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CLASSIFICATION OF UNIVALENT HARMONIC MAPPINGS ON THE UNIT DISK WITH HALF-INTEGER COEFFICIENTS

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Abstract

Let $\mathcal{S}$ denote the set of all univalent analytic functions f of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ on the unit disk $|z| < 1$. In 1946, Friedman ['Two theorems on Schlicht functions', *Duke Math. J.* **13** (1946), 171–177] found that the set $\mathcal{S}_{\mathbb{Z}}$ of those functions in $\mathcal{S}$ which have integer coefficients consists of only nine functions. In a recent paper, Hiranuma and Sugawa ['Univalent functions with half-integer coefficients', *Comput. Methods Funct. Theory* **13**(1) (2013), 133–151] proved that the similar set obtained for functions with half-integer coefficients consists of only 21 functions; that is, 12 more functions in addition to these nine functions of Friedman from the set $\mathcal{S}_{\mathbb{Z}}$. In this paper, we determine the class of all normalized sense-preserving univalent harmonic mappings f on the unit disk with half-integer coefficients for the analytic and co-analytic parts of f. It is surprising to see that there are only 27 functions out of which only six functions in this class are not conformal. This settles the recent conjecture of the authors. We also prove a general result, which leads to a new conjecture.

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1. Introduction and main results

Let $\mathbb{D} = \{z : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $\mathcal{S}$ the class of all normalized analytic and univalent mappings f on $\mathbb{D}$ with the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

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In 1946, Friedman [9] proved that if $f \in \mathcal{S}$ has integer coefficients, then f is one of the nine functions from $\mathcal{S}_{\mathbb{Z}}$, where

$$\mathcal{S}_{\mathbb{Z}} = \left\{ z, \frac{z}{1 \pm z}, \frac{z}{1 \pm z^2}, \frac{z}{(1 \pm z)^2}, \frac{z}{1 \pm z + z^2} \right\}. \tag{1.1}$$

We refer to [18] (see also [25]) for a simpler proof of this result. Observe that each $f \in \mathcal{S}_{\mathbb{Z}}$ maps $\mathbb{D}$ onto a domain starlike with respect to the origin and, hence, functions in $\mathcal{S}_{\mathbb{Z}}$ are starlike in $\mathbb{D}$. In [15], Jenkins presented a different proof of this result and extended it also to functions with coefficients in an imaginary quadratic extension of the rational numbers; see also [27, 29]. In order to deal with the harmonic analog of this result, we consider the class $\mathcal{S}_H$ of all complex-valued, harmonic, sense-preserving univalent mappings $f = h + \overline{g}$ in $\mathbb{D}$, with the normalization $f(0) = 0 = h(0) = f_z(0) - 1$. Thus, h and g are analytic in $\mathbb{D}$, so that

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \tag{1.2}$$

where we write for convenience $a_0 = 0$ and $a_1 = 1$. Often h and g are referred to as the analytic and co-analytic parts of f, respectively. Also, let

$$\mathcal{S}_H^0 = \{f \in \mathcal{S}_H : f_{\overline{z}}(0) = 0\},$$

so that $\mathcal{S}_H^0 \subset \mathcal{S}_H$ and $\mathcal{S} = \{f = h + \overline{g} \in \mathcal{S}_H^0 : g(z) \equiv 0\}$. The important facts about $\mathcal{S}_H$ and $\mathcal{S}_H^0$ are that both are normal whereas only the latter one is compact with respect to the topology of local uniform convergence; see [2, 7].

Just as the class $\mathcal{S}$ has been a central object in the study of univalent function theory, $\mathcal{S}_H^0$ plays a vital role in the study of harmonic univalent mappings (see [2, 7]). We recall that (see [17]) a necessary and sufficient condition for a complex-valued harmonic function $f = h + \overline{g}$ to be locally univalent and sense-preserving in $\mathbb{D}$ is that the Jacobian $J_f(z)$ is positive in $\mathbb{D}$, where

$$J_f(z) = |f_z(z)|^2 - |f_{\overline{z}}(z)|^2 = |h'(z)|^2 - |g'(z)|^2.$$

If f is sense-preserving, then the complex dilatation $\omega := g'/h'$ is analytic in $\mathbb{D}$ and maps $\mathbb{D}$ into $\mathbb{D}$. The Bieberbach conjecture had been a driving force to develop the theory of univalent functions for a long time [6, 10, 20] and was finally solved in the affirmative by Louis de Branges in 1985. On the other hand, the corresponding coefficient conjecture for the class $\mathcal{S}_H^0$ has not been solved even for the second coefficient of the analytic part h of f [2, 7], although the analog coefficient inequalities have been proved, for example, for the classes of harmonic convex functions, harmonic starlike functions, harmonic close-to-convex functions, and typically real-harmonic functions, respectively.

We say that a harmonic function $f = h + \overline{g}$ belongs to $\mathcal{S}_H(\frac{1}{2}\mathbb{Z})$, that is, $f \in \mathcal{S}_H$ and has half-integer coefficients, if all the Taylor coefficients a_n of h and b_n of g are half-integers. Here and hereafter, a half-integer will mean half of an integer. Clearly, an integer is a half-integer in our context. In a recent paper, the present authors in [21] obtained the following surprising result as an analog of the result of Friedman.

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Theorem A. *If $f = h + \overline{g} \in \mathcal{S}_H$ have integer coefficients, then f is one of the nine functions from $\mathcal{S}_{\mathbb{Z}}$, where $\mathcal{S}_{\mathbb{Z}}$ is given by (1.1).*

The proof of Theorem A uses the subordination result due to Rogosinski [24]. We now recall the recent result of Hiranuma and Sugawa [14, Theorem 1.2], which extends the result of Friedman for functions in $\mathcal{S}$ that have half-integer coefficients.

Theorem B. *Suppose that all the Taylor coefficients a_n of a function f in $\mathcal{S}$ are half-integers. Then f is either one of the nine functions from $\mathcal{S}_{\mathbb{Z}}$ given by (1.1) or else one of the 12 functions from $\mathcal{F}$ given by*

$$\mathcal{F} = \left\{z \pm \frac{z^2}{2}, \frac{z(2 \pm z)}{2(1 \pm z)}, \frac{z(2 \pm z^2)}{2(1 \pm z^2)}, \frac{z(2 \pm z)}{2(1 - z^2)}, \frac{z(2 \pm z)}{2(1 \pm z)^2}, \frac{z(2 \pm z + z^2)}{2(1 \pm z + z^2)}\right\}. \tag{1.3}$$

The proof of Theorem B involves a lot of technical details. In view of Theorem B, it is natural to ask for an analog of Theorem B if we replace 'integers' by 'half-integers' in the assumption of Theorem A. This has led to investigation of functions in $\mathcal{S}_H^0(\frac{1}{2}\mathbb{Z})$ that have half-integer coefficients and, as a consequence of it, the present authors in [21, Conjecture 1] proposed a conjecture. One of the aims of this article is to prove this conjecture. We now state the result here.

Theorem 1.1. *Let $f \in \mathcal{S}_H^0(\frac{1}{2}\mathbb{Z})$. Then f is one of the following 27 functions from $\mathcal{S}_{\mathbb{Z}} \cup \mathcal{F} \cup \mathcal{F}_0$, where $\mathcal{S}_{\mathbb{Z}}$ and $\mathcal{F}$ are given by (1.1) and (1.3), respectively, and $\mathcal{F}_0$ is given by*

$$\left\{\mathrm{Re}\left(\frac{z}{(1 \mp z)^2}\right) + i\,\mathrm{Im}\left(\frac{z}{1 \mp z}\right), \mathrm{Re}\left(\frac{z}{1 \mp z}\right) + i\,\mathrm{Im}\left(\frac{z}{(1 \mp z)^2}\right), z \pm \overline{\frac{z^2}{2}}\right\}.$$

We remark that, in the proof of Theorem 1.1, functions in $\mathcal{F}_0$ are represented by $f_4(z), -f_4(-z), f_2(z), -f_2(-z), f_5(z), -f_5(-z)$, respectively.

We emphasize that these are the only six functions in $\mathcal{S}_H^0(\frac{1}{2}\mathbb{Z})$ that are not conformal. We see that these six functions play the role of extremal functions in different subclasses of $\mathcal{S}_H^0$.

A harmonic function f in $\mathbb{D}$ is said to be convex (respectively starlike, close-to-convex) if f is univalent and maps $\mathbb{D}$ onto a convex (respectively starlike with respect to the origin, close-to-convex) domain (see [6, 7, 10, 20]).

In [19], the authors pointed out that each $f \in \mathcal{S}_{\mathbb{Z}}$ is not only starlike in $\mathbb{D}$ but also belongs to the class $\mathcal{U}$ of normalized analytic functions in $\mathbb{D}$ satisfying the condition

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^2 - 1\right| < 1$$

for $|z| < 1$. It is proved in [14] that functions in $\mathcal{F} \backslash \{(z(2 + z + z^2)/2(1 + z + z^2)), (z(2 - z + z^2)/2(1 - z + z^2))\}$ are close-to-convex. On the other hand, it is easy to see that the two univalent functions $(z(2 + z + z^2)/2(1 + z + z^2))$ and $(z(2 - z + z^2)/2(1 - z + z^2))$ are neither close-to-convex nor belong to $\mathcal{U}$.

We denote by $\mathcal{CV}(1)$ (respectively $\mathcal{CV}(i)$) the class of univalent harmonic functions convex *in the direction of the real axis* (respectively *in the direction of the imaginary axis*). Functions in these classes are referred to as convex in real direction and convex in imaginary direction, respectively. These classes are obtained by taking, respectively, $\alpha = 0$ and $\alpha = \pi/2$, in Definition 2.1 (see Section 2). Moreover, the classes $\mathcal{CV}(1)$ and $\mathcal{CV}(i)$ have special roles in geometric function theory and each function in these geometric classes is characterized by its analytic and co-analytic parts (see Lemma C with $\alpha = 0, \pi/2$). In [21, Theorems 3 and 4], the present authors proved that the number of univalent harmonic mappings with half-integer coefficients that are either convex in real direction or convex in imaginary direction is finite. Indeed, the finiteness result is true even in a more general situation. For a subset E of the set $\mathbb{R}$ of real numbers, let $\mathcal{H}(E)$ denote the set of all normalized harmonic functions on $\mathbb{D}$ of the form

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}$$

such that a_n, $b_n \in E$ for all $n \geq 1$. Set $\mathcal{S}_H^0(E) = \mathcal{S}_H^0 \cap \mathcal{H}(E)$. Denote by $U(a, r)$ the interval $(a - r, a + r)$. If $E \cap U(a, r_0) = \{a\}$ for every $a \in E$ for some constant $r_0 > 0$ which is independent of the point a, then we will say that E is *uniformly discrete* (with bound r_0). We denote

$$\mathcal{S}_{H,\mathcal{CV}}^0(E) = \mathcal{S}_H^0(E) \cap (\mathcal{CV}(i) \cup \mathcal{CV}(1)).$$

THEOREM 1.2. *Suppose that $E \subset \mathbb{R}$ is uniformly discrete. Then $\mathcal{S}_{H,\mathcal{CV}}^0(E)$ consists of only finitely many functions.*

Theorem B depends heavily on the area theorem due to Gronwall [12] and the characterization of univalence of a normalized analytic function in terms of the Grunsky matrix. Unfortunately, there is no corresponding area theorem for the harmonic case along the lines of the proof of Theorem B and, so, it becomes necessary to consider a suitable method to obtain a proof of Theorem 1.1.

We briefly describe the organization of the paper. In Section 2, we will recall necessary lemmas that are required for the proofs of Theorems 1.1 and 1.2. In Section 3, we present a proof of Theorem 1.1; the proof uses coefficient estimates of typically real analytic functions and a result of Rogosinski on subordination. In Section 4, we present the proof of Theorem 1.2.

We end the section with a conjecture.

CONJECTURE 1.3. Suppose that $E \subset \mathbb{R}$ is uniformly discrete. Then $\mathcal{S}_H^0(E)$ consists of finitely many functions.

2. Lemmas

We will first need some background information. We begin with the following definition.

Definition 2.1. A domain $D \subset \mathbb{C}$ is called convex in the direction α $(0 \leq \alpha < \pi)$ if every line parallel to the line through 0 and $e^{i\alpha}$ has a connected intersection with D. A univalent harmonic function f in $\mathbb{D}$ is said to be *convex in the direction* α if $f(\mathbb{D})$ is convex in the direction α.

Obviously, every function that is convex in the direction α $(0 \leq \alpha < \pi)$ is necessarily close-to-convex, but the converse is not true. Clearly, a convex function is convex in every direction. The class of functions convex in one direction has been studied by many mathematicians (see for example [3, 4, 13, 16, 26]) as a subclass of functions introduced by Robertson [22].

In proving our main theorems, we will need a number of known lemmas. The first lemma is popularly known as Clunie and Sheil-Small's shear construction theorem [2, Theorem 5.3], which, in particular, produces a univalent harmonic function that maps $\mathbb{D}$ onto a domain that is convex in the direction α.

Lemma C (Method of shearing). *A harmonic function $f = h + \overline{g}$ locally univalent in $\mathbb{D}$ is a univalent mapping of $\mathbb{D}$ onto a domain convex in the direction α $(0 \leq \alpha < \pi)$ if and only if $h - e^{2i\alpha}g$ is a conformal univalent mapping of $\mathbb{D}$ onto a domain convex in the direction α.*

In particular, a locally univalent harmonic mapping $f = h + \overline{g}$ is convex in the direction of the real axis (respectively imaginary axis) if and only if $h - g$ (respectively $h + g$) is convex in the direction of the real axis (respectively imaginary axis). Greiner [11] has constructed numerous examples using the method of shearing.

The next lemma is about the coefficient estimates for univalent harmonic mappings. The coefficient conjecture for functions in $\mathcal{S}_H^0$ proposed by Clunie and Sheil-Small [2] (see also [7]) is unsolved, although the same has been verified for a number of subclasses, for example mappings that are close-to-convex, starlike, and convex in one direction. In the full class $\mathcal{S}_H^0$, however, only the sharp elementary inequality $|b_2| \leq \frac{1}{2}$ has been verified.

Lemma D [7, page 87, Theorem]. *For all functions $f \in \mathcal{S}_H^0$, the sharp inequality $|b_2| \leq \frac{1}{2}$ holds, with equality if and only if $\omega(z) = e^{i\alpha}z$ for some real α.*

The notion of subordination is an important property in analytic function theory; see [6]. For *analytic functions* f and g in $\mathbb{D}$, we say that f is subordinate to g, written $f(z) \prec g(z)$ or simply $f \prec g$, if there exists a Schwarz function φ (that is, φ is analytic in $\mathbb{D}$ with $\varphi(0) = 0$ and $|\varphi(z)| < 1$ for $z \in \mathbb{D}$) such that $f(z) = g(\varphi(z))$. The condition implies that $f(0) = g(0)$ and $|f'(0)| \leq |g'(0)|$. If, in addition, g is univalent, then $f \prec g$ if and only if $f(\mathbb{D}) \subset g(\mathbb{D})$ and $f(0) = g(0)$.

A number of results of Rogosinski are crucial in the proof of Theorem 1.1. We begin with the following result due to Rogosinski [24] (see also Duren [6, page 195, Theorem 6.4]).

Lemma E. *If $g(z) = \sum_{n=1}^{\infty} b_n z^n$ is analytic in $\mathbb{D}$ and $g \prec f$ for some convex function from $f \in \mathcal{S}$, then $|b_n| \leq 1$ for $n \geq 1$.*

A function f harmonic in $\mathbb{D}$ is said to be typically real on $\mathbb{D}$ if it assumes real values on the real axis and nonreal values elsewhere. Let $\mathcal{T}$ denote the class of all typically real functions f analytic in $\mathbb{D}$ such that $f(0) = 0 = f'(0) - 1$. It is easy to see that if $f \in \mathcal{T}$, then $\mathrm{Im}\{f(z)\} > 0$ when $\mathrm{Im}\{z\} > 0$ and $\mathrm{Im}\{f(z)\} < 0$ when $\mathrm{Im}\{z\} < 0$. Moreover, functions in $\mathcal{T}$ are not necessarily univalent in $\mathbb{D}$. In [1], Bshouty *et al.* discussed typically real harmonic mappings (see also [2] and [7, Section 6.6]). The analog of $\mathcal{T}$ for the harmonic case is the class $\mathcal{T}_H$ of sense-preserving typically real harmonic functions $f = h + \overline{g}$ such that $h(0) = g(0) = 0$, $h'(0) = 1$, and $f(r) > 0$ for $0 < r < 1$. As in the analytic case, a typically real harmonic function need not be univalent. Moreover, every $f \in \mathcal{S}_H$ with real coefficients is typically real and belongs to $\mathcal{T}_H$. See [7, Section 6.6] for further details on this class. The subclass of $\mathcal{T}_H$ for which $g'(0) = 0$ is denoted by $\mathcal{T}_H^0$. A family of typically real harmonic polynomials that has some interesting geometric properties has been discussed for example in [28] (see also [5]).

The following inequality is due to Rogosinski [23] (see also [6, page 57, Theorem 2.21]).

Lemma F. *If $f \in \mathcal{T}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then $|a_{n+2} - a_n| \le 2$ for $n = 0, 1, 2, \ldots$.*

A closer examination of the proof of Lemma F gives the following result (see also [6, page 58, Corollary]).

Lemma G. *If $f \in \mathcal{T}$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then $|a_n| \le n$ for $n = 2, 3, \ldots$. Strict inequality occurs for all even n unless f is the Koebe function $k(z) = z/(1-z)^2$ or its real rotation $-k(-z)$. Strict inequality occurs for all odd n unless f is a convex combination of these two functions.*

The final lemma due to FitzGerald [8] gives another necessary condition for the coefficients for typically real analytic functions.

Lemma H. *Suppose that $f \in \mathcal{T}$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then the coefficients of f satisfy the inequality $a_n^2 \le 1 + a_3 + \cdots + a_{2n-1}$ for $n = 2, 3, \ldots$.*

3. The proof of Theorem 1.1

Let $f = h + \overline{g} \in \mathcal{S}_H^0$, where h and g have the standard normalization given by (1.2), and a_n, b_n are half-integers. By Lemma D, $|b_2| \le 1/2$. Since b_2 is a half-integer, we must have $b_2 = 0$, $b_2 = 1/2$, or $b_2 = -1/2$.

Case 3.1. The case $b_2 = 0$.

Now, we claim that $g(z) \equiv 0$ in $\mathbb{D}$. Set $\varphi = h - g$. Then $\varphi'(z) = h'(z) - g'(z) \neq 0$, since f is sense-preserving in $\mathbb{D}$. Suppose on the contrary that g is not identically zero. Because f is sense-preserving, we have $|h'| = |g' + \varphi'| > |g'|$ and, therefore,

$$\left|\frac{g'}{\varphi'} + 1\right| > \left|\frac{g'}{\varphi'}\right|, \quad \text{that is,}\ \mathrm{Re}\left\{\frac{g'(z)}{\varphi'(z)}\right\} > -\frac{1}{2} \quad \text{for } z \in \mathbb{D}.$$

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In terms of subordination, we may rewrite the last inequality as

$$\frac{g'(z)}{\varphi'(z)} \prec \frac{z}{1-z} \quad \text{for } z \in \mathbb{D}. \tag{3.1}$$

Let $n_0 = \min\{n : b_n \neq 0\}$ and observe that

$$\varphi'(z) = 1 + \sum_{n=2}^{\infty} n(a_n - b_n)z^{n-1} \neq 0 \quad \text{in } \mathbb{D},$$

so that $1/\varphi'$ can be written in power series as

$$\frac{1}{\varphi'(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}.$$

Then $b_{n_0} \neq 0$ for $n_0 > 2$ and, therefore, we may write

$$\frac{g'(z)}{\varphi'(z)} = n_0 b_{n_0} z^{n_0-1} + \sum_{n=n_0}^{\infty} d_n z^n \quad \text{for } z \in \mathbb{D}.$$

By Lemma E and (3.1), we deduce that $|n_0 b_{n_0}| \leq 1$. Since b_{n_0} is a half-integer and $n_0 > 2$, it follows that $b_{n_0} = 0$, which is a contradiction. Thus, we conclude that $g(z) \equiv 0$. Hence, f reduces to an analytic function in $\mathcal{S}_H^0(\frac{1}{2}\mathbb{Z})$, and it follows from Theorem B that $f \in \mathcal{S}_{\mathbb{Z}} \cup \mathcal{F}$.

CASE 3.2. The case $b_2 = \frac{1}{2}$.

Since $b_2 = 1/2$, by Lemma D, we deduce that $\omega(z) = e^{i\alpha}z$. By the condition $g'(z) = \omega(z)h'(z)$, we must have $e^{i\alpha} = 1$ and, hence, $\omega(z) = z$. As a consequence, h and g are related by $g'(z) = zh'(z)$, which gives the Taylor coefficients of g, in terms of the coefficients of h, as

$$b_n = \frac{(n-1)a_{n-1}}{n} \quad (n \geq 2), \tag{3.2}$$

which is a half-integer.

Since $f = h + \overline{g} \in \mathcal{S}_H^0(\frac{1}{2}\mathbb{Z})$, it follows from [2, page 22, 6.2] and [2, page 22, 6.3] that f and $h - g$ are typically real functions with half-integer coefficients. But then, by [2, page 23, Theorem 6.4],

$$|a_n| \leq \tfrac{1}{6}(n+1)(2n+1) \tag{3.3}$$

for $n = 2, 3, \ldots$. Since $h - g$ is typically real, by using Lemma F, we obtain

$$|(a_{n+2} - b_{n+2}) - (a_n - b_n)| \leq 2 \tag{3.4}$$

for $n = 0, 1, 2, \ldots$, where we assume that $a_0 = b_0 = b_1 = 0$ and $a_1 = 1$. Also, by Lemma H,

$$(a_n - b_n)^2 \leq 1 + (a_3 - b_3) + \cdots + (a_{2n-1} - b_{2n-1}) \tag{3.5}$$

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for $n = 2, 3, \ldots$. Equations (3.2)–(3.5) will be used frequently in the proof of Subcase 3.4.

By Lemma G, we observe that $h(z) - g(z)$ is a function in the set A of the one-parameter family of functions given by

$$A := \{k_t(z) = tk(z) - (1-t)k(-z) \text{ with } t \in [0, 1]\},$$

so that $k_0(z) = -k(-z)$ and $k_1(z) = k(z)$, or the Taylor coefficients of $h(z) - g(z)$ satisfy the strict inequality

$$|a_n - b_n| < n \quad \text{for } n = 2, 3, \ldots.$$

Here $k(z) = z/(1-z)^2$. In the following, we divide this case into two subcases.

Subcase 3.3. Let $h(z) - g(z)$ belong to A.

Solving $g'(z) = zh'(z)$ together with $\varphi(z) = h(z) - g(z)$ gives us the harmonic function $f(z)$ in a convenient form (since $h'(z) = \varphi'(z)/(1-z)$):

$$f(z) = h(z) + \overline{g(z)} = 2\,\mathrm{Re}\,h(z) - \overline{\varphi(z)} \quad \text{with } h(z) = \int_0^z \frac{\varphi'(t)}{1-t}\,dt. \tag{3.6}$$

Evaluating the integral in (3.6) with $\varphi(z) = k_1(z) = k(z) := z/(1-z)^2$ yields the harmonic function

$$f_1(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} + \overline{\frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}},$$

which is indeed the well-known harmonic Koebe function (with the dilatation $\omega(z) = z$). The function $f_1(z)$ is convex in the real direction but has no half-integer coefficients.

As in the previous case, it follows easily that for $\varphi(z) = k_0(z) = -k(-z)$,

$$f_2(z) = \frac{z(2+z)}{2(1+z)^2} + \overline{\frac{z^2}{2(1+z)^2}} = \mathrm{Re}\left(\frac{z}{1+z}\right) + i\,\mathrm{Im}\left(\frac{z}{(1+z)^2}\right).$$

Applying Lemma C with $\alpha = \pi/2$, it can be easily seen that the function $f_2(z)$ is convex in the real direction and has half-integer coefficients.

When $\varphi(z) = tk(z) - (1-t)k(-z)$, $t \in (0, 1)$, the analytic part $h(z)$ in (3.6) takes the form

$$h(z) = t\frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} + (1-t)\frac{z(2+z)}{2(1+z)^2}$$

and a computation quickly gives the Taylor coefficients of h as

$$a_n(t) = \frac{t}{6}(n+1)(2n+1) + (1-t)(-1)^{n+1}\frac{n+1}{2}.$$

Also,

$$g(z) = t\frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} + (1-t)\frac{z^2}{2(1+z)^2}$$

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with its coefficients

$$b_n(t) = \frac{t}{6}(n-1)(2n-1) + (1-t)(-1)^n\frac{n-1}{2}.$$

Writing $a_n(t)$ and $b_n(t)$ as

$$a_n(t) = \frac{n+1}{2}\left[\frac{2t}{3}\left(n + \frac{1+3(-1)^n}{2}\right) + (-1)^{n+1}\right]$$

and

$$b_n(t) = \frac{n-1}{2}\left[\frac{2t}{3}\left(n - \frac{1+3(-1)^n}{2}\right) + (-1)^{n}\right],$$

it follows easily that the corresponding harmonic function f does not have half-integer coefficients when $t \neq \frac{3}{4}$. Now, we need to deal with the case $t = \frac{3}{4}$. Thus, if $t = \frac{3}{4}$, then we obtain the corresponding harmonic function f_3 in the form

$$\begin{aligned} f_3(z) &= h_3(z) + \overline{g_3(z)} \\ &= \frac{3}{4}\left(\frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}\right) + \frac{z(2+z)}{8(1+z)^2} + \overline{\frac{3}{4}\left(\frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}\right) + \frac{z^2}{8(1+z)^2}} \end{aligned}$$

or, equivalently, $f_3(z) = \mathrm{Re}(h_3(z) + g_3(z)) + i\,\mathrm{Im}(h_3(z) - g_3(z))$, so that

$$h_3(z) + g_3(z) = \frac{3}{4}\left(\frac{z + \frac{1}{3}z^3}{(1-z)^3}\right) + \frac{1}{4}\left(\frac{z}{1+z}\right)$$

and

$$h_3(z) - g_3(z) = \frac{3}{4}\left(\frac{z}{(1-z)^2}\right) + \frac{1}{4}\left(\frac{z}{(1+z)^2}\right),$$

so that $\mathrm{Re}\{((1-z^2)/z)(h_3(z) - g_3(z))\} > 0$ in $\mathbb{D}$. Moreover, a computation gives the Taylor coefficients of h_3 and g_3 as

$$a_n(3/4) = \begin{cases} \dfrac{(n+1)^2}{4} & \text{when } n \text{ is odd,} \\ \dfrac{n(n+1)}{4} & \text{when } n \text{ is even} \end{cases}$$

and

$$b_n(3/4) = \begin{cases} \dfrac{(n-1)^2}{4} & \text{when } n \text{ is odd,} \\ \dfrac{n(n-1)}{4} & \text{when } n \text{ is even,} \end{cases}$$

showing that the function f_3 has half-integer coefficients in the case $t = \frac{3}{4}$. We observe that $f_3(z) = \overline{f_3(\overline{z})}$ for all $z \in \mathbb{D}$ and $f_3 = h_3 + \overline{g_3}$ belongs to the class $\mathcal{T}_H^0$ of the normalized sense-preserving typically real harmonic functions introduced in Section 2. We shall now prove that f_3 is not univalent in $\mathbb{D}$. Indeed, the analytic part h_3 of f_3 has the derivative

$$h_3'(z) = \frac{N(z)}{(1-z)^4(1+z)^3}, \quad N(z) = 1 + 2z + 6z^2 + 2z^3 + z^4.$$

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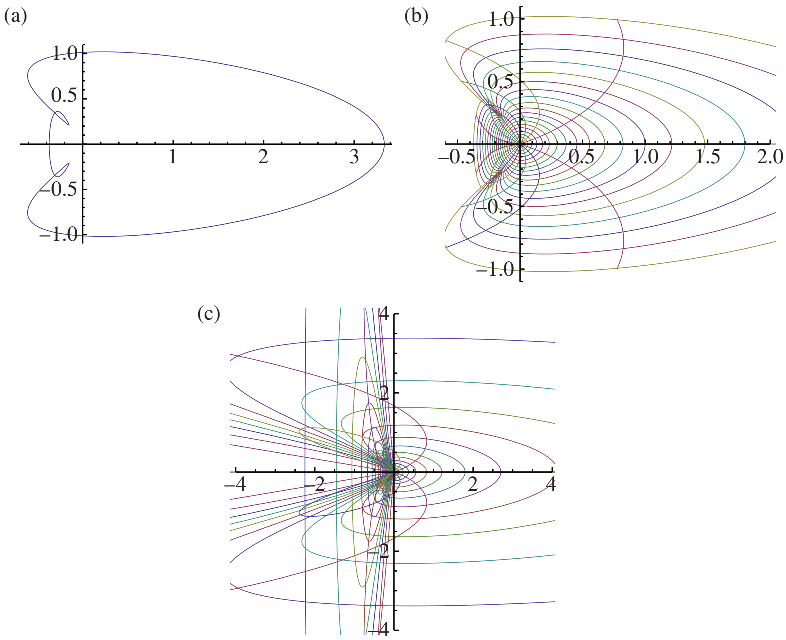

FIGURE 1. Images of the circle of radius $1/2$ and the open disks of radii $1/2$ and 1 under $f_3(z)$.

The numerator $N(z)$ has two zeros in the unit disk, which are given by

$$\begin{aligned} z_1 &= \frac{-1+i\sqrt{3}}{2} + \sqrt{\frac{-3-i\sqrt{3}}{3}} \\ &= \frac{-1+i\sqrt{3}}{2} + \sqrt[4]{3}\Big(\frac{\sqrt{6}-\sqrt{2}}{4} - i\sqrt{\frac{2+\sqrt{3}}{2}}\Big) \\ &\approx -0.159375 - 0.405204i \end{aligned}$$

and $z_2 = \overline{z_1}$. Thus, by Lewy's theorem, f_3 cannot be univalent in $\mathbb{D}$.

Images of the circle of radius $1/2$ and the open disks of radii $1/2$ and 1 under $f_3(z)$ are shown in Figure 1(a)–(c).

SUBCASE 3.4. The case $|a_n - b_n| < n$ for $n = 2, 3, \ldots$.

CLAIM 3.5. Either $a_n = 0$ or $a_n = (n+1)/2$ for $n = 2, 3, \ldots$.

We prove the claim by using the principle of induction. In view of the complexity of the proof, it is required to prove this claim first for $n = 2, 3, 4, 5$ and then for $n \geq 6$.

The case $n = 2$. We begin to prove the case $n = 2$. According to (3.3) and the assumption, we must have $|a_2| \leq \frac{5}{2}$ and $|a_2 - b_2| < 2$. Since $(2a_2)/3$ has to be a half integer by (3.2),

$$a_2 \in \{0, 3/2, -3/2\}.$$

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It follows from the inequality $|a_2 - b_2| < 2$ with $b_2 = \frac{1}{2}$ that either $a_2 = 0$ or $a_2 = \frac{3}{2}$, since $a_2 = -3/2$ is not possible. From (3.2),

$$b_3 = \begin{cases} 0 & \text{when } a_2 = 0, \\ 1 & \text{when } a_2 = 3/2. \end{cases} \tag{3.7}$$

The case $n = 3$. Inequality (3.3) for $n = 3$ gives $|a_3| \le \frac{14}{3}$, and (3.2) for $n = 4$ implies that $(3a_3)/2$ is an integer. Also, $|a_3 - b_3| < 3$ with $b_3 = 0$ or $b_3 = 1$, from which we obtain that either $a_3 = 0$ or $a_3 = 2$. Again from (3.2), a computation gives

$$b_4 = \begin{cases} 0 & \text{when } (a_3, b_3) = (0, 0), \\ 3/2 & \text{when } (a_3, b_3) = (2, 0) \text{ or } (a_3, b_3) = (2, 1) \end{cases}$$

and, moreover, $b_4 = -3/2$ when $(a_3, b_3) = (-2, 0)$, which is clearly not possible by the condition (3.4) for $n = 1$.

The case $n = 4$. From the case $n = 3$, we see that there are only two choices for b_4, namely 0 or 3/2. Now, by (3.2) and (3.3), we have $(4a_4/5) \in \frac{1}{2}\mathbb{Z}$ and $|a_4| \le \frac{15}{2}$, respectively.

In the case $b_4 = \frac{3}{2}$, the inequality $|a_4 - b_4| < 4$ is equivalent to $a_4 \in (-5/2, 11/2)$ with $2a_4 \in \mathbb{Z}$ and $(4a_4/5) \in \frac{1}{2}\mathbb{Z}$. This gives

$$a_4 \in \{0, 5/2, 5\}.$$

But the inequality (3.4) with $n = 2$ yields that $a_4 = 5$ is not possible.

In the case $b_4 = 0$, the inequality $|a_4 - b_4| < 4$ reduces to $a_4 \in (-4, 4)$ with $2a_4 \in \mathbb{Z}$. This gives

$$a_4 \in \{0, \pm 1/2, \pm 1, \pm 3/2, \pm 2, \pm 5/2, \pm 3, \pm 7/2\}$$

and, because $(4a_4/5) \in \frac{1}{2}\mathbb{Z}$, the choices of a_4 reduce to

$$a_4 \in \{0, \pm 5/2\}.$$

Next, we will prove that $a_4 = -\frac{5}{2}$ is also not possible (with $b_4 = 0$), so that

$$a_4 \in \{0, 5/2\}$$

and thus, the claim for $n = 4$ holds. Thus, it suffices to show that $a_4 \ne -\frac{5}{2}$. Suppose on the contrary that $a_4 = -\frac{5}{2}$. Then $b_5 = -2$ (with $b_4 = 0$), by (3.2) for $n = 5$.

By using (3.2) for $n = 4$, we obtain $a_3 = 0$. Consequently, (3.5) for $n = 2$ (with $b_2 = 1/2$ and $a_3 = 0$) gives

$$(a_2 - 1/2)^2 \le 1 - b_3,$$

which, because of (3.7), implies that $a_2 = 0$ (observe that for $a_2 = 3/2$ and $b_3 = 1$ the last inequality does not hold) and, hence, $b_3 = 0$. Further, it follows from the inequality (3.4) for $n = 3$ that

$$|(a_5 - b_5) - (a_3 - b_3)| = |(a_5 + 2) - (0 - 0)| = |a_5 + 2| \le 2$$

and also the fact that $2b_6 = (5a_5/3) \in \mathbb{Z}$ gives $a_5 = 0$ or $a_5 = -3$.

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We see that $a_5 = -3$ is not possible. Indeed, if $a_5 = -3$, then $b_6 = -\frac{5}{2}$. Also (see (3.4) with $n = 4$),

$$|(a_6 - b_6) - (a_4 - b_4)| = |a_6 + (5/2) - (-5/2 - 0)| = |a_6 + 5| \le 2$$

and $(12a_6/7) \in \mathbb{Z}$ (since $2b_7 \in \mathbb{Z}$), so we have $a_6 = -\frac{7}{2}$ or $a_6 = -7$.

We shall now show that the function corresponding to the case $a_6 = -\frac{7}{2}$ is not univalent in $\mathbb{D}$, whereas the case $a_6 = -7$ is not possible.

First we let $a_6 = -\frac{7}{2}$. Our previous assumption is $a_5 = -3$. Then, by using induction, we will prove that $a_n = -(n+1)/2$ and $b_n = -(n-1)/2$ with $n = 6, 7, 8, \ldots$. Suppose that $a_m = -(m+1)/2$ for some $m \ge 6$. Then

$$b_{m+1} = \frac{ma_m}{m+1} = -\frac{m}{2}.$$

Using (3.4) with $n = m + 1$ gives

$$|(a_{m+1} - b_{m+1}) - (a_{m-1} - b_{m-1})| = |a_{m+1} + m/2 + 1| \le 2;$$

that is,

$$-m - 6 \le 2a_{m+1} \le -m + 2.$$

Note that $2b_{m+2} = ((2(m+1)a_{m+1})/(m+2)) \in \mathbb{Z}$. It follows that $a_{m+1} = -(m+2)/2$. Hence, we obtain the following function (with $a_5 = -3$, $b_6 = -5/2$; $a_4 = -5/2$, $b_5 = -2$; $a_3 = 0 = b_4$; $a_2 = 0 = b_3$):

$$\begin{aligned} f_{4,1}(z) &= h_{4,1}(z) + \overline{g_{4,1}(z)} \\ &= z - \sum_{n=4}^{\infty} \frac{n+1}{2} z^n + \overline{\left(\frac{1}{2}z^2 - \sum_{n=5}^{\infty} \frac{n-1}{2} z^n\right)} \\ &= 2z + \frac{3}{2}z^2 + 2z^3 - \frac{1}{2}\left(\frac{z}{(1-z)^2} + \frac{z}{1-z}\right) \\ &\quad + \overline{\left(z^2 + z^3 + \frac{3}{2}z^4 - \frac{1}{2}\left(\frac{z}{(1-z)^2} - \frac{z}{1-z}\right)\right)}. \end{aligned} \tag{3.8}$$

We next show that $f_{4,1}(z)$ is not univalent in $\mathbb{D}$. In order to prove this, we first observe that $f_{4,1}(z) = \overline{f_{4,1}(\overline{z})}$ for all $z \in \mathbb{D}$ and, therefore,

$$\operatorname{Re} f_{4,1}(re^{i\theta}) = \operatorname{Re} f_{4,1}(re^{-i\theta}) \quad \text{for each } r \in (0,1) \text{ and } \theta \in (0, 2\pi).$$

Thus, to show that $f_{4,1}(z)$ is not univalent in $\mathbb{D}$, it suffices to show that there exist an $r_1 \in (0,1)$ and a $\theta_1 \in (0, 2\pi)$ such that

$$\operatorname{Im} f_{4,1}(r_1 e^{i\theta_1}) = 0 = -\operatorname{Im} f_{4,1}(r_1 e^{-i\theta_1}).$$

Since

$$h_{4,1}(z) - g_{4,1}(z) = 2z + \frac{1}{2}z^2 + z^3 - \frac{3}{2}z^4 - \frac{z}{1-z},$$

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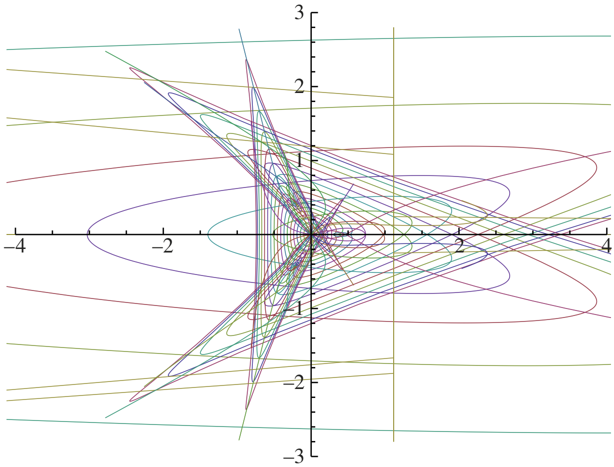

FIGURE 2. The graph of f_4.

from the definition of $f_{4,1}$, it follows by setting $z = re^{i\theta} \in \mathbb{D}$ that

$$\begin{aligned}\operatorname{Im} f_{4,1}(re^{i\theta}) &= \operatorname{Im}(h_{4,1}(re^{i\theta}) - g_{4,1}(re^{i\theta})) \\ &= 2r\sin\theta + \frac{r^2\sin 2\theta}{2} + r^3\sin 3\theta - \frac{3r^4\sin 4\theta}{2} - \frac{r\sin\theta}{1 + r^2 - 2r\cos\theta}\end{aligned}$$

and, thus, $\operatorname{Im} f_{4,1}(re^{i\pi/6}) = r\phi_{4,1}(r)$, where

$$\phi_{4,1}(r) = 1 + \frac{\sqrt{3}}{4}r + r^2 - \frac{3\sqrt{3}}{4}r^3 - \frac{1}{2(1 - \sqrt{3}\,r + r^2)}.$$

We see that $r_1 \approx 0.500966$ is the root of the equation $\phi_{4,1}(r) = 0$ in the interval $(0, 1)$. Thus,

$$\operatorname{Im} f_{4,1}(r_1e^{i\pi/6}) = 0 = -\operatorname{Im} f_{4,1}(r_1e^{-i\pi/6}),$$

showing that the function $f_{4,1}(z)$ is not univalent in $\mathbb{D}$. As an alternate proof of this fact, we refer to Remarks 3.7(a) below. The graph of $f_{4,1}$ is also shown in Figure 2.

Now, we will prove that $a_6 \neq -7$. Suppose on the contrary that $a_6 = -7$. Then (with $b_7 = -6$, $a_5 = -3$, and $b_5 = -2$), it follows from the inequality

$$|(a_7 - b_7) - (a_5 - b_5)| = |a_7 + 6 - (-3 + 2)| = |a_7 + 7| \leq 2$$

that $a_7 \in [-9, -5] \cap \frac{1}{2}\mathbb{Z}$,

$$a_7 \in \{-9, -17/2, -8, -15/2, -7, -13/2, -6, -11/2, -5\}.$$

This leads to $a_7 = -8$, since $2b_8 = \frac{7}{4}a_7 \in \mathbb{Z}$. Setting $a_7 = -8$ gives $b_8 = -7$. By using the inequality

$$|(a_8 - b_8) - (a_6 - b_6)| = |a_8 + 7 - (-7 + 2.5)| = |a_8 + 11.5| \leq 2,$$

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we deduce that $-13.5 \le a_8 \le -9.5$, which never implies that $a_8 = -9$. This contradiction shows that $a_6 \neq -7$.

Thus, $a_5 = 0$ and, therefore, $b_6 = 0$. Using (3.4) with $n = 4$,

$$|(a_6 - b_6) - (a_4 - b_4)| = |a_6 + (5/2)| \le 2,$$

which, because $(12a_6/7) \in \mathbb{Z}$, gives $a_6 = -\frac{7}{2}$ and, thus, $b_7 = -3$ by (3.2). Again,

$$|(a_7 - b_7) - (a_5 - b_5)| = |a_7 + 3 - (0 + 2)| = |a_7 + 1| \le 2,$$

which, because $(7a_7/4) \in \mathbb{Z}$, gives $a_7 = 0$. Finally, it follows that

$$\begin{aligned}(a_4 - b_4)^2 = \tfrac{25}{4} &> 1 + (a_3 - b_3) + (a_5 - b_5) + (a_7 - b_7)\\ &= 1 + (0 - 0) + (0 + 2) + (0 + 3) = 6,\end{aligned}$$

which contradicts (3.5) for $n = 4$. This contradiction shows that $a_4 \neq -\frac{5}{2}$. Hence, we have either $a_4 = 0$ or $a_4 = \frac{5}{2}$. Consequently,

$$b_5 = \begin{cases} 0 & \text{when } a_4 = 0,\\ 2 & \text{when } a_4 = 5/2.\end{cases}$$

The case $n = 5$. By using the fact that $(5a_5/6) \in \frac{1}{2}\mathbb{Z}$, and also $|a_5 - b_5| < 5$ with $b_5 = 0$ or $b_5 = 2$, it follows that $a_5 \in \{0, 3, -3, 6\}$. But the inequality (3.4) with $n = 3$ yields that $a_5 = 6$ does not hold. Next, suppose that $a_5 = -3$. Then, the inequality $|a_5 - b_5| < 5$ with $b_5 = 0$ or $b_5 = 2$ gives $b_5 = 0$. Using (3.5) with $n = 3$,

$$(a_3 - b_3)^2 < 1 + (a_3 - b_3) + (-3 - 0),$$

which is impossible because $(a_3 - b_3)^2 - (a_3 - b_3) + 2 = (a_3 - b_3 - 1/2)^2 + 7/4 > 0$. Hence, $a_5 = -3$ cannot occur. Note that

$$b_6 = \begin{cases} 0 & \text{when } a_5 = 0,\\ 5/2 & \text{when } a_5 = 3.\end{cases}$$

The case $n \ge 6$. We assume that $a_m = 0$ or $a_m = (m + 1)/2$ for $2 \le m \le n$. Then

$$b_{n+1} = \begin{cases} 0 & \text{when } a_n = 0,\\ n/2 & \text{when } a_n = (n + 1)/2.\end{cases}$$

Since $|a_{n+1} - b_{n+1}| < n + 1$, and also $(2(n + 1)a_{n+1})/(n + 2)$ is an integer, it follows that

$$a_{n+1} \in \{0, (n + 2)/2, -(n + 2)/2, n + 2\}.$$

If $a_{n+1} = -(n + 2)/2$, then by using $|a_{n+1} - b_{n+1}| < n + 1$ we deduce that $b_{n+1} = 0$, which implies that $a_n = 0$. Thus, it follows from (3.4) that

$$a_{n-1} = 0 \quad \text{and} \quad b_{n-1} = \frac{n - 2}{2}.$$

Then $b_n = 0$, $a_{n-2} = (n-1)/2$ and hence we obtain $b_{n-2} = (n-3)/2$, $a_{n-3} = (n-2)/2$, since

$$|(a_n - b_n) - (a_{n-2} - b_{n-2})| \leq 2 \quad \text{(see (3.4))}.$$

Therefore (see (3.4)),

$$|(a_{n-1} - b_{n-1}) - (a_{n-3} - b_{n-3})| = |(n-2) - b_{n-3}| > 2, \quad \text{since } n \geq 6.$$

This contradiction shows that $a_{n+1} \neq -(n+2)/2$. Now we will prove that $a_{n+1} \neq n+2$. Suppose not. Then $a_{n+1} = n+2$. We recall that

$$|(a_{n+1} - b_{n+1}) - (a_{n-1} - b_{n-1})| \leq 2 \quad \text{(see (3.4))}, \tag{3.9}$$

where $b_{n+1} = 0$ or $n/2$; $a_{n-1} = 0$ or $n/2$; and $b_{n-1} = 0$ or $(n-2)/2$. When $b_{n+1} = 0$, it can be easily seen that the inequality (3.9) does not hold. If $b_{n+1} = n/2$, then one has $a_{n+1} - b_{n+1} = (n/2) + 2$ and so, we need only to discuss the case $a_{n-1} = n/2$ and $b_{n-1} = 0$, since in other cases, it is easy to verify that (3.9) does not hold. In the case $a_{n-1} = n/2$ and $b_{n-1} = 0$, we have $a_n = (n+1)/2$, $b_n = (n-1)/2$, $a_{n-2} = 0$. Thus, by using the inequality

$$|(a_n - b_n) - (a_{n-2} - b_{n-2})| = |1 + b_{n-2}| \leq 2 \quad (n \geq 6),$$

we have $b_{n-2} = 0$, which shows that $a_{n-3} = 0$. Hence,

$$|(a_{n-1} - b_{n-1}) - (a_{n-3} - b_{n-3})| = |(n/2) + b_{n-3}| \geq 3,$$

which contradicts (3.4). Therefore, $a_{n+1} \neq n+2$. Hence, $a_n = 0$ or $a_n = (n+1)/2$ for $n \geq 6$. Then, by induction, the claim follows.

The remaining part of the proof of Case 1 is divided into three subcases first (together with the fact that $b_2 = 1/2$).

Case (i). $a_2 = 3/2$.

Case (ii). $a_2 = a_3 = 0$.

Case (iii). $a_2 = 0$ and $a_3 = 2$.

Case (i). If $a_2 = 3/2$, then $b_3 = 1$ and so, by using the inequality

$$(a_2 - b_2)^2 \leq 1 + (a_3 - b_3),$$

we obtain that $a_3 = 2$. Thus, $b_4 = 3/2$ and, from the inequality (see (3.4) with $n = 2$)

$$|(a_4 - b_4) - (a_2 - b_2)| = |a_4 - (5/2)| \leq 2,$$

it follows that $a_4 = 5/2$. Again, the inequality (see (3.4) with $n = 3$)

$$|(a_5 - b_5) - (a_3 - b_3)| = |a_5 - 3| \leq 2$$

yields $a_5 = 3$. Using the method of induction and the inequalities (see (3.4))

$$|(a_n - b_n) - (a_{n-2} - b_{n-2})| \leq 2 \quad (n \geq 6),$$

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we have that $a_n = (n+1)/2$. Thus, we have ended up with the harmonic function

$$\begin{aligned} f_4(z) &= h_4(z) + \overline{g_4(z)} \\ &= z + \sum_{n=2}^{\infty} \frac{n+1}{2} z^n + \overline{\sum_{n=2}^{\infty} \frac{n-1}{2} z^n} \\ &= \frac{1}{2}\Big(\frac{z}{(1-z)^2} + \frac{z}{1-z}\Big) + \frac{1}{2}\overline{\Big(\frac{z}{(1-z)^2} - \frac{z}{1-z}\Big)} \\ &= \mathrm{Re}\Big(\frac{z}{(1-z)^2}\Big) + i\,\mathrm{Im}\Big(\frac{z}{1-z}\Big). \end{aligned}$$

Recall again that $zh_4'(z) = g_4'(z)$ and $h_4(z) - g_4(z) = z/(1-z)$, by Lemma C (with $\alpha = 0$); it follows that f_4 is univalent in $\mathbb{D}$ and maps $\mathbb{D}$ onto a domain convex in the real direction.

Case (ii). Let $a_2 = 0$ and $a_3 = 0$. Then, $b_3 = 0$, $b_4 = 0$, and, from the inequality

$$|(a_4 - b_4) - (a_2 - b_2)| = |a_4 + (1/2)| \le 2 \quad \text{(see (3.4) with } n = 2),$$

it follows that $a_4 = 0$ and, hence, by (3.2), $b_5 = 0$. Then the inequality (3.4) with $n = 3$ becomes

$$|(a_5 - b_5) - (a_3 - b_3)| = |a_5| \le 2,$$

which gives $a_5 = 0$. By using the induction and the inequalities (3.4), we obtain that $a_n = 0$ for all $n \ge 6$. Thus,

$$f_5(z) = z + \overline{\frac{z^2}{2}},$$

which is obviously univalent in $\mathbb{D}$.

Case (iii). Let $a_2 = 0$ and $a_3 = 2$. Then, $b_3 = 0$, $b_4 = 3/2$, and the inequality (see (3.5))

$$(a_3 - b_3)^2 \le 1 + (a_3 - b_3) + (a_5 - b_5)$$

reduces to $1 \le a_5 - b_5$, which gives $a_5 = 3$ (and therefore $b_6 = 5/2$). Now, there are two possibilities for a_4, that is, either $a_4 = 0$ or $a_4 = 5/2$.

In the case $a_4 = 0$, the inequality (see (3.4) with $n = 4$)

$$|(a_6 - b_6) - (a_4 - b_4)| = |a_6 - 1| \le 2$$

implies that $a_6 = 0$. Similarly (see (3.4) with $n = 5$),

$$|(a_7 - b_7) - (a_5 - b_5)| = |a_7 - 3| \le 2$$

yields $a_7 = 4$ and (see (3.4) with $n = 6$)

$$|(a_8 - b_8) - (a_6 - b_6)| = |a_8 - 1| \le 2$$

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shows that $a_8 = 0$. By using the induction and the inequalities (3.4), we can easily obtain that

$$a_{2n} = 0 \quad \text{and} \quad a_{2n-1} = n \text{ for all } n \geq 1.$$

Therefore,

$$\begin{aligned} f_6(z) &= z + \sum_{n=2}^{\infty} n z^{2n-1} + \overline{\frac{1}{2}z^2 + \sum_{n=2}^{\infty} \frac{2n-1}{2} z^{2n}} \\ &= \frac{z}{(1-z^2)^2} + \overline{\frac{z^2(1+z^2)}{2(1-z^2)^2}}. \end{aligned}$$

We next show that the function f_6 is not univalent in $\mathbb{D}$. Indeed, the analytic part h_6 of f_6, namely, $h_6(z) = z/(1-z^2)^2$, has the derivative

$$h_6'(z) = \frac{1+3z^2}{(1-z^2)^3}$$

and, thus, $h_6'(i/\sqrt{3}) = 0$. Again, by Lewy's theorem, f_6 cannot be univalent in $\mathbb{D}$. In Figure 3, we have drawn the images of the rays $re^{i(\pi/3)}$ and $re^{i(2\pi/3)}$ under $f_6(z)$ for $0 < r \leq 1$. Moreover, from Figure 3, we can see that there are three pairs of points (r, s) other than $(0, 0)$ such that $f_6(re^{i(\pi/3)}) = f_6(se^{i(2\pi/3)})$. In Figures 4 and 5, we have also drawn $f_6(\mathbb{D}_r)$ and $f_6(C_r)$, the images of the disk of radius r and the image of the circle of radius r for different values of r under f_6.

In the case $a_4 = 5/2$ (and, hence, $b_5 = 2$), the inequality (see (3.4) with $n = 4$)

$$|(a_6 - b_6) - (a_4 - b_4)| = |a_6 - (7/2)| \leq 2$$

shows that $a_6 = 7/2$ (and, hence, $b_7 = 3$). Thus, the inequality (see (3.4) with $n = 5$)

$$|(a_7 - b_7) - (a_5 - b_5)| = |a_7 - 4| \leq 2$$

clearly implies that $a_7 = 4$. Finally, by using the induction and the inequalities (3.4), we can easily see that $a_n = (n+1)/2$ with $n \geq 8$. Thus, we end up with the harmonic function $f_7(z) = h_7(z) + \overline{g_7(z)}$, where

$$h_7(z) = z + \sum_{n=3}^{\infty} \frac{n+1}{2} z^n = \frac{1}{2}\left(\frac{z}{(1-z)^2} + \frac{z}{1-z}\right) - \frac{3}{2}z^2$$

and

$$g_7(z) = \frac{1}{2}z^2 + \sum_{n=4}^{\infty} \frac{n-1}{2} z^n = \frac{1}{2}\left(\frac{z}{(1-z)^2} - \frac{z}{1-z}\right) - z^3.$$

We claim that the function f_7 is not univalent in $\mathbb{D}$. As in the case of $f_{4,1}$, it suffices to show that

$$\operatorname{Im} f_7(re^{i\theta}) = 0 = -\operatorname{Im} f_7(re^{-i\theta})$$

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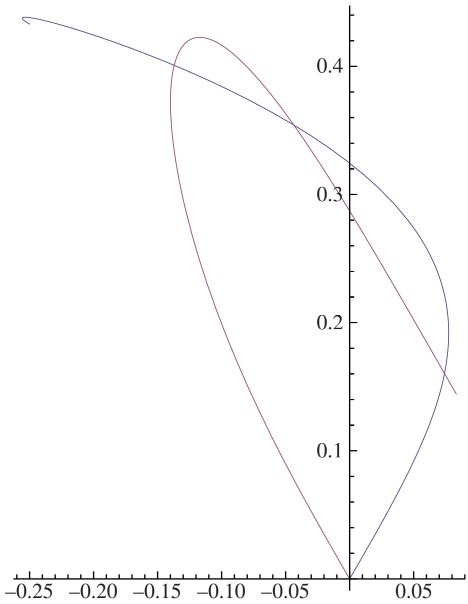

FIGURE 3. Images of the rays $re^{i(\pi/3)}$ and $re^{i(2\pi/3)}$ under $f_6(z)$.

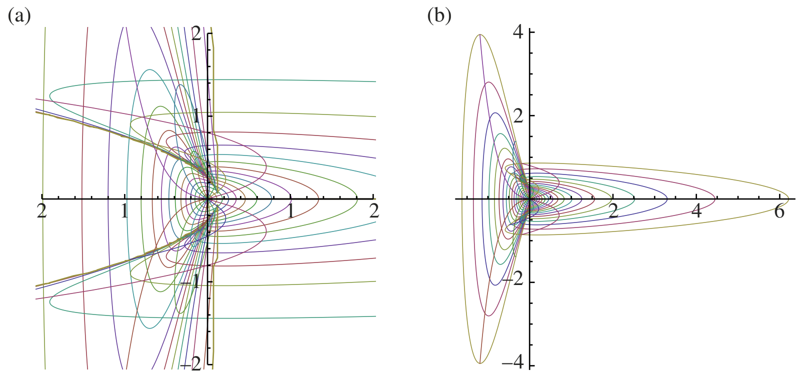

FIGURE 4. The images $f_6(\mathbb{D})$ and $f_6(\mathbb{D}_{3/4})$.

for some $r \in (0, 1)$ and a $\theta \in (0, \pi)$. Indeed,

$$\begin{aligned}\operatorname{Im} f_7(re^{i\theta}) &= \operatorname{Im}(h_7(re^{i\theta}) - g_7(re^{i\theta})) \\ &= -\frac{3}{2}r^2 \sin 2\theta + r^3 \sin 3\theta + \frac{r \sin\theta}{1 + r^2 - 2r\cos\theta}\end{aligned}$$

and, in particular,

$$\operatorname{Im} f_7(re^{i\pi/2}) = \frac{r(1 - r^2 - r^4)}{1 + r^2},$$

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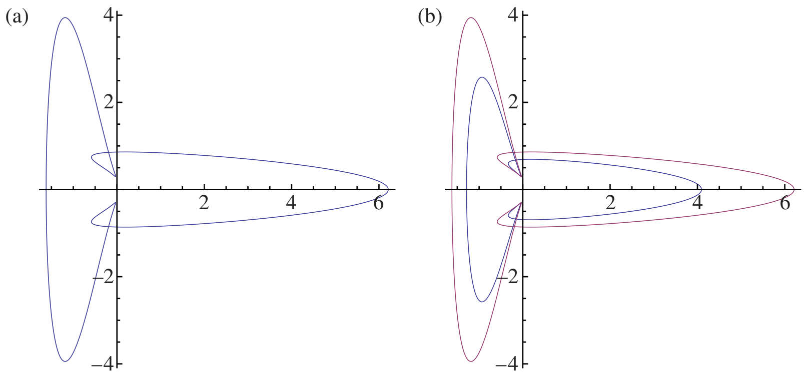

Figure 5. The image curves $f_6(|z| = 3/4)$ and $f_6(|z| = 7/10)$.

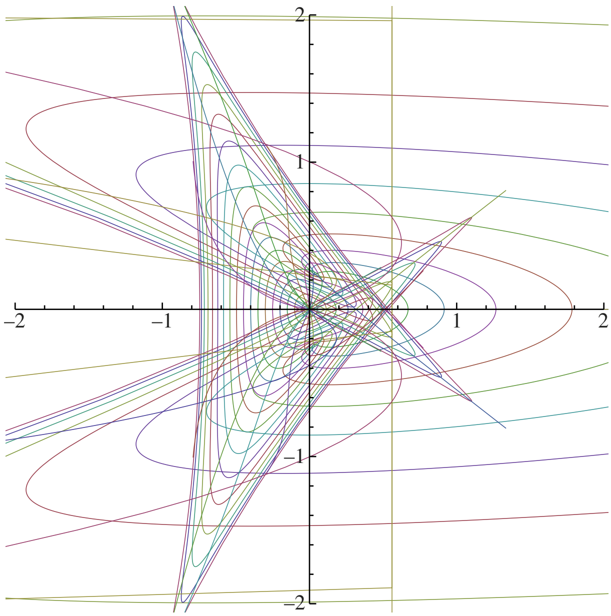

Figure 6. The image of the unit disk under f_7.

which shows that

$$\operatorname{Im} f_7(r_0 e^{i\pi/2}) = 0 = -\operatorname{Im} f_7(r_0 e^{-i\pi/2}),$$

where $r_0 = \sqrt{(\sqrt{5}-1)/2} \approx 0.786151$ is the root of the equation $1 - r^2 - r^4 = 0$ in the interval $(0, 1)$. Hence, the function $f_7(z)$ is not univalent in $\mathbb{D}$ (see also Remarks 3.7(b) below for an alternate proof of it). The graph of f_7 under the unit disk is shown in Figure 6.

Case 3.6. The case $b_2 = -\frac{1}{2}$.

Let $F(z) = -f(-z) = H(z) + \overline{G(z)} = z + \sum_{n=2} A_n z^2 + \overline{\sum_{n=2} B_n z^n}$. Then $F \in \mathcal{S}_H^0$ and A_n, B_n are half-integers with $B_2 = 1/2$. From the case $b_2 = 1/2$,

$$F \in \left\{ \mathrm{Re}\left(\frac{z}{(1-z)^2}\right) + i\,\mathrm{Im}\left(\frac{z}{1-z}\right), \mathrm{Re}\left(\frac{z}{1+z}\right) + i\,\mathrm{Im}\left(\frac{z}{(1+z)^2}\right), z + \overline{\frac{z^2}{2}} \right\},$$

which, by transferring in terms of f, shows that

$$f \in \left\{ \mathrm{Re}\left(\frac{z}{(1+z)^2}\right) + i\,\mathrm{Im}\left(\frac{z}{1+z}\right), \mathrm{Re}\left(\frac{z}{1-z}\right) + i\,\mathrm{Im}\left(\frac{z}{(1-z)^2}\right), z - \overline{\frac{z^2}{2}} \right\}.$$

The proof of Theorem 1.1 is complete.

REMARKS 3.7. Here are alternate approaches to show that the functions $f_{4,1} = h_{4,1} + \overline{g_{4,1}}$ and $f_7 = h_7 + \overline{g_7}$ are not univalent in $\mathbb{D}$.

(a) Suppose on the contrary that $f_{4,1}$ is univalent in $\mathbb{D}$. Since the coefficients of $f_{4,1}$ are all real, $f_{4,1}$ is typically real (see [7, Section 6.6]) and, hence, the analytic function $h_{4,1}(z) - g_{4,1}(z)$ must be typically real. Now, from (3.8) we note that

$$h_{4,1}(z) - g_{4,1}(z) = 2z + \frac{1}{2}z^2 + z^3 - \frac{3}{2}z^4 - \frac{z}{1-z},$$

so that $\mathrm{Re}\,\psi_{4,1}(z) > 0$ holds in $\mathbb{D}$ (see [6, Theorem 2.20]), where

$$\psi_{4,1}(z) = \frac{1-z^2}{z}(h_{4,1}(z) - g_{4,1}(z)) = 1 - \frac{1}{2}z - z^2 - 2z^3 - z^4 + \frac{3}{2}z^5.$$

But it is easy to verify that $\mathrm{Re}\,\psi_{4,1}(z) > 0$ does not hold in $\mathbb{D}$, which is a contradiction. Hence, $f_{4,1}$ does not belong to the class $\mathcal{S}_H^0$.

(b) As an alternate approach to show that $f_7(z)$ is not univalent in $\mathbb{D}$, we begin to observe that

$$h_7(z) - g_7(z) = -\frac{3}{2}z^2 + z^3 + \frac{z}{1-z}.$$

Suppose on the contrary that f_7 is univalent in $\mathbb{D}$. Then, because f_7 is typically real, $h_7 - g_7$ is a typically real analytic function (see [7, page 103]) and, thus, $\mathrm{Re}\,\psi_7(z) > 0$ holds in $\mathbb{D}$ (see [6, Theorem 2.20]), where

$$\psi_7(z) = \frac{1-z^2}{z}(h_7(z) - g_7(z)) = 1 - \frac{1}{2}z + z^2 + \frac{3}{2}z^3 - z^4.$$

But it is easy to verify that $\mathrm{Re}\,\psi_7(z) > 0$ does not hold in $\mathbb{D}$. Thus, f_7 cannot be univalent in $\mathbb{D}$.

4. The proof of Theorem 1.2

Let

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=2}^{\infty} b_n z^n} \in \mathcal{S}^0_{H,\mathcal{CV}}(E)$$

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and $\varphi = h - g$. As before, since f is sense-preserving, it follows that

$$|h'(z)| = |g'(z) + \varphi'(z)| > |g'(z)| \quad \text{for } z \in \mathbb{D},$$

which implies that

$$\frac{g'(z)}{\varphi'(z)} \prec \frac{z}{1-z} \quad \text{for } z \in \mathbb{D}.$$

Hence, there is a Schwarz function $\omega_1(z) = \sum_{n=1}^{\infty} c_n z^n$ such that

$$\frac{g'(z)}{\varphi'(z)} = \frac{\omega_1(z)}{1-\omega_1(z)} \quad \text{for } z \in \mathbb{D}$$

and, therefore,

$$g'(z) = \frac{\omega_1(z)}{1-\omega_1(z)}\varphi'(z) \quad \text{for } z \in \mathbb{D}. \tag{4.1}$$

Next, we consider

$$F(z) = H(z) + \overline{G(z)} = z + \sum_{n=2}^{\infty} A_n z^n + \overline{\sum_{n=2}^{\infty} B_n z^n} \in \mathcal{S}^0_{H,\mathcal{CV}}(E)$$

and define $\Phi = H - G$. Similarly, there is a Schwarz function $\omega_2(z) = \sum_{n=1}^{\infty} C_n z^n$ such that

$$G'(z) = \frac{\omega_2(z)}{1-\omega_2(z)}\Phi'(z). \tag{4.2}$$

Also, we write

$$\frac{1}{h(z)-g(z)} = \frac{1}{z} + e_0 + e_1 z + \cdots .$$

Claim 4.1. Suppose that $a_n = A_n$ and $b_n = B_n$ for $n = 2, 3, \ldots, N$,

$$2\sqrt{\frac{1-\sum_{n=1}^{N-2} n|e_n|^2}{N-1}} < r_0, \tag{4.3}$$

$$\frac{2\sqrt{1-\sum_{n=1}^{N} |c_n|^2}}{N+1} < r_0, \tag{4.4}$$

and

$$\frac{2\sqrt{1-\sum_{n=1}^{N} |C_n|^2}}{N+1} < r_0, \tag{4.5}$$

where $r_0 > 0$ is a bound of the uniformly discrete set E. If both f and F are either convex in real direction or convex in imaginary direction, then $f = F$.

Without loss of generality, we assume that f and F are convex in real direction. Then, by the shearing lemma, both $h - g$ and $H - G$ are univalent and convex in real direction. By [14, Lemma 2.1], $h - g = H - G$.

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Now we use the principle of induction to prove Claim 4.1. Assume that $a_n = A_n$ and $b_n = B_n$ for $n = 1, \ldots, m$ with $m \geq N$. We let

$$C_{m+1} =: a_{m+1} - A_{m+1} = b_{m+1} - B_{m+1}$$

and, also, let

$$\frac{1}{1-\omega_1(z)} = 1 + \sum_{n=1}^{\infty} d_n z^n \quad \text{and} \quad \frac{1}{1-\omega_2(z)} = 1 + \sum_{n=1}^{\infty} D_n z^n.$$

Then

$$\frac{\omega_1(z)}{1-\omega_1(z)} = \sum_{n=1}^{\infty} d_n z^n = \Big(\sum_{n=1}^{\infty} c_n z^n\Big)\Big(\sum_{n=0}^{\infty} d_n z^n\Big)$$

and similarly

$$\frac{\omega_2(z)}{1-\omega_2(z)} = \sum_{n=1}^{\infty} D_n z^n = \Big(\sum_{n=1}^{\infty} C_n z^n\Big)\Big(\sum_{n=0}^{\infty} D_n z^n\Big).$$

These two relations imply that

$$\begin{cases} d_1 = c_1, \\ d_k = c_1 d_{k-1} + c_2 d_{k-2} + \cdots + c_{k-1} d_1 + c_k \quad \text{for } k = 2, \ldots, m+1 \end{cases} \tag{4.6}$$

and similarly

$$\begin{cases} D_1 = C_1, \\ D_k = C_1 D_{k-1} + C_2 D_{k-2} + \cdots + C_{k-1} D_1 + C_k \quad \text{for } k = 2, \ldots, m+1. \end{cases} \tag{4.7}$$

By using (4.1) and (4.2),

$$\begin{cases} 2b_2 = d_1, \\ (k+1)b_{k+1} = kd_1(a_k - b_k) + (k-1)d_2(a_{k-1} - b_{k-1}) \\ \qquad\qquad + \cdots + 2d_{k-1}(a_2 - b_2) + d_k \quad \text{for } k = 2, \ldots, m \end{cases}$$

and similarly

$$\begin{cases} 2B_2 = D_1 \\ (k+1)B_{k+1} = kD_1(A_k - B_k) + (k-1)D_2(A_{k-1} - B_{k-1}) \\ \qquad\qquad + \cdots + 2D_{k-1}(A_2 - B_2) + D_k \quad \text{for } k = 2, \ldots, m. \end{cases}$$

Therefore, we see that $d_n = D_n$ for $n = 1, \ldots, m-1$ and

$$d_m - D_m = (m+1)(b_{m+1} - B_{m+1}) = (m+1)C_{m+1}.$$

It follows from (4.6) and (4.7) that $c_n = C_n$ for $n = 1, \ldots, m-1$ and

$$c_m - C_m = d_m - D_m = (m+1)C_{m+1}.$$

For the Schwarz functions $\omega_1(z)$ and $\omega_2(z)$, it follows from (4.4) and (4.5) that

$$|c_m - C_m| = (m+1)|C_{m+1}| < |c_m| + |C_m| < (m+1)\frac{r_0}{2} + (m+1)\frac{r_0}{2} = (m+1)r_0,$$

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which implies that

$$|C_{m+1}| = |a_{m+1} - A_{m+1}| = |b_{m+1} - B_{m+1}| < r_0.$$

Hence,

$$a_{m+1} = A_{m+1} \quad \text{and} \quad b_{m+1} = B_{m+1}.$$

Suppose that E is uniformly discrete with bound r_0 and N is a natural number sufficiently large enough that

$$1 < \frac{(N-1)r_0^2}{4} \quad \text{and} \quad 1 < \frac{(N+1)r_0}{2}.$$

We note that the conditions (4.3), (4.4), and (4.5) are fulfilled whatever the e_n, c_n, and C_n are. Since f is univalent with real coefficients, f is a typically real univalent harmonic function and therefore by the well-known coefficient estimates (see [2, page 23, Theorem 6.4])

$$|a_n| \le \frac{(n+1)(2n+1)}{6} \quad \text{and} \quad |b_n| \le \frac{(n-1)(2n-1)}{6}.$$

Hence, we have only finitely many choices of $a_2, \ldots, a_N, b_2, \ldots, b_N$ as the coefficients of functions in $\mathcal{S}^0_{H,C\mathcal{V}}(E)$. Once $a_2, \ldots, a_N, b_2, \ldots, b_N$ are specified, by Claim 4.1, there is at most one candidate for such a function $f \in \mathcal{S}^0_{H,C\mathcal{V}}(E)$. The proof is complete.

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