. Preliminaries

The reader is referred to [10] and [4] for a detailed account of the theory of covariant systems, crossed products and twisted crossed products. A covariant representation of a covariant system (A, G, α) is a pair (π, u) where π is a $*$-representation of A on a Hilbert space H, u is a unitary representation of G on H and the equation $\pi(\alpha_g(x)) = u(g)\pi(x)u(g)^*$ holds for all $x \in A$, $g \in G$. Given a covariant system (A, G, α) the crossed product $C^*(A, G, \alpha)$ is a C^*-algebra constructed in such a way that its $*$-representations $\pi \times u$ are in bijection with the covariant representations (π, u) of (A, G, α). We define actions of G on $\widehat{A}$ and $\operatorname{Prim}(A)$ respectively by $(g.\pi)(a) = \pi(\alpha_{g^{-1}}(a))$ and $g.\ker(\pi) = \ker(g.\pi)$.

A twisted covariant system (A, G, α, τ) consists of a covariant system (A, G, α) together with a strictly continuous homomorphism τ of a closed normal subgroup N_τ of G into the group of unitaries in the multiplier algebra $M(A)$ of A such that

(i) $\tau(n)a\tau(n^{-1}) = \alpha_n(a)$ for all $n \in N_\tau$, $a \in A$; and
(ii) $\tau(gng^{-1}) = \widetilde{\alpha}_g(\tau(n))$ for all $n \in N_\tau$, $g \in G$.

Given a twisted covariant system (A, G, α, τ) let I_τ be the intersection of the kernels of all the $*$-representations $\pi \times u$ of $C^*(A, G, \alpha)$ having the property that $u(n) = \widetilde{\pi}(\tau(n))$ for all $n \in N_\tau$ (that is, (π, u) preserves the twist τ). The C^*-algebra $C^*(A, G, \alpha)/I_\tau$ is denoted by $C^*(A, G, \alpha, \tau)$ and referred to as the twisted crossed product of the twisted covariant system (A, G, α, τ). Note that $(\pi, u) \to \pi \times u$ is a bijection between the covariant representations of (A, G, α) which preserve τ and the $*$-representations of $C^*(A, G, \alpha, \tau)$. These construction are motivated by the following examples.

Example 1: Let G be a separable locally compact group, N a closed normal subgroup of G, define an action of G on $L^1(N)$ by $(gf)(n) = f(g^{-1}ng)$ and lift it to an action α of G on $C^*(N)$. For each $n \in N$ let $\tau(n)\colon L^1(N) \to L^1(N)$ be $(\tau(n)f)(\zeta) = f(n^{-1}\zeta)$ and extend $\tau(n)$ to a unitary multiplier (also denoted by $\tau(n)$) of $C^*(N)$. Then $(C^*(N), G, \alpha, \tau)$ forms a twisted covariant system and the twisted crossed product $C^*(C^*(N), G, \alpha, \tau)$ is $*$-isomorphic to $C^*(G)$ [4, Proposition 1]. It is through this identification that Green's analysis [4] of the structure of twisted crossed products will be applied to group C^*-algebras.

Example 2: Let G be a separable locally compact group, σ a 2-cocycle on G and define G^σ to be the group whose elements are ordered pairs (t, g) with $t \in \mathbf{T}$, $g \in G$ and multiplication defined as $(t, g)(t'g') = (tt'\overline{\sigma}(g, g'), gg')$. Noting that G^σ is not necessarily a locally compact group under the product topology on $\mathbf{T} \times G$ we topologise G^σ as follows: Endow G^σ with the product of the Borel structures on $\mathbf{T}$ and G and note that the product of the respective left Haar measures is itself left invariant for G^σ. By [7, Theorem 7.1] G^σ admits a unique locally compact topology under which it is a locally compact topological group whose natural Borel structure coincides with that G^σ. Equip G^σ with this topology. Given a σ-representation L of G define an ordinary representation L^0 of G^σ by $L^0(t, g) = t\,L(g)$. Then $L \to L^0$ is injective and has for its range the set of all ordinary representations of G^σ which reduce on T to the trivial representation $(t, e) \to t$. Denote by $\mathbf{T} \times \{e\}$ the closed normal subgroup $\{(t, e) \mid t \in \mathbf{T}\}$ of G^σ and noting that $\operatorname{Aut}(\mathbf{C})$ is trivial and $M(\mathbf{C}) = \mathbf{C}$, define $\alpha\colon G^\sigma \to \operatorname{Aut}(\mathbf{C})$ to be trivial and $\tau\colon \mathbf{T} \times \{e\} \to \mathbf{T}$ by $\tau(t, e) = t$. Then $(\mathbf{C}, G^\sigma, \alpha, \tau)$ is a twisted covariant system and a covariant pair (id, u) preserves the twist if and only if $u(t, e) = t$ for all $\tau \in \mathbf{T}$. Furthernore $C^*(\mathbf{C}, G^\sigma, \alpha, \tau)$ is $*$-isomorphic to $C^*(G, \sigma)$.

3. The main results

Lemma 1. *Let G be a separable locally compact group which admits a non-trivial compact normal subgroup K. Then $\operatorname{Prim}(C^*(G))$ may be expressed as a disjoint union of clopen subsets.*

Proof: The reader is assumed to be familiar with the definitions presented in [4, p.221]. We treat $C^*(G)$ as a twisted crossed product $C^*(C^*(K), G, \alpha, \tau)$, (see Example 1 for the definitions of α and τ) and denote by Q the space of all G-quasi orbits in $\operatorname{Prim}(C^*(K))$ endowed with the quotient topology arising from the canonical surjection $T\colon \operatorname{Prim}(C^*(K)) \to Q$. Since $\operatorname{Prim}(C^*(K))$ is discrete, [3, 18.4.3] G quasi-orbits are in fact G-orbits in this case. Denote by $I(C^*(K))$ the set of all ideals in $C^*(K)$ equipped with the topology having as an open subbase the family $\{S_I \mid I \in I(C^*(K))\}$ where $S_I = \{J \in I(C^*(K)) \mid I \not\subset J\}$. Then $\Phi\colon Q \to I(C^*(K))$ given by $\Phi(q) = \bigcap_{x \in q} x$ is a homeomorphism onto its range [4, Lemma p.221] and since Q is discrete, $(C^*(K), G, \alpha, \tau)$ is quasi-regular [4, Corollary 19]. Thus the restriction map Res (see [4, Proposition 9] and its following discussion) takes $\operatorname{Prim}(C^*(C^*(K), G, \alpha, \tau))$ into $\Phi(Q)$, and $\phi^{-1} \circ \operatorname{Res}$ is a continuous map from $\operatorname{Prim}(C^*(C^*(K), G, \alpha, \tau))$ into Q. Denote by $[\operatorname{Prim}(C^*(C^*(K), G, \alpha, \tau))]^q$ (or equivalently $[\operatorname{Prim}(C^*(G))]^q$) the subset of $\operatorname{Prim}(C^*(C^*(K), G, \alpha, \tau))$ living on the orbit $q \in Q$. To complete the proof we need only note that $[\operatorname{Prim}(C^*(G))]^q = \left(\phi^{-1} \circ \operatorname{Res}\right)^{-1}(q)$ and that Q is discrete. Hence $\operatorname{Prim}(C^*(G)) = \dot{\bigcup}_{q \in Q} [\operatorname{Prim}(C^*(G))]^q$ decomposes $\operatorname{Prim}(C^*(G))$ as a disjoint union of clopen subsets. $\square$

Theorem 2. *Let G be a separable locally compact group which admits a com-*

https://doi.org/10.1017/S0004972700012375 Published online by Cambridge University Press

pact normal subgroup K. Then

$$C^*(G) \cong \bigoplus_{G.\pi \in \widehat{K}/G} C^*(G_\pi/K, \overline{w}_\pi) \otimes \mathbf{K}(H_\pi) \otimes \mathbf{K}(L^2(G/G_\pi))$$

where G_π is the stabiliser subgroup of G at $\pi \in \widehat{K}$, w_π is the associated Mackey obstruction 2-cocycle (see [**8**, Theorem 8.2] *for a description of w_π) and $\mathbf{K}(H_\pi)$ is the C^*-algebra of compact operators on the Hilbert space of π.*

PROOF: We begin by noting that since K is of Type 1, [**3**, 15.1.4] $\widehat{K}$ may be identified with $\text{Prim}(C^*(K))$ and $\widehat{K}/G$ with Q (see the proof of Lemma 1 for a description of Q). For each G-orbit $q = G.\pi$ in $\widehat{K}$ let $I_q = \{\bigcap \sigma \mid \sigma \in [\text{Prim}(C^*(G))]^q\}$. Then $\text{Prim}(C^*(G)/I_q) = [\text{Prim}(C^*(G))]^q$ and it follows, from Lemma 1, that $C^*(G) \cong \bigoplus_{q \in \widehat{K}/G} (C^*(G)/I_q)$. Since $\text{Prim}(C^*(K))$ is discrete, all the G-orbits in $\text{Prim}(C^*(K))$ are locally closed, hence K is regularly embedded in G. Furthermore for each $q \in Q$, $C^*(G)/I_q$ is the subquotient of $C^*(G)$ determined by the locally closed (in fact clopen) subset $[\text{Prim}(C^*(G))]^q$ of $\text{Prim}(C^*(G))$ living on the orbit q (see [**5**, Remark 2.14]. The result now follows from ([**5**, Theorem 3.1] and its preceding discussion) and Example 2. ∎

We refer to the above decomposition as the decomposition of $C^*(G)$ based upon the compact subgroup K. If G is compact, the decomposition of $C^*(G)$ based upon G itself yields the version of the Peter-Weyl theorem discussed in the introduction. As a corollary of Theorem 2 we have the following well known result.

COROLLARY 3. *Let G be a separable locally compact group, σ a 2-cocycle on G, G^σ the extension of $\mathbf{T}$ by G presented as Example 2. Then*

$$C^*(G^\sigma) \cong \bigoplus_{n \in \mathbf{Z}} C^*(G, \sigma^n).$$

PROOF: We base a decomposition of $C^*(G^\sigma)$ upon the compact, central subgroup $\mathbf{T} \times \{e\}$ of G^σ. Noting that $\text{Prim}(C^*(\mathbf{T})) \cong \mathbf{Z}$, $G^\sigma/\mathbf{T} \cong G$, and the Mackay 2-cocycle on $G^\sigma/\mathbf{T}$ determined by $n \in \mathbf{Z}$ is $(\overline{\sigma})^n$ we have by Theorem 2

$$\begin{aligned} C^*(G^\sigma) &\cong \bigoplus_{n \in \mathbf{Z}} C^*(G, \sigma^n) \otimes \mathbf{C} \otimes \mathbf{K}(L^2(e)) \\ &\cong \bigoplus_{n \in \mathbf{Z}} C^*(G, \sigma^n). \end{aligned}$$

We close the paper with an application of our main result to the special case of $[FD]^-$ groups, that is separable locally compact groups satisfying a short exact sequence $\{e\} \to K \to G \to G/K \to \{e\}$ where K is compact and G/K is abelian. ∎

Corollary 4. *Let G be a separable locally compact Type 1 $[FD]^-$ group. Then $C^*(G)$ is a direct sum of homogeneous C^*-algebras.*

Proof: Let K be a compact subgroup of G such that G/K is abelian. Basing a decomposition of $C^*(G)$ upon K we need only verify that $C^*(G_\pi/K, \overline{w}_\pi) \otimes \mathrm{K}(H_\pi) \otimes \mathrm{K}(L^2(G/G_\pi))$ is homogeneous for each orbit $G.\pi$ in $\widehat{K}$ (notation as in Theorem 2). Since $C^*(G_\pi/K, \overline{w}_\pi) \otimes \mathrm{K}(H_\pi) \otimes \mathrm{K}(L^2(G/G_\pi))$ is isomorphic to a subquotient of $C^*(G)$ and G is of Type 1, $C^*(G_\pi/K, \overline{w}_\pi)$ is also of Type 1 implying that $\overline{w}_\pi$ is a Type 1 2-cocycle on the abelian group G_π/K. It now follows from [**1**, Theorems 3.1 and 3.3] that all irreducible $\overline{w}_\pi$-representations of G_π/K are of the same dimension. This completes the proof. ∎

The hierarchy diagram 1 of [**9**, p.698] illustrates the position that $[FD]^-$ groups occupy in the theory of non-compact, non-abelian groups.

Corollary 5. *Suppose that G is a separable locally compact group which admits a compact neighbourhood of the identity invariant under all inner automorphisms (such a group is called in $[IN]$ group) and that $\widehat{G}$ is Hausdorff. Then $G \in [FD]^- \cap$ Type 1 and hence $C^*(G)$ is a direct sum of homogeneous C^*-algebras.*

Proof: We denote by $[FC]^-$ the class of groups with pre-compact conjugacy classes. Then $[FD] \subseteq [FC]$ [**9**, Diagram 1] and if G is of Type 1 the reverse inclusion holds [**6**, Proposition 3.1]. The result now follows from [**6**, Theorem 5.2] and Corollary 4.

It has been conjectured that a separable locally compact group G has the property that $\widehat{G}$ is Hausdorff if and only if $G \in [FD]^- \cap$ Type 1 [**2**, Section 5]. Should this prove to be correct the condition $G \in [IN]$ may be omitted from Corollary 5. Note finally that the class $[IN]$ contains all Moore groups, [**9**, Diagram 1] hence Corollary 5 is a generalisation of [**11**, Proposition 4]. ∎

References

[1] L. Baggett and A. Kleppner, 'Multiplier representations of abelian groups', *J. Funct. Anal.* **14** (1973), 299–324.

[2] L. Baggett and T. Sund, 'The Hausdorff dual problem for connected groups', *J. Funct. Anal.* **43** (1981), 60–68.

[3] J. Dixmier, *C^*-algebras* (North-Holland, Amsterdam, 1977).

[4] P. Green, 'The local structure of twisted covariance algebras', *Acta Math.* **140** (1978), 191–250.

[5] P. Green, 'The structure of imprimitivity algebras', *J. Funct. Anal.* **36** (1980), 88–104.

[6] J.R. Liukkonen, 'Dual spaces of groups with pre-compact conjugacy classes', *Trans. Amer. Math. Soc.* **180** (1973), 85–108.

[7] G.W. Mackay, 'Borel structures in groups and their duals', *Trans. Amer. Math. Soc.* **85** (1957), 134–165.

[8] G.W. Mackay, 'Unitary representations of group extensions I', *Acta Math.* **99** (1958), 265–311.

[9] T.W. Palmer, 'Classes of nonabelian, noncompact locally compact groups', *Rocky Mountain J. Math.* **8** (1978), 683–741.

[10] G. Pedersen, *C^*-algebras and their automorphism groups* (Academic Press, 1979).

[11] I. Raeburn, 'On group C^*-algebras of bounded representation dimension', *Trans. Amer. Math. Soc.* **272** (1982), 629–644.

Department of Mathematics
The University of New South Wales
Kensington NSW 2033
Australia