On the Solubility of Linear Algebraic Equations (*Contd.*)— L from P129

THEOREM II.

A system of n non-homogeneous linear equations in n variables has one and only one solution if the homogeneous system obtained from the given system by putting all the constant terms equal to zero has no solution except the null solution.

This may be proved independently by similar reasoning to that given for Theorem I., or it may be deduced from that theorem. We follow the latter method.

Take the case of $n = 3$.

Let the equations be

$$\left.\begin{aligned} a_1x + b_1y + c_1z + d_1 = 0, \\ a_2x + b_2y + c_2z + d_2 = 0, \\ a_3x + b_3y + c_3z + d_3 = 0. \end{aligned}\right\} \qquad (22)$$

The degenerate homogeneous system is

$$\left.\begin{aligned} a_1x + b_1y + c_1z = 0, \\ a_2x + b_2y + c_2z = 0, \\ a_3x + b_3y + c_3z = 0. \end{aligned}\right\} \qquad (23)$$

Consider also the allied homogeneous system

$$\left.\begin{aligned} a_1x + b_1y + c_1z + d_1w = 0, \\ a_2x + b_2y + c_2z + d_2w = 0, \\ a_3x + b_3y + c_3z + d_3w = 0. \end{aligned}\right\} \qquad (24)$$

By Theorem I., the system (24) has at least one non-null solution (X, Y, Z, W). Here X, Y, Z are not all zero; for this would imply $W \neq 0$, and therefore $d_1 = d_2 = d_3 = 0$, so that (22) would not be non-homogeneous.

Then further, W cannot be zero, or else (X, Y, Z) would be a non-null solution of (23).

$$\left(\frac{X}{W}, \frac{Y}{W}, \frac{Z}{W}\right) \text{ is one solution of (22).}$$

There cannot be another; for if (x_1, y_1, z_1) and (x_2, y_2, z_2) are two distinct solutions of (22), then obviously $(x_1 - x_2, y_1 - y_2, z_1 - z_2)$ is a non-null solution of (23).

Cor. If the degenerate homogeneous system has a non-null solution, the non-homogeneous system has either no solution, or an infinite number of solutions.

(22) may have no solution; *e.g.* all the coefficients may vanish except the d's. But if (22) have a solution (x_1, y_1, z_1), and (X, Y, Z)

is the non-null solution of (23), then $(x_1+\lambda X, y_1+\lambda Y, z_1+\lambda Z)$ is a solution of (22), where λ is arbitrary.

Example 4. *Theorem.—If A/PQ be a rational proper fraction whose denominator contains two integral factors P and Q, which are algebraically prime to each other, then we can always, and in one way only, decompose A/PQ into the sum of two proper fractions $P'/P+Q'/Q$.*

Let the degrees of P, Q be p, q, then the degree of A is $p+q-1$ at most. Assume for P′ and Q′ general forms of degrees $p-1$ and $q-1$ with undetermined coefficients. The equation

$$A=P'Q+Q'P$$

is then equivalent to $p+q$ non-homogeneous linear equations in the $p+q$ coefficients of P′ and Q′.

These equations have one solution and one only, unless the degenerate homogeneous system has a non-null solution. This would imply for the corresponding values of P′, Q′,

$$0=P'Q+Q'P,$$
$$\text{or} \quad P/Q=-P'/Q',$$

i.e. P, Q are not algebraically prime to each other, contrary to the hypothesis.

Example 5. *Theorem.—A rational proper fraction of the form B/A^n, where A and B are rational integral functions, can be thrown into the form*

$$X_1/A+X_2/A^2+\ldots+X_n/A^n,$$

where $X_1, X_2, \ldots X_n$ *are of lower degree than A.*

Proceed as in Ex. 4. If the degree of A is p, we get np non-homogeneous linear equations for the np coefficients of the functions X. These have a unique solution, for otherwise functions X not all zero would exist, giving

$$X_1/A+X_2/A^2+\ldots+X_n/A^n\equiv 0 \ldots\ldots\ldots\ldots (25)$$

But it is easy to show that this cannot be, for (25) gives

$$X_n=-A(X_{n-1}+AX_{n-2}+\ldots+A^{n-2}X_1),$$

which, since X_n is of lower degree than A, implies $X_n=0$.

Substitute $X_n=0$ in (25) and we find similarly $X_{n-1}=0$, and so on; *i.e.* all the functions X must be zero if (25) holds.

On the two preceding theorems depends the decomposability of a rational fraction into partial fractions.

JOHN DOUGALL