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L^2 EXTENSION FOR JETS OF HOLOMORPHIC SECTIONS OF A HERMITIAN LINE BUNDLE

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Abstract. Let (X, ω) be a weakly pseudoconvex Kähler manifold, $Y \subset X$ a closed submanifold defined by some holomorphic section of a vector bundle over X, and L a Hermitian line bundle satisfying certain positivity conditions. We prove that for any integer $k \geq 0$, any section of the jet sheaf $L \otimes \mathcal{O}_X/\mathcal{J}_Y^{k+1}$, which satisfies a certain L^2 condition, can be extended into a global holomorphic section of L over X whose L^2 growth on an arbitrary compact subset of X is under control. In particular, if Y is merely a point, this gives the existence of a global holomorphic function with an L^2 norm under control and with prescribed values for all its derivatives up to order k at that point. This result generalizes the L^2 extension theorems of Ohsawa-Takegoshi and of Manivel to the case of jets of sections of a line bundle. A technical difficulty is to achieve uniformity in the constant appearing in the final estimate. To this end, we make use of the exponential map and of a Rauch-type comparison theorem for complete Riemannian manifolds.

0.1. Introduction

Let (X, ω) be a weakly pseudoconvex Kähler manifold, and $Y \subset X$ a closed smooth hypersurface. In their ground-breaking paper [OT87], T. Ohsawa and K. Takegoshi proved that every holomorphic function f on Y which satisfies a weighted L^2 condition can be extended to a global holomorphic function F on X whose weighted L^2 norm is bounded above by a uniform constant multiplied by the weighted L^2 norm of the original f on Y. T. Ohsawa subsequently generalized this result in a series of papers ([Ohs88], [Ohs94], [Ohs95]). A far-reaching geometric-oriented generalization was given by L. Manivel ([Man93]) for submanifolds Y of arbitrary codimension and sections of holomorphic line bundles satisfying appropriate positivity conditions, instead of merely functions. Since then, the Ohsawa-Takegoshi-Manivel L^2 extension theorem has grown into a major tool of algebraic geometry and complex analysis. Its scope extended over such vastly different areas as regularization of currents ([Dem92]), invari-

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ance of plurigenera ([Siu98], [Siu02]), very ampleness and freeness criteria ([AS95], [Siu93], [Dem96]), and afforded new deep insights into the structure of multiplier ideal sheaves and plurisubharmonic functions ([DEL00], [DK01]). It also found applications in questions related to the Minimal Model Programme ([Kol97]).

However, a good deal has yet to be done. It has been convincingly shown by K. Diederich and E. Mazzilli ([DM00]) that the case of extensions from singular subvarieties Y is a subtle question, sometimes with an unexpected outcome, which still needs further probing. Relations between the L^2 extension problem and Skoda's L^2 division theorem ([Sko78]) were revealed by T. Ohsawa's works [Ohs02] and [Ohs04], but the general case has yet to be grasped. In particular, a unified theory of L^2 extension and division for holomorphic functions is needed.

Our point of view in the present article is different, shifting from subvarieties to unreduced subschemes. Motivated by geometric and complex analytic questions in keeping with those just mentioned, we have undertaken to obtain an L^2 extension result from a certain type of unreduced subschemes of the given ambient manifold X. In analytic terms, we prove the existence of holomorphic extensions for line bundle sections which have, in addition to the properties granted by the now classical Ohsawa-Takegoshi-Manivel theorem, prescribed partial derivatives (or jets) along a given submanifold Y up to an arbitrary pregiven order $k \in \mathbb{N}$. The result is new even in the simple case when Y is a point in a bounded pseudoconvex open set $\Omega \subset \mathbb{C}^n$. It asserts the existence of a holomorphic function f on Ω having prescribed values for all its derivatives up to order k at the given point, and an L^2 norm under control.

Setting

Let (X, ω) be a weakly pseudoconvex Kähler manifold of complex dimension n, and $Y \subset X$ a closed submanifold defined as

$$Y = \{x \in X\,;\, s(x) = 0,\, \Lambda^r(ds)(x) \neq 0\},$$

for some section $s \in H^0(X,\, E)$, assumed to be generically transverse to the zero section, of some Hermitian holomorphic vector bundle E of rank $r \geq 1$ over X. On the other hand, let L be a holomorphic line bundle over X equipped with a Hermitian fibre metric which satisfies an appropriate positivity condition.

Let $\mathcal{I}_Y$ be the sheaf of germs of holomorphic functions on X which vanish on Y. For any integer $k \geq 0$, let $\mathcal{O}_X/\mathcal{I}_Y^{k+1}$ be the nonlocally free sheaf of k-jets which are "transversal" to Y. Its fibre at an arbitrary point $y \in Y$ consists of all Taylor series at y truncated to order k in the vertical directions. We aim here at extending transversal k-jets of sections (over Y) of the line bundle of holomorphic L-valued $(n, 0)$-forms, namely sections $f \in H^0(X, \Lambda^n T^\star X \otimes L \otimes \mathcal{O}_X/\mathcal{I}_Y^{k+1})$. Equivalently, this amounts to extending sections from the unreduced scheme $Y^{(k+1)}$ defined by the quotient sheaf $\mathcal{O}_X/\mathcal{I}_Y^{k+1}$, to the ambient manifold X.

Construction of relevant metrics on jets

The first obstacle to overcome before even stating the result is to define a relevant intrinsic Sobolev-type $L^2_{(k)}$ norm of a k-jet. Since the jet sheaf $\mathcal{O}_X/\mathcal{I}_Y^{k+1}$ is not locally free, we make the following ad hoc inductive definition. Let $f \in H^0(X, \Lambda^n T^\star X \otimes L \otimes \mathcal{O}_X/\mathcal{I}_Y^{k+1})$. The holomorphic line bundle $L' := \Lambda^n T^\star_X \otimes L$ over X is canonically equipped with a fibre metric induced by the fibre metric of L and the reference metric ω on X. Let ∇ be the Chern connection associated with this metric of L', and $\nabla = \nabla^{1,0} + \nabla^{0,1}$ its decomposition into its $(1, 0)$ and $(0, 1)$ parts. Fix an arbitrary point $y \in Y$, and let U be a Stein neighbourhood in X giving rise to a surjective morphism $H^0(U, L') \to H^0(U, L' \otimes \mathcal{O}_X/\mathcal{I}_Y^{k+1})$ of local section spaces. Let $\tilde{f} \in H^0(U, L')$ be an arbitrary local lifting of f. Consider now the C^∞ vector bundle morphism $T^\star X_{|Y} \to N^\star_{Y/X}$ which is the ω-orthogonal C^∞ splitting of the exact sequence

$$0 \longrightarrow N^\star_{Y/X} \longrightarrow T^\star X_{|Y} \longrightarrow T^\star Y \longrightarrow 0.$$

Let $\nabla^{1,0}\tilde{f} \in H^0(U, L' \otimes T^\star X)$. Set $\nabla^1 \tilde{f} \in C^\infty(U, L' \otimes N^\star_{Y/X})$, the projection of $\nabla^{1,0}\tilde{f}$ under the surjective bundle morphism $L' \otimes T^\star X_{|Y} \to L' \otimes N^\star_{Y/X}$. Assume that $\nabla^{j-1}\tilde{f} \in C^\infty(U, L' \otimes S^{j-1} N^\star_{Y/X})$ has been constructed. Then $\nabla^{1,0}(\nabla^{j-1}\tilde{f}) \in C^\infty(U, L' \otimes S^{j-1} N^\star_{Y/X} \otimes T^\star X)$. We use here the same symbol $\nabla^{1,0}$ to designate the $(1, 0)$-type component of the Chern connection on $L' \otimes S^{j-1} N^\star_{Y/X}$ equipped with the induced metric. Set $\nabla^j \tilde{f} \in C^\infty(U, L' \otimes S^j N^\star_{Y/X})$, the projection of $\nabla^{1,0}(\nabla^{j-1}\tilde{f})$ under the surjective bundle morphisms

$$L' \otimes S^{j-1} N^\star_{Y/X} \otimes T^\star X \longrightarrow L' \otimes S^{j-1} N^\star_{Y/X} \otimes N^\star_{Y/X} \longrightarrow L' \otimes S^j N^\star_{Y/X}.$$

We have thus inductively constructed $\nabla^j \tilde{f} \in C^\infty(U,\, L' \otimes S^j N^\star_{Y/X})$ for all nonnegative integers j. The associated pointwise norms $|\tilde{f}|^2(y), \ldots, |\nabla^k \tilde{f}|^2(y)$ are well defined at every point $y \in Y$ with respect to the induced metrics.

Definition 0.1.1. For any transversal k-jet $f \in H^0(U,\, \Lambda^n T^\star X \otimes L \otimes \mathcal{O}_X/\mathcal{J}_Y^{k+1})$ and any weight function $\rho > 0$ on U, we define, at every point $y \in Y \cap U$, the pointwise ρ-weighted norm associated to the section s, by:

$$|f|^2_{s,\rho,(k)}(y) := |\tilde{f}|^2(y) + \frac{|\nabla^1 \tilde{f}|^2}{|\Lambda^r(ds)|^{2\frac{1}{r}}\, \rho^{2(r+1)}}(y) + \cdots + \frac{|\nabla^k \tilde{f}|^2}{|\Lambda^r(ds)|^{2\frac{k}{r}}\, \rho^{2(r+k)}}(y),$$

and the $L^2_{(k)}$ weighted norm by:

$$\|f\|^2_{s,\rho,(k)} = \int_Y |f|^2_{s,\rho,(k)}\, |\Lambda^r(ds)|^{-2}\, dV_{Y,\omega}.$$

Example 0.1.2. Consider the case where $X = \Omega$ is a bounded pseudoconvex open subset of $\mathbb{C}^n$ containing 0, $z = (z_1, \ldots, z_n)$ is the coordinate on $\mathbb{C}^n$, and $Y = \{z_1 = \cdots = z_r = 0\} \cap \Omega$. Take $E = \Omega \times \mathbb{C}^r$, equipped with the trivial flat metric, $L = \Omega \times \mathbb{C}$, and $s = \left(\frac{z_1}{e\, diam\, \Omega}, \ldots, \frac{z_r}{e\, diam\, \Omega}\right)$. For all $z \in \Omega$, $|s(z)|^2 = \frac{1}{e^2} \frac{|z_1|^2 + \cdots + |z_r|^2}{(diam\, \Omega)^2} \leq \frac{1}{e^2}$. The jet f is then defined by holomorphic functions a_α, $|\alpha| \leq k$, on Y, and its weighted $L^2_{(k)}$ norm is given by:

$$\begin{aligned} &\int_Y |f|^2_{s,\rho,(k)}\, |\Lambda^r(ds)|^{-2}\, dV_{Y,\omega} \\ &\quad = \int_Y \frac{|a_0|^2}{|\Lambda^r(ds)|^2}\, dV_{Y,\omega} + \sum_{|\alpha|=1} \int_Y \frac{|a_\alpha|^2}{|\Lambda^r(ds)|^{2\frac{r+1}{r}}\, \rho^{2(r+1)}}\, dV_{Y,\omega} \\ &\qquad + \cdots + \sum_{|\alpha|=k} \int_Y \frac{1}{(\alpha!)^2} \frac{|a_\alpha|^2}{|\Lambda^r(ds)|^{2\frac{r+k}{r}}\, \rho^{2(r+k)}}\, dV_{Y,\omega}. \end{aligned}$$

It should be noticed that the norm $|f|^2_{s,\rho,(k)}(y)$ of the k-jet f at the point $y \in Y$ is independent of the choice of the local lifting $\tilde{f}$. Indeed, if $\hat{f} \in H^0(U,\, L')$ is another lifting of $f_{|U} \in H^0(U,\, L' \otimes \mathcal{O}_X/\mathcal{J}_Y^{k+1})$, then $\tilde{f}$ and $\hat{f}$ have the same transversal k-jet on $U \cap Y$ (equal to $f_{|U}$). This implies that $\nabla^j \tilde{f} = \nabla^j \hat{f}$ at every point in $U \cap Y$, for all integers $j = 0, \ldots, k$.

NOTATION 0.1.3. (a) For a transversal k-jet $f \in H^0(U,\, \Lambda^n T^\star_X \otimes L \otimes \mathcal{O}_X/\mathcal{J}^{k+1}_Y)$, denote $\nabla^j f := (\nabla^j \tilde{f})_{|U\cap Y}$, for all $j = 0, \ldots, k$ and an arbitrary lifting $\tilde{f} \in H^0(U,\, \Lambda^n T^\star_X \otimes L)$ of f.

(b) For every integer $k \geq 0$, set

$$J^k : H^0(X,\, \Lambda^n T^\star_X \otimes L) \longrightarrow H^0(X,\, \Lambda^n T^\star_X \otimes L \otimes \mathcal{O}_X/\mathcal{J}^{k+1}_Y)$$

the cohomology group morphism induced by the projection $\mathcal{O}_X \to \mathcal{O}_X/\mathcal{J}^{k+1}_Y$.

Statement of results: geometric setting

We can now state the jet extension theorem. In the case of a compact ambient manifold X, the final L^2 estimate of the extension F_k with prescribed k-order jet along Y is obtained over the whole of X. In the general noncompact case, the boundary of X is avoided by estimating the extension on an arbitrary relatively compact open subset $\Omega \subset X$. If $\Omega \subset\subset X$ is such a subset, we define an associated weight function $\rho = \rho_\Omega > 0$ by

$$\rho(y) = \frac{1}{\|Ds_y^{-1}\| \sup_{\xi\in\Omega}(\|D^2 s_\xi\| + \|Ds_\xi\|)},$$

where D stands for the Chern connection of E. It is with respect to this weight function that the Sobolev-type norm on jets is considered throughout the paper.

THEOREM 0.1.4. (Main theorem) *Let X be a complex weakly pseudoconvex manifold of complex dimension n, equipped with a Kähler metric ω, L a Hermitian holomorphic line bundle, E a Hermitian holomorphic vector bundle of rank $r \geq 1$ over X, and $s \in H^0(X, E)$ a section assumed to be generically transverse to the zero section. Set*:

$$Y := \{x \in X\ ;\ s(x) = 0,\ \Lambda^r(ds)(x) \neq 0\},$$

a subvariety of X of codimension r. Also assume that, for an integer $k \geq 0$, the $(1,1)$-form $i\Theta(L) + (r+k)\, id'd'' \log|s|^2$ involving the curvature of L is semipositive on X, and that there exists a continuous function $\alpha \geq 1$ such that the following two inequalities are satisfied on X:

(*a*) $i\Theta(L) + (r+k)\, id'd'' \log|s|^2 \geq \alpha^{-1} \dfrac{\{i\Theta(E)s,\, s\}}{|s|^2}$,

(*b*) $|s| \leq e^{-\alpha}$.

Then, for every relatively compact open subset $\Omega \subset X$*, and every* k*-jet* $f \in H^0(X,\, \Lambda^n T_X^\star \otimes L \otimes \mathcal{O}_X/\mathcal{J}_Y^{k+1})$ *satisfying*

$$\int_Y |f|^2_{s,\rho,(k)}\, |\Lambda^r(ds)|^{-2}\, dV_{Y,\omega} < +\infty,$$

there exists $F_k \in H^0(X,\, \Lambda^n T_X^\star \otimes L)$ *such that* $J^k F_k = f$ *and*

$$\int_\Omega \frac{|F_k|^2}{|s|^{2r}(-\log|s|)^2}\, dV_{X,\omega} \leq C_r^{(k)} \int_Y |f|^2_{s,\rho,(k)}\, |\Lambda^r(ds)|^{-2}\, dV_{Y,\omega},$$

where $C_r^{(k)} > 0$ *is a constant depending only on* r*,* k*,* E*, and* $\sup_\Omega \|i\Theta(L)\|$.

Remarks. (a) The case when $k = 0$ is the Ohsawa-Takegoshi-Manivel L^2 extension theorem. The above theorem is new for $k \geq 1$.

(b) The section $s \in H^0(X,\, E)$ induces a nowhere zero section $\Lambda^r(ds)$ of the vector bundle $\Lambda^r(T_X/T_Y)^\star \otimes \det E$, and its norm $|\Lambda^r(ds)|$ is computed with respect to the induced metric on this vector bundle. The notation $\|i\Theta(L)\|$ stands for the norm of the curvature tensor of L viewed as a $(1,1)$-form on X. As with the Ohsawa-Takegoshi-Manivel extension theorem, only the curvature hypothesis (a) is essential among the inequalities satisfied by s, but it is now, significantly, dependent on k. Indeed, if (a) holds for a choice of the function $\alpha \geq 1$, we can always achieve (b) by multiplying the metric of E by a sufficiently small weight $e^{-\chi\circ\psi}$, where ψ is a plurisubharmonic exhaustion of X and χ is a real convex increasing function. Property (a) still holds after multiplying the metric of L by the weight $e^{-(r+k+\alpha_0^{-1})\chi\circ\psi}$, where $\alpha_0 = \inf_{x\in X} \alpha(x)$.

The following theorem is a special case of the main theorem for a bounded pseudoconvex open set $\Omega \subset \mathbb{C}^n$.

THEOREM 0.1.5. *Let* $\Omega \subset \mathbb{C}^n$ *be a bounded pseudoconvex open set, and* $Y \subset \Omega$ *a closed nonsingular subvariety defined by some section* $s \in H^0(X,\, E)$ *of a Hermitian holomorphic vector bundle* E *of rank* $r \geq 1$ *with bounded curvature form. Assume that* $|s| \leq e^{-1}$ *on* Ω.

Then, for any nonnegative integer k *and any plurisubharmonic function* φ *on* Ω*, there exists a constant* $C_r^{(k)} > 0$ *depending only on* E*, on* Ω*, and on the modulus of continuity of* φ*, such that for every holomorphic section* f *of* $\mathcal{O}_\Omega/\mathcal{J}_Y^{k+1}$ *satisfying*

$$\int_Y |f|^2_{s,\rho,(k)}\, |\Lambda^r(ds)|^{-2}\, e^{-\varphi}\, dV_Y < +\infty,$$

there exists a holomorphic function F_k *on* Ω *such that* $J^k F_k = f$ *and*

$$\int_\Omega \frac{|F_k|^2}{|s|^{2r}(-\log|s|)^2}\, e^{-\varphi}\, dV_{\Omega'} \le C_r^{(k)} \int_Y |f|^2_{s,\rho,(k)}\, |\Lambda^r(ds)|^{-2}\, e^{-\varphi}\, dV_Y.$$

Local analytic setting

The case of a singleton $Y = \{z_0\}$ is of special interest. The jet f at z_0 is given by complex numbers $a_\alpha \in \mathbb{C}$, $|\alpha| \le k$, $\alpha = (\alpha_1, \dots, \alpha_n)$. Take $s = (e\, diam\, \Omega)^{-1}(z - z_0)$, viewed as a section of the trivial vector bundle $E = \Omega \times \mathbb{C}^n$. It is clear that $|s| \le e^{-1}$, and that:

$$\int_Y |f|^2_{s,\rho,(k)}\, |\Lambda^n(ds)|^{-2}\, e^{-\varphi} = \left(\sum_{|\alpha|\le k} |a_\alpha|^2\right) e^{-\varphi(z_0)}.$$

Since $-\log|s| = \frac{1}{\varepsilon}\log|s|^{-\varepsilon} \le \frac{1}{\varepsilon}|s|^{-\varepsilon}$, for all $\varepsilon > 0$, we may replace $|s|^{2n}(-\log|s|)^2$ in the denominator by $|s|^{2(n-\varepsilon)}$. We thus get the following.

Corollary 0.1.6. *Let* $\Omega \subset \mathbb{C}^n$ *be a bounded pseudoconvex open set, and let* $z_0 \in \Omega$ *be a point. Then, for every positive integer* k *and every plurisubharmonic function* φ *on* Ω, *there exists a constant* $C_n^{(k)} > 0$ *depending only on the modulus of continuity of* φ, *with the following property. For all complex numbers* a_α, $|\alpha| \le k$, *there exists a holomorphic function* f *on* Ω *such that*

$$f(z_0) = a_0,\ \frac{\partial^\alpha f}{\partial z^\alpha}(z_0) = a_\alpha,\ 1 \le |\alpha| \le k,\ \text{and}$$

$$\int_\Omega \frac{|f|^2}{|z - z_0|^{2(n-\varepsilon)}}\, e^{-\varphi(z)}\, dV_\Omega(z) \le \frac{C_n^{(k)}}{\varepsilon^2\,(diam\,\Omega)^{2(n-\varepsilon)}}\left(\sum_{|\alpha|\le k} |a_\alpha|^2\right) e^{-\varphi(z_0)}.$$

To avoid confusion, it is worth entering a caveat. The final constants in the above statements depend on the modulus of continuity of the weight function φ, and this dependence seems to be inevitable in this setting. If applications with singular weights are intended, special attention should be paid to getting smooth regularizing weight functions with the same modulus of continuity before recovering the same estimate for the singular weight in the limit.

Article layout

We will split the proofs of Theorems 0.1.4 and 0.1.5 into two parts. In the first part, the qualitative one, we make use of techniques of the original papers of Ohsawa and Takegoshi ([OT87], [Ohs88]), cast into a more geometric mould by Manivel ([Man93]), and subsequently simplified by Demailly ([Dem00]), that we appropriately fit into our generalized situation. The main idea, harking back to Ohsawa and Takegoshi ([OT87]), is to use a "weight bumping" technique to concentrate the curvature of the line bundle L on a tubular neighbourhood of the submanifold Y. This leads to defining a new curvature operator and to proving L^2 estimates modified accordingly which are analogous to those of Hörmander. The main tool is a Bochner-Kodaira-Nakano-type inequality due to Ohsawa and Takegoshi ([OT87]) and later improved by Ohsawa ([Ohs95]). This step is performed in Section 0.3 and is common to the proofs of Theorems 0.1.4 and 0.1.5. Our method in this section parallels previous methods with the necessary modifications.

The second half of the proofs of Theorems 0.1.4 and 0.1.5, the quantitative one, introduces new ideas. The main goal is to achieve uniformity for the constant appearing in the final L^2 estimate. Here we deal separately with Theorems 0.1.4 and 0.1.5. In Section 0.4, we apply Cauchy's inequalities to get a control of the growth of the k-jet of a holomorphic function in terms of the growth of this very function, and we thus complete the proof of Theorem 0.1.5. The proof of Theorem 0.1.4 is more involved. In order to get intrinsic L^2 estimates independent of the radii of local holomorphic coordinate patches on X, we make use of the exponential map to carry the situation over to the tangent space to X at a point. In Section 0.5, the Jacobi field technique will enable us to get a Riemannian geometric result related to the Rauch comparison theorem. In Section 0.6, building on this comparison theorem, we get the final estimate in the main theorem thanks to Gårding's lemma on the solutions of elliptic systems.

Prospects

We hope the results of this paper will find applications in complex analysis and algebraic geometry. Here is a very brief outline of some possible developments. The main interest of the extension theorem lies in its quantitative part and was mainly intended as an effective device for producing sections for Hermitian line bundles with positivity properties. It could thus be useful in solving questions related to the Fujita conjecture as a continuation of such works as [Siu93], [AS95], or [Dem96]. It is, indeed, tempting to think that producing global holomorphic functions, or sections with effective global bounds, out of much simpler initial data defined merely at a point or on a line, could prove efficient in problems where an effective control of the objects involved is needed. In fact, we were originally motivated by an attempt at getting a regularization of closed positive currents, with an additional control of the Monge-Ampère masses for the regularizing currents, to extend Demailly's regularization-of-currents theorem ([Dem92]). Such an undertaking is likely to have interesting geometric consequences, such as singular Morse inequalities or bigness criteria for line bundles.

The present results treat the case of holomorphic extensions from a special type of unreduced subschemes, namely those which consist of several layers of the same submanifold. A further step would be to obtain holomorphic extensions from more general unreduced subschemes.

0.2. Ingredients

We list here the main preliminary results underlying the proof of the original Ohsawa-Takegoshi theorem. They will be needed again in our proof. For proofs and details see, for instance, Demailly's paper [Dem00].

The main idea in the proof of the Ohsawa-Takegoshi extension theorem ([OT87], [Ohs88]) was to derive and use a modified version of the Bochner-Kodaira-Nakano inequality. This version was subsequently improved by Ohsawa ([Ohs95]) in the following form.

Proposition 0.2.1. (Main curvature inequality) *Let (X,ω) be a Kähler manifold with a nonnecessarily complete Kähler metric, let (E,h) be a Hermitian vector bundle on X, and let $\eta,\lambda>0$ be C^∞ functions on X.*

Then, for every $u\in\mathcal{D}(X,\,\Lambda^{p,q}T^\star_X\otimes E)$, we have:

$$\begin{aligned}&\|(\eta^{\frac{1}{2}}+\lambda^{\frac{1}{2}})D''^\star u\|^2+\|\eta^{\frac{1}{2}}D''u\|^2+\|\lambda^{\frac{1}{2}}D'u\|^2+2\,\|\lambda^{-\frac{1}{2}}\,d'\eta\wedge u\|^2\\ &\qquad\geq\langle\!\langle[\eta\, i\Theta(E)-id'd''\eta-i\lambda^{-1}\,d'\eta\wedge d''\eta,\,\Lambda_\omega]u,\,u\rangle\!\rangle.\end{aligned}$$

In the particular case of (n,q)-forms, the forms $D'u$ and $d'\eta\wedge u$ vanish as having bidegree $(n+1,q)$. Then the above inequality reads:

$$\begin{aligned}&\|(\eta^{\frac{1}{2}}+\lambda^{\frac{1}{2}})D''^\star u\|^2+\|\eta^{\frac{1}{2}}D''u\|^2\\ &\qquad\geq\langle\!\langle[\eta\, i\Theta(E)-id'd''\eta-i\lambda^{-1}\,d'\eta\wedge d''\eta,\,\Lambda]u,\,u\rangle\!\rangle.\end{aligned}$$

This key curvature inequality enables one to infer the following L^2 existence theorem which parallels Hörmander's L^2 existence theorem ([Hör65], [Hör66]) for a modified curvature operator.

Proposition 0.2.2. *Let (X,ω) be a Kähler manifold. The metric ω may not be complete but X is assumed to carry a complete Kähler metric. Given a Hermitian vector bundle (E,h) and smooth bounded functions $\eta, \lambda > 0$ on X, consider the curvature operator*

$$B := B^{n,q}_{E,\omega,\eta,\lambda} := [\eta\, i\Theta(E) - id'd''\eta - i\lambda^{-1}\, d'\eta \wedge d''\eta,\, \Lambda_\omega],$$

acting on the sections of the vector bundle $\Lambda^{n,q}T^\star_X \otimes E$, for some $q \geq 1$, and assume that B is positive definite at every point of X.

Then, for all $g \in L^2(X,\, \Lambda^{n,q}T^\star_X \otimes E)$ such that $D''g = 0$, and

$$\int_X \langle B^{-1}g,\, g\rangle\, dV_\omega < +\infty,$$

there exists $f \in L^2(X,\, \Lambda^{n,q-1}T^\star_X \otimes E)$ such that $D''f = g$ and

$$\int_X (\eta + \lambda)^{-1}\, |f|^2\, dV_\omega \leq 2\int_X \langle B^{-1}g,\, g\rangle\, dV_\omega.$$

In the course of the proof of the jet extension theorem we shall need to apply the above proposition for a modified fibre metric of the line bundle under consideration, which is obtained by multiplying the original smooth metric by the weight $|s|^{-2(r+k)}$ with singularities along $Y = \{s = 0\}$. To avoid the singularities, we shall restrict to $X\backslash Y$. The following standard lemma ensures that $X\backslash Y$ still carries a complete Kähler metric.

Lemma 0.2.3. (see, for instance, [Dem82]) *Let (X,ω) be a Kähler weakly pseudoconvex manifold, ψ a plurisubharmonic exhaustion, and $X_c = \{x \in X\ ;\ \psi(x) < c\}$, for $c \in \mathbb{R}$. Let $Y = \{s = 0\} \subset X$ be an analytic subset defined by a section $s \in H^0(X,\, E)$ of a Hermitian vector bundle (E,h) over X.*

Then, for all $c \in \mathbb{R}$, $X_c\backslash Y$ carries a complete Kähler metric.

0.3. Proof of Theorem 0.1.4

Assume that the set $\Sigma = \{s = 0,\, \Lambda^r(ds) = 0\}$ of singularities of Y is empty, which means that Y is a smooth closed subvariety of X. This restriction can be lifted through a standard argument like in [Dem00, 4.8, p. 12]. We argue by induction on $k \geq 0$. The case $k = 0$ is the Ohsawa-Takegoshi theorem. Assume the theorem has been proved for $k - 1$. Consider the short exact sequence of sheaves:

$$0 \longrightarrow S^k N^\star_{Y/X} \longrightarrow \mathcal{O}_X/\mathcal{J}_Y^{k+1} \longrightarrow \mathcal{O}_X/\mathcal{J}_Y^k \longrightarrow 0$$

and let $J^{k-1}f \in H^0(X,\, \Lambda^n T^\star_X \otimes L \otimes \mathcal{O}_X/\mathcal{J}^k_Y)$ be the image of $f \in H^0(X,\, \Lambda^n T^\star_X \otimes L \otimes \mathcal{O}_X/\mathcal{J}^{k+1}_Y)$ under the induced cohomology group morphism. By the induction hypothesis, there exists $F_{k-1} \in H^0(X,\, \Lambda^n T^\star_X \otimes L)$ such that

$$\begin{aligned} &J^{k-1}F_{k-1} = J^{k-1}f \quad \text{and} \\ &\int_\Omega \frac{|F_{k-1}|^2}{|s|^{2r}(-\log|s|)^2}\, dV_\omega \leq C_r^{(k-1)} \int_Y |f|^2_{s,\rho,(k-1)}\, |\Lambda^r(ds)|^{-2}\, dV_{Y,\omega}, \end{aligned}$$

where $C_r^{(k-1)} > 0$ is a constant as in the statement of Theorem 0.1.4. Thus the image of $f - J^kF_{k-1} \in H^0(X,\, \Lambda^n T^\star_X \otimes L \otimes \mathcal{O}_X/\mathcal{J}^{k+1}_Y)$ in $H^0(X,\, \Lambda^n T^\star_X \otimes L \otimes \mathcal{O}_X/\mathcal{J}^k_Y)$ is $J^{k-1}f - J^{k-1}F_{k-1} = 0$. This allows for the jet $f - J^kF_{k-1}$ to be viewed as a global holomorphic section (on Y) of the sheaf $\Lambda^n T^\star_X \otimes L \otimes S^k N^\star_{Y/X} = \Lambda^n T^\star_X \otimes L \otimes S^k E^\star_{|Y}$.

A C^∞ extension of the jet. We start off by constructing an extension $\tilde{f} \in C^\infty(X,\, \Lambda^n T^\star_X \otimes L)$ of the holomorphic k-jet $f \in H^0(X,\, \Lambda^n T^\star_X \otimes L \otimes \mathcal{O}_X/\mathcal{J}^{k+1}_Y)$ by means of a partition of unity. Consider a covering of Y by coordinate patches $U_i \subset X$ on which the vector bundles E and $\Lambda^n T^\star_X \otimes L$ are trivial. Let e_i be a nonvanishing holomorphic section of $\Lambda^n T^\star_X \otimes L_{|U_i}$, and $s_1, \ldots, s_r$ holomorphic functions on U_i such that $s_{|U_i} = (s_1, \ldots, s_r)$ in a trivialization of $E_{|U_i}$. The functions $s_1, \ldots, s_r$ define holomorphic coordinates on U_i transversal to Y. Let $z'_{(i)} = (z^{(i)}_{r+1}, \ldots, z^{(i)}_n)$ be holomorphic coordinates on $Y \cap U_i$, and write the restriction jet f as $f_{|Y\cap U_i} = w_i \otimes e_{i|Y\cap U_i}$, with $w_i \in H^0(U_i,\, \mathcal{O}_X/\mathcal{J}^{k+1}_Y)$. The local k-jet w_i is given by holomorphic functions $a^{(i)}_\alpha(z'_{(i)})$ on $Y \cap U_i$, indexed over multi-indices $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$, with $|\alpha| \leq k$. Set

$$\hat{f}_i(s,\, z'_{(i)}) := \left(\sum_{|\alpha| \leq k} a_\alpha(z'_{(i)})\, s^\alpha \right) \otimes e_i \in H^0(U_i,\, \Lambda^n T^\star_X \otimes L).$$

Then $\frac{\partial^\alpha \hat{f}_i}{\partial s^\alpha}(0, z'_{(i)}) = a^{(i)}_\alpha(z'_{(i)})$, for all α, $|\alpha| \leq k$, and $\hat{f}_i$ defines thus a local holomorphic extension of the jet f from $U_i \cap Y$ to U_i. Let $\theta_i \in \mathcal{D}(U_i)$ be a partition of unity such that $\sum \theta_i \equiv 1$ on a neighbourhood of Y. Then

$$\tilde{f} := \sum_i \theta_i \hat{f}_i \in C^\infty(X, \Lambda^n T^\star_X \otimes L)$$

defines a C^∞ extension of the jet f. Furthermore, we have:

$$D''\tilde{f} = \sum_i d''\theta_i \wedge \hat{f}_i, \quad D''\tilde{f} = 0 \quad \text{on } Y,$$

since all $\hat{f}_i$ assume the same value at every point of Y and $\sum_i d''\theta_i = 0$ on Y. Likewise, for any multi-index $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r$, $|\alpha| \leq k$, if we derive locally $D''\tilde{f}$ along the directions $s = (s_1, \dots, s_r)$ transversal to Y, we get:

$$D^\alpha(D''\tilde{f}) = \sum_{\beta \leq \alpha} \sum_i \binom{\alpha}{\beta} D^\beta(d''\theta_i) \wedge D^{\alpha-\beta}\hat{f}_i = 0 \quad \text{on } Y,$$

since for fixed $\alpha - \beta$, all the $D^{\alpha-\beta}\hat{f}_i$ assume the same value at every point in Y (as k-order extensions of the same transversal jet f). As the subvariety $Y = \{s = 0\}$ is assumed to be smooth, the Taylor expansion of $D''\tilde{f}$ near Y shows that the C^∞extension of f we have just constructed satisfies:

$$|D''\tilde{f}| = O(|s|^{k+1}) \quad \text{in a neighbourhood of } Y.$$

Weight construction; weight bumping technique. Here we reuse the auxiliary functions considered in the original proof of the Ohsawa-Takegoshi theorem, and repeat the computations of [OT87], [Man93] and [Dem00] with an additional k. Since we hardly know $\tilde{f}$ away from Y, we take a truncation with support in a tubular neighbourhood of Y. Let

$$G^{(k-1)}_\varepsilon := \theta\left(\frac{|s|^2}{\varepsilon^2}\right)(\tilde{f} - F_{k-1}) \in C^\infty(X, \Lambda^n T^\star_X \otimes L),$$

where $\theta : \mathbb{R} \to \mathbb{R}$ is a C^∞ function such that $\theta \equiv 1$ on $]-\infty, \frac{1}{2}]$, and $\operatorname{Supp}\theta \subset]-\infty, 1[$. It is clear that $\operatorname{Supp} G^{(k-1)}_\varepsilon \subset \{|s| < \varepsilon\}$. We shall solve the equation:

$$(\star) \qquad D''u_\varepsilon = D''G^{(k-1)}_\varepsilon,$$

with the extra condition that $\frac{|u_\varepsilon|^2}{|s|^{2(r+k)}} \in L^1_{\mathrm{loc}}$ in a neighbourhood of Y. This condition ensures that u_ε, as well as all its jets of order $\leq k$, vanish on Y. Let ψ be a plurisubharmonic exhaustion of X, and set $X_c = \{\psi < c\} \subset\subset X$, for all real c. The ideal thing would be to solve the equation $(\star)$ on X. For technical reasons which will become apparent later, we shall solve the equation $(\star)$ on $X_c \backslash Y_c$ which is still complete Kähler thanks to Lemma 0.2.3. The desired holomorphic extension of the jet f will then be $G^{(k-1)}_\varepsilon - u_\varepsilon + F_{k-1}$. The final solution will be obtained by passing to the limit with $c \to \infty$ and $\varepsilon \to 0$.

Consider now the following auxiliary functions (as in [OT87], [Man93], [Dem00]):

$$\sigma_\varepsilon := \log(|s|^2 + \varepsilon^2), \quad \eta_\varepsilon := \varepsilon - \chi_0(\sigma_\varepsilon), \quad \lambda_\varepsilon := \frac{\chi_0'(\sigma_\varepsilon)^2}{\chi_0''(\sigma_\varepsilon)},$$

where $\chi_0 :]-\infty, 0] \to]-\infty, 0]$, $\chi_0(t) = t - \log(1-t)$, for all $t \leq 0$, having the following properties: $\chi(t) \leq t,\ 1 \leq \chi_0' \leq 2,\ \chi''(t) = \frac{1}{(1-t)^2}$.

The function η_ε is close to $+\infty$ near Y and decays upon getting away from Y. It allows therefore for concentrating the curvature of L on a small neighbourhood of Y. We define a new curvature operator:

$$B_\varepsilon := [\eta_\varepsilon(i\Theta(L) + (r+k)\, id'd'' \log|s|^2) - id'd''\eta_\varepsilon - \lambda_\epsilon^{-1}\, id'\eta_\varepsilon \wedge d''\eta_\varepsilon,\, \Lambda],$$

and prove the estimate:

$$B_\varepsilon \geq \frac{\varepsilon^2}{2|s|^2}(d''\eta_\varepsilon)(d''\eta_\varepsilon)^\star,$$

as operators acting on the (n,q)-forms. Easy computations yield, in terms of the canonical sesquilinear pairing $\{\ ,\ \}$ of vector bundle valued forms:

$$d'\sigma_\varepsilon = \frac{\{D's,\, s\}}{|s|^2 + \varepsilon^2}, \qquad d''\sigma_\varepsilon = \frac{\{s,\, D's\}}{|s|^2 + \varepsilon^2},$$

$$d'd''\sigma_\varepsilon = \frac{\{D's,\, D's\}}{|s|^2 + \varepsilon^2} + \frac{\{s,\, D''D's\}}{|s|^2 + \varepsilon^2} - \frac{\{D's,\, s\} \wedge \{s,\, D's\}}{(|s|^2 + \varepsilon^2)^2}.$$

On the other hand, $\Theta(E) = D^2 = D'D'' + D''D'$, and since $D''s = 0$, owing to s being holomorphic, we see that $D''D's = \Theta(E)s$. This finally yields:

$$id'd''\sigma_\varepsilon = \frac{i\{D's,\, D's\}}{|s|^2 + \varepsilon^2} - \frac{i\{D's,\, s\} \wedge \{s,\, D's\}}{(|s|^2 + \varepsilon^2)^2} - \frac{\{i\Theta(E)s,\, s\}}{|s|^2 + \varepsilon^2}.$$

We now use Lagrange's inequality: $i\{D's,\, D's\} \geq \frac{i\{D's, s\}\wedge\{s, D's\}}{|s|^2}$ to get:

$$\begin{aligned} id'd''\sigma_\varepsilon &\geq \frac{\varepsilon^2}{|s|^2}\frac{i\{D's,\, s\}\wedge\{s,\, D's\}}{(|s|^2+\varepsilon^2)^2} - \frac{\{i\Theta(E)s,\, s\}}{|s|^2+\varepsilon^2} \\ &= \frac{\varepsilon^2}{|s|^2} id'\sigma_\varepsilon \wedge d''\sigma_\varepsilon - \frac{\{i\Theta(E)s,\, s\}}{|s|^2+\varepsilon^2}. \end{aligned}$$

On the other hand, $d'\eta_\varepsilon = -\chi_0'(\sigma_\varepsilon)\, d'\sigma_\varepsilon$, $d''\eta_\varepsilon = -\chi_0'(\sigma_\varepsilon)\, d''\sigma_\varepsilon$, and

$$\begin{aligned} -id'd''\eta_\varepsilon &= \chi_0'(\sigma_\varepsilon)\, id'd''\sigma_\varepsilon + \chi_0''(\sigma_\varepsilon)\, id'\sigma_\varepsilon \wedge d''\sigma_\varepsilon \\ &\geq \left(\frac{\varepsilon^2}{2|s|^2} + \frac{\chi_0''(\sigma_\varepsilon)}{\chi_0'(\sigma_\varepsilon)^2}\right) id'\eta_\varepsilon \wedge d''\eta_\varepsilon - 2\frac{\{i\Theta(E)s,\, s\}}{|s|^2+\varepsilon^2}. \end{aligned}$$

Let us multiply now the original metric of L by the weight $|s|^{-2(r+k)}$; the curvature of this new metric satisfies the inequality

$$i\Theta(L) + (r+k)\, id'd''\log|s|^2 \geq \alpha^{-1}\,\frac{\{i\Theta(E)s,\, s\}}{|s|^2+\varepsilon^2},$$

thanks to hypothesis (a). Indeed, the inequality still holds with the denominator $|s|^2+\varepsilon^2$ instead of $|s|^2$, owing to the semipositivity of the left-hand term. On the other hand, $|s| \leq e^{-\alpha} \leq e^{-1}$, which entails $\sigma_\varepsilon \leq 0$ for ε small, and

$$\eta_\varepsilon \geq \varepsilon - \sigma_\varepsilon \geq \varepsilon - \log(e^{-2\alpha}+\varepsilon^2).$$

In addition, we have: $\eta_\varepsilon \geq 2\alpha$, for $\varepsilon < \varepsilon(c)$ small enough. This, along with the previous inequalities, implies:

$$\begin{aligned} &\eta_\varepsilon(i\Theta(L) + (r+k)\, id'd''\log|s|^2) - id'd''\eta_\varepsilon - \frac{\chi_0''(\sigma_\varepsilon)}{\chi_0'(\sigma_\varepsilon)^2}\, id'\eta_\varepsilon \wedge d''\eta_\varepsilon \\ &\geq \frac{\varepsilon^2}{2|s|^2}\, id'\eta_\varepsilon \wedge d''\eta_\varepsilon, \end{aligned}$$

on X_c. Set $\lambda_\varepsilon = \frac{\chi_0'(\sigma_\varepsilon)^2}{\chi_0''(\sigma_\varepsilon)}$, and get the lower curvature estimate we were looking for:

$$\begin{aligned} B_\varepsilon &:= [\eta_\varepsilon(i\Theta(L) + (r+k)\, id'd''\log|s|^2) - id'd''\eta_\varepsilon - \lambda_\varepsilon^{-1}\, id'\eta_\varepsilon \wedge d''\eta_\varepsilon,\, \Lambda] \\ &\geq \left[\frac{\varepsilon^2}{2|s|^2}\, id'\eta_\varepsilon \wedge d''\eta_\varepsilon,\, \Lambda\right] = \frac{\varepsilon^2}{2|s|^2}(d''\eta_\varepsilon)(d''\eta_\varepsilon)^\star, \end{aligned}$$

as operators acting on the (n,q)-forms.

$\bar{\partial}$-resolution with L^2 estimates. In this section the estimation details for jets are new, although the idea of resolution is classic. We shall now solve the equation ($\star$) using Proposition 0.2.2. To avoid the singularities of the weight $|s|^{-2(r+k)}$ along Y, we will be working on the relatively compact open subset $X_c \setminus Y_c$, where $Y_c = Y \cap X_c = Y \cap \{\psi < c\}$, instead of working on X itself. We first need verify that the a priori L^2 condition required in Proposition 0.2.2 is satisfied. Easy computations show that:

$$\begin{aligned} D'' G_\varepsilon^{(k-1)} &= g_\varepsilon^{(1)} + g_\varepsilon^{(2)}, \quad \text{where} \\ g_\varepsilon^{(1)} &= \left(1 + \frac{|s|^2}{\varepsilon^2}\right) \theta'\left(\frac{|s|^2}{\varepsilon^2}\right) d''\sigma_\varepsilon \wedge (\tilde{f} - F_{k-1}), \\ g_\varepsilon^{(2)} &= \theta\left(\frac{|s|^2}{\varepsilon^2}\right) D''(\tilde{f} - F_{k-1}). \end{aligned}$$

Since $g_\varepsilon^{(2)}$ converges uniformly to 0 on every compact when ε tends to 0, it will have no contribution in the limit. Indeed, $\mathrm{Supp}(g_\varepsilon^{(2)}) \subset \{|s| < \varepsilon\}$ and $|g_\varepsilon^{(2)}| = O(|s|^{k+1})$, since we have previously shown that $|D''\tilde{f}| = O(|s|^{k+1})$ in a neighbourhood of Y. This implies that:

$$\int_{X_c \setminus Y_c} \langle B_\varepsilon^{-1} g_\varepsilon^{(2)}, g_\varepsilon^{(2)} \rangle \, |s|^{-2(r+k)} \, dV_{X,\omega} = O(\varepsilon),$$

if B_ε is locally uniformly bounded below in a neighbourhood of Y. If this is not the case, we solve the approximate equation $D''u + \delta^{\frac{1}{2}} h = g_\varepsilon$, where $\delta > 0$ is small (see [Dem00, Remark 3.2], for the details). Since there is no essential extra difficulty in this case, we may assume, for the sake of perspicuity, that we have the desired lower bound for B_ε.

The estimation of $g_\varepsilon^{(1)}$ is different from the case of previous proofs of extensions without jets. We get the following:

$$\begin{aligned} &\int_{X_c \setminus Y_c} \langle B_\varepsilon^{-1} g_\varepsilon^{(1)}, g_\varepsilon^{(1)} \rangle \, |s|^{-2(r+k)} \, dV_{X,\omega} \\ &\qquad \leq 8 \int_{X_c \setminus Y_c} |\tilde{f} - F_{k-1}|^2 \, \theta'\left(\frac{|s|^2}{\varepsilon^2}\right)^2 |s|^{-2(r+k)} \, dV_{X,\omega}. \end{aligned}$$

Indeed,

$$\begin{aligned} g_\varepsilon^{(1)} &= -\left(1 + \frac{|s|^2}{\varepsilon^2}\right) \theta'\left(\frac{|s|^2}{\varepsilon^2}\right) \chi_0'(\sigma_\varepsilon)^{-1} \, d''\eta_\varepsilon \wedge (\tilde{f} - F_{k-1}), \\ B_\varepsilon^{-1} &\leq \frac{2|s|^2}{\varepsilon^2} \, (d''\eta_\varepsilon)^{\star -1} (d''\eta_\varepsilon)^{-1}, \end{aligned}$$

and therefore:

$$\begin{aligned}&\langle B_\varepsilon^{-1}(d''\eta_\varepsilon\wedge u),\,(d''\eta_\varepsilon\wedge u)\rangle\\&\quad\leq\frac{2|s|^2}{\varepsilon^2}\langle (d''\eta_\varepsilon)^{-1\star}(d''\eta_\varepsilon)^{-1}(d''\eta_\varepsilon\wedge u),\,(d''\eta_\varepsilon\wedge u)\rangle\\&\quad=\frac{2|s|^2}{\varepsilon^2}\,\langle u,\,u\rangle=\frac{2|s|^2}{\varepsilon^2}|u|^2.\end{aligned}$$

Furthermore, $\frac{2|s|^2}{\varepsilon^2}\leq 2$ and $\big(1+\frac{|s|^2}{\varepsilon^2}\big)\chi_0'(\sigma_\varepsilon)^{-1}\leq 2$, on $\operatorname{Supp} g_\varepsilon^{(1)}\subset\{|s|<\varepsilon\}$. This implies

$$\langle B_\varepsilon^{-1}g_\varepsilon^{(1)},\,g_\varepsilon^{(1)}\rangle\leq 8\,\theta'\bigg(\frac{|s|^2}{\varepsilon^2}\bigg)^2|\tilde f-F_{k-1}|^2.$$

If $z=(z_1,\ldots,z_r)$ is an arbitrary local holomorphic coordinate system transversal to Y, we have

$$\frac{|s|^{2r}}{|\Lambda^r(ds)|^2}=\frac{|z|^{2r}}{|\Lambda^r(dz)|^2},$$

the norms of the sections $\Lambda^r(ds)\in H^0(X,\,\Lambda^r(T_X/T_Y)^\star\otimes\det E)$ and $\Lambda^r(dz)\in H^0(U,\,\Lambda^r(T_X/T_Y)^\star)$ being computed with respect to the metrics induced on the respective vector bundles by ω and by the given metric on E.

The integrand of the last integral estimating $g_\varepsilon^{(1)}$ can be locally written, after the change of variable $s\rightsquigarrow\varepsilon\,s$, as

$$\begin{aligned}&\frac{|(\tilde f-F_{k-1})(\varepsilon\,s,z')|^2}{\varepsilon^{2(r+k)}|s|^{2(r+k)}}\,\frac{\theta'(|s|^2)^2}{|\Lambda^r(ds)|^{2\frac{r+k}{r}}}\,dV_\omega(\varepsilon\,s,z')\\&\quad=\frac{|(\tilde f-F_{k-1})(\varepsilon\,s,z')|^2}{\varepsilon^{2k}|s|^{2(r+k)}}\,\frac{\theta'(|s|^2)^2}{|\Lambda^r(ds)|^{2\frac{r+k}{r}}}\,dV_\omega(s,z').\end{aligned}$$

Since $J^{k-1}f-J^{k-1}F_{k-1}=0$, the Taylor series development yields:

$$(\tilde f-F_{k-1})(\varepsilon\,s,z')=\sum_{|\alpha|+|\beta|\geq k}\frac{\varepsilon^{|\alpha|+|\beta|}}{(\alpha+\beta)!}\,\frac{\partial^{\alpha+\beta}(\tilde f-F_{k-1})}{\partial s^\alpha\partial\bar s^\beta}(0,z')\,s^\alpha\bar s^\beta$$

$$
\begin{aligned}
&= \varepsilon^k \Bigg(\sum_{|\alpha|=k} \frac{1}{\alpha!} \frac{\partial^\alpha(\tilde f - F_{k-1})}{\partial s^\alpha}(0,z')\, s^\alpha \\
&\qquad + \sum_{|\alpha|+|\beta|\geq k+1} \frac{\varepsilon^{|\alpha|+|\beta|-k}}{(\alpha+\beta)!} \frac{\partial^{\alpha+\beta}(\tilde f - F_{k-1})}{\partial s^\alpha \partial \bar s^\beta}(0,z')\, s^\alpha \bar s^\beta \Bigg) \\
&= \varepsilon^k (f - J^k F_{k-1})(z') + O(|\varepsilon s|^{k+1}) \\
&= \varepsilon^k\, \nabla^k (f - J^k F_{k-1})(z') + O(|\varepsilon s|^{k+1}).
\end{aligned}
$$

The first sum ranges only on multi-indices α and β such that if $|\alpha|+|\beta| = k$, then $|\alpha| = k$.

This shows that $\frac{|(\tilde f - F_{k-1})(\varepsilon s, z')|^2}{\varepsilon^{2k}}$ converges to $|\nabla^k(f - J^k F_{k-1})(z')|^2$, (see Notation 0.1.3), uniformly on every compact, when $\varepsilon \to 0$.

We have thus proved that:

$$
\begin{aligned}
&\int_{X_c \setminus Y_c} \langle B_\varepsilon^{-1} g_\varepsilon^{(1)},\, g_\varepsilon^{(1)} \rangle\, |s|^{-2(r+k)} dV_{X,\varepsilon} \\
&\qquad \leq 8 \int_{X_c \setminus Y_c} |\tilde f - F_{k-1}|^2\, \Theta'\left(\frac{|s|^2}{\varepsilon^2}\right)^2 |s|^{-2(r+k)}\, dV_{X,\varepsilon} \\
&\qquad \to 8\, C_{r,k} \int_{Y_c} \frac{|\nabla^k (f - J^k F_{k-1})|^2}{|\Lambda^r(ds)|^{2\frac{r+k}{r}}}\, dV_{Y,\omega},
\end{aligned}
$$

where

$$
C_{r,k} := \int_{z\in\mathbb{C}^r,\, |z|\leq 1} \theta'(|z|^2)^2\, \frac{i\Lambda^r(dz)\wedge \Lambda^r(d\bar z)}{|z|^{2(r+k)}}.
$$

It is worth noticing that $|\nabla^k(f - J^k F_{k-1})| = |f - J^k F_{k-1}|$, where $|f - J^k F_{k-1}|$ is the norm of the section:

$$
f - J^k F_{k-1} \in H^0(Y,\, \Lambda^n T_X^\star \otimes L \otimes S^k N_{Y/X}^\star)
$$

with respect to the metric induced on $S^k N_{Y/X}^\star$ by the reference metric ω on X. Indeed, $S^k N_{Y/X}$ is a subbundle of $(S^k T_X)_{|Y}$; we merely take the metric induced on $S^k N_{Y/X}$ by restriction.

The L^2 condition required beforehand in Proposition 0.2.2 is thus satisfied. The solution $u_{c,\varepsilon}$ to the equation ($\star$) $D'' u_{c,\varepsilon} = D'' G_\varepsilon^{(k+1)} = g_\varepsilon^{(1)} + g_\varepsilon^{(2)}$

on $X_c\backslash Y_c$ satisfies then the estimate:

$$\begin{aligned}
(1)\qquad & \int_{X_c\backslash Y_c} \frac{|u_{c,\varepsilon}|^2}{|s|^{2(r+k)}(-\log(|s|^2+\varepsilon^2))^2}\, dV_{X,\varepsilon} \\
& \leq \int_{X_c\backslash Y_c} \frac{|u_{c,\varepsilon}|^2}{(\eta_\varepsilon+\lambda_\varepsilon)|s|^{2(r+k)}}\, dV_{X,\omega} \\
& \leq 2\int_{X_c\backslash Y_c} \langle B_\varepsilon^{-1} g_\varepsilon,\, g_\varepsilon\rangle\, |s|^{-2(r+k)}\, dV_{X,\omega} \\
& \leq 16\, C_{r,k} \int_{Y_c} \frac{|\nabla^k(f-J^kF_{k-1})|^2}{|\Lambda^r(ds)|^{2\frac{r+k}{r}}}\, dV_{Y,\omega} + O(\varepsilon).
\end{aligned}$$

Indeed, we have used the following obvious estimates (cf. [Dem00, 4.6]):

$$\begin{aligned}
&\sigma_\varepsilon = \log(|s|^2+\varepsilon^2) \leq \log(e^{-2\alpha}+\varepsilon^2) \leq -2\alpha + O(\varepsilon^2) \leq -2 + O(\varepsilon^2), \\
&\eta_\varepsilon = \varepsilon - \chi_0(\sigma_\varepsilon) \leq (1+O(\varepsilon))\sigma_\varepsilon^2, \\
&\lambda_\varepsilon = \frac{\chi_0'(\sigma_\varepsilon)^2}{\chi_0''(\sigma_\varepsilon)} = (1-\sigma_\varepsilon)^2 + (1-\sigma_\varepsilon) \leq (3+O(\varepsilon))\sigma_\varepsilon^2, \\
&\eta_\varepsilon + \lambda_\varepsilon \leq (4+O(\varepsilon))\sigma_\varepsilon^2 \leq (4+O(\varepsilon))(-\log(|s|^2+\varepsilon^2))^2.
\end{aligned}$$

The extension of f to $X_c\backslash Y_c$ is then given by:

$$F_{c,\varepsilon}^{(k)} := G_\varepsilon^{(k-1)} - u_{c,\varepsilon} + F_{k-1}.$$

Locally, near an arbitrary point of Y, this means that all partial derivatives of order $\leq k$ of $F_{c,\varepsilon}^{(k)}$ are prescribed by f. The function $G_\varepsilon^{(k-1)}$ is C^∞ on a tubular neighbourhood of Y and $\operatorname{Supp} G_\varepsilon^{(k-1)} \subset \{|s| < \varepsilon\}$. This implies that:

$$(2)\qquad \int_{X_c} \frac{|G_\varepsilon^{(k-1)}|^2}{(|s|^2+\varepsilon^2)^r(-\log(|s|^2+\varepsilon^2))^2}\, dV_{X,\omega} \leq \frac{\text{Const}}{(\log\varepsilon)^2}.$$

Since

$$\begin{aligned}
& \int_{X_c\backslash Y_c} \frac{|u_{c,\varepsilon}|^2}{|s|^{2r}(-\log(|s|^2+\varepsilon^2))^2}\, dV_{X,\omega} \\
& \quad \leq \int_{X_c\backslash Y_c} \frac{|u_{c,\varepsilon}|^2}{|s|^{2(r+k)}(-\log(|s|^2+\varepsilon^2))^2}\, dV_{X,\omega},
\end{aligned}$$

(1), (2), and the induction hypothesis made on the L^2 norm of F_{k-1}, imply the estimate:

$$\begin{aligned}
&\int_{X_c\setminus Y_c} \frac{|F^{(k)}_{c,\varepsilon}|^2}{(|s|^2+\varepsilon^2)^r(-\log(|s|^2+\varepsilon^2))^2}\, dV_{X,\omega} \\
&\quad\le 16\, C_{r,k} \int_{Y_c} \frac{|\nabla^k(f - J^k F_{k-1})|^2}{|\Lambda^r(ds)|^{2\frac{r+k}{r}}}\, dV_{Y,\omega} \\
&\qquad\qquad + \int_{X_c} \frac{|F_{k-1}|^2}{|s|^{2r}(-\log|s|)^2}\, dV_{X,\omega} + \frac{\mathrm{Const}}{(\log\varepsilon)^2} \\
&\quad\le 16\, C_{r,k} \int_{Y_c} \frac{|\nabla^k(f - J^k F_{k-1})|^2}{|\Lambda^r(ds)|^{2\frac{r+k}{r}}}\, dV_{Y,\omega} \\
&\qquad\qquad + C^{(k-1)}_r \int_{Y} |f|^2_{s,\,\rho,\,(k-1)}\, |\Lambda^r(ds)|^{-2}\, dV_{Y,\omega} + \frac{\mathrm{Const}}{(\log\varepsilon)^2} \\
&\quad\le C'^{(k)}_r \int_{Y} |f|^2_{s,\,\rho,\,k}\, |\Lambda^r(ds)|^{-2}\, dV_{Y,\omega} \\
&\qquad\qquad + 16\, C_{r,k} \int_{Y_c} \frac{|\nabla^k(J^k F_{k-1})|^2}{|\Lambda^r(ds)|^{2\frac{r+k}{r}}}\, dV_{Y,\omega} + \frac{\mathrm{Const}}{(\log\varepsilon)^2},
\end{aligned}$$

where $C'^{(k)}_r = C^{(k-1)}_r + 16\, C_{r,k}$.

We also have $D'' F^{(k)}_{c,\varepsilon} = 0$ on $X_c\setminus Y_c$, by construction. This relation extends from $X_c\setminus Y_c$ to X_c because $F^{(k)}_{c,\varepsilon}$ is L^2_{loc} in a neighbourhood of Y_c. The extension is granted by the following standard lemma on the $\bar\partial$ operator (see, for instance, [Dem82]).

Lemma 0.3.1. *Let Ω be an open subset of $\mathbb{C}^n$ and Y an analytic subset of Ω. Let v be a $(p, q-1)$-form with L^2_{loc} coefficients, and w a (p,q)-form with L^1_{loc} coefficients such that $d''v = w$ on $\Omega\setminus Y$ (in the sense of distributions). Then $d''v = w$ on Ω.*

The ellipticity of the operator $\bar\partial$ in bidegree $(0,0)$ ensures that $u_{c,\varepsilon}$ is C^∞. Consequently, $F^{(k)}_{c,\varepsilon}$ is C^∞ as well.

We have thus obtained a family of solutions $(F^{(k)}_{c,\varepsilon})_\varepsilon$ with corresponding L^2 estimates on the relatively compact open subset X_c of X. By extracting a weak limit when $\varepsilon \to 0$, we thus get a solution $F^{(k)}_c$ and an L^2 estimate of it on the relatively compact open subset X_c, for all $c > 0$.

This completes the qualitative part of the proofs of Theorems 0.1.4 and 0.1.5, and estimates in part the solutions. The final estimates will be obtained in the subsequent sections.

0.4. Estimation of the solution in Theorem 0.1.5

In order to get the final estimates in Theorems 0.1.4 and 0.1.5, it remains to estimate

$$\int_{Y_c} \frac{|\nabla^k(J^k F_{k-1})|^2}{|\Lambda^r(ds)|^{2\frac{r+k}{r}}}\, dV_{Y,\omega}.$$

In this section we will complete the proof of Theorem 0.1.5. Here the analysis is simplified by the ambient manifold being an open subset $\Omega \subset \mathbb{C}^n$. We will use the Cauchy inequalities (or, equivalently, Parseval's formula). In the more general case of Theorem 0.1.4, such an approach would yield a constant depending on the radii of the local holomorphic coordinate balls of X. Since this is an uncontrollable quantity, we will avoid such arbitrariness in the subsequent sections by means of the exponential map replacing locally the ambient manifold X by its tangent space at a point.

Let ω be the standard Kähler metric on Ω. Since the curvature of E is assumed to be bounded, there exists a constant $M > 0$ such that $i\Theta(E) \leq M\omega \otimes \mathrm{Id}_E$. Set $L = \Omega \times \mathbb{C}$, equipped with the metric of weight $e^{-\varphi - A|z|^2}$, with a constant $A \gg 0$. If we set $\alpha \equiv 1$, the curvature hypothesis (a) in Theorem 0.1.4 reads:

$$id'd''\varphi + A\, id'd''|z|^2 + (r+k)\, id'd''\log|s|^2 \geq \frac{\{i\Theta(E)s,\, s\}}{|s|^2}.$$

Since $id'd''\varphi \geq 0$, $id'd''\log|s|^2 \geq -\frac{\{i\Theta(E)s,\, s\}}{|s|^2}$, and $\frac{\{i\Theta(E)s,\, s\}}{|s|^2} \leq M\omega$, this relation is satisfied as soon as A has been chosen large enough. This choice of A depends on the bound M of the curvature tensor of E.

Let $\psi : \Omega \to \mathbb{R}$ be a C^∞ plurisubharmonic exhaustion of Ω, namely a function such that the sublevel sets $\Omega_c := \{\psi < c\}$ are relatively compact in Ω for all $c > 0$. We may assume that $\Omega' = \Omega_c$ for some c, and denote $Y_c := Y \cap \Omega_c$. Consider now a covering of Y_c by open subsets U_j, $j = 1, \ldots, p$, such that on every U_j there exist local holomorphic coordinates $z = (z', z'')$, $z' = (z_1, \ldots, z_r)$, $z'' = (z_{r+1}, \ldots, z_n)$ for which $Y \cap U_j = \{z' = 0\}$. Pick such a U_j and assume that $U_j = B'(0,\, \rho) \times B''(0,\, \rho) \subset B(0,\, \rho\sqrt{2})$, where $B'(0,\, \rho)$ is the ball of radius ρ of $\mathbb{C}^r$, $B''(0,\, \rho)$ is the ball of radius ρ of $\mathbb{C}^{n-r}$, and $B(0,\, \rho\sqrt{2})$ is the ball of radius $\rho\sqrt{2}$ of $\mathbb{C}^n$. The jet $\nabla^k(J^k F_{k-1})$ can be written on U_j as $\sum_{|\alpha|=k} \frac{1}{\alpha!} \frac{\partial^\alpha F_{k-1}}{\partial {z'}^\alpha}(0, z''){z'}^\alpha$, and its norm is given by

$$|\nabla^k (J^k F_{k-1})|^2 = \sum_{|\alpha|=k} \left| \frac{\frac{\partial^\alpha F_{k-1}}{\partial {z'}^\alpha}(0, z'')}{\alpha!} \right|^2 e^{-2\varphi(0,z'')-2A|z''|^2}.$$

Parseval's formula applied for $z' \in B'(0,\, \rho)$ gives

$$\begin{aligned} \frac{\mathrm{Const}}{\rho^{2r}} \int_{z' \in B'(0,\, \rho)} |F_{k-1}(z', z'')|^2 \, d\lambda(z') &= \sum_{\alpha} \left| \frac{\frac{\partial^\alpha F_{k-1}}{\partial {z'}^\alpha}(0, z'')}{\alpha!} \right|^2 \frac{\rho^{2|\alpha|}}{2r + 2|\alpha|} \\ &\geq \sum_{|\alpha|=k} \left| \frac{\frac{\partial^\alpha F_{k-1}}{\partial {z'}^\alpha}(0, z'')}{\alpha!} \right|^2 \frac{\rho^{2k}}{2(r+k)}, \end{aligned}$$

where Const is a universal constant. Consequently,

$$\begin{aligned} \frac{\displaystyle\sum_{|\alpha|=k} \left| \frac{\frac{\partial^\alpha F_{k-1}}{\partial {z'}^\alpha}(0, z'')}{\alpha!} \right|^2 e^{-2\varphi(0,z'')-2A|z''|^2}}{|\Lambda^r(ds)(0, z'')|^{2\frac{r+k}{r}}} &\leq \mathrm{Const}\, \frac{2(r+k)}{\rho^{2(r+k)}} \\ &\times \int_{z' \in B'(0,\, \rho)} \|F_{k-1}(z', z'')\|^2 \, \frac{e^{2(\varphi(z',z'')-\varphi(0,z''))}\, e^{2A|z'|^2}}{|\Lambda^r(ds)(0, z'')|^{2\frac{r+k}{r}}} \, d\lambda(z'), \end{aligned}$$

for all $z'' \in B''(0,\, \rho)$, where we have denoted by

$$\|F_{k-1}(z', z'')\|^2 := |F_{k-1}(z', z'')|^2 \, e^{-2\varphi(z',z'')} \, e^{-2A(|z'|^2+|z''|^2)},$$

the norm of F_{k-1} regarded as a section of the line bundle L. Due to a notation inconsistency, this vector bundle norm $\| \;\; \|$ is the same as the one we had denoted by $| \;\; |$ in the induction hypothesis (see start of Section 0.3). Let ε be a modulus of continuity for φ, namely a function such that

$$|\varphi(z', z'') - \varphi(0, z'')| \leq \varepsilon(|z'|), \quad \forall (z', z'') \in \bigcup_{j=1}^{p} U_j,$$

and $\varepsilon(\delta) \downarrow 0$ when $\delta \downarrow 0$.

Since $\varepsilon(|z'|) \leq \varepsilon(\rho)$ for $z' \in B'(0,\, \rho)$, the previous estimate entails

$$\frac{\displaystyle\sum_{|\alpha|=k} \left| \frac{\frac{\partial^\alpha F_{k-1}}{\partial z'^\alpha}(0, z'')}{\alpha!} \right|^2 e^{-2\varphi(0,z'')-2A|z''|^2}}{|\Lambda^r(ds)(0, z'')|^{2\frac{r+k}{r}}} \leq \mathrm{Const}\, \frac{2(r+k)}{\rho^{2(r+k)}}\, e^{2(\varepsilon(\rho)+A\rho^2)} \sup_{(z',z'')\in U_j} \frac{|s(z', z'')|^{2r}\,(-\log|s(z', z'')|)^2}{|\Lambda^r(ds)(0, z'')|^{2\frac{r+k}{r}}} \times \int_{z'\in B'(0,\,\rho)} \frac{\|F_{k-1}(z', z'')\|^2}{|s(z', z'')|^{2r}\,(-\log|s(z', z'')|)^2}\, d\lambda(z'),$$

for all $z'' \in B''(0,\, \rho)$. A topological property of Y ensures that there exists a nonnegative integer N such that the covering $(U_j)_j$ of Y_c can be chosen in such a way that $\#\{j\ ;\ U_j \ni y\} \leq N$. An integration with respect to z'' in the previous inequality, a summation over j, and obvious upper bounds yield

$$\int_{Y_c} \frac{|\nabla^k(J^k F_{k-1})|^2}{|\Lambda^r(ds)|^{2\frac{r+k}{r}}}\, dV_{Y,\omega} \leq C_{r,k}\, N M(c)\, \frac{1}{\rho^{2(r+k)}}\, e^{2(\varepsilon(\rho)+A\rho^2)} \int_{\Omega'} \frac{\|F_{k-1}\|^2}{|s|^{2r}\,(-\log|s|)^2}\, dV_{X,\omega},$$

if

$$M(c) := \sup_{(z',z'')\in\Omega'} \frac{|s(z', z'')|^{2r}\,(-\log|s(z', z'')|)^2}{|\Lambda^r(ds)(0, z'')|^{2\frac{r+k}{r}}}$$

and $C_{r,k} := \mathrm{Const}\, 2(r+k)$. The radius ρ of the local holomorphic coordinate charts on which the submanifold Y can be straightened is explicitly given by the following elementary lemma which is a refinement of the local inversion theorem expressing the “size” of the ball on which we get a local diffeomorphism.

LEMMA 0.4.1. *Let E and F be Banach spaces, U an open subset of E, and $f : U \to F$ a C^1 map such that its differential map $df_a : E \to F$ at a point $a \in U$ is a bicontinuos isomorphism.*

Then the open neighbourhood V of a, given by the local inversion theorem, on which f is a diffeomorphism onto its image, contains the ball $B(a,\, \rho)$, where

$$\rho = \frac{1}{6(\|df_a^{-1}\|)(\sup_{\xi\in U}\|d^2 f_\xi\|)}.$$

The elementary proof of this lemma is left to the reader. It can be easily inferred from the proof of the local inversion theorem. As the submanifold Y is defined by the section $s \in H^0(X, E)$, this lemma accounts for the explicit formula of the weight function ρ featuring in the statements of Theorems 0.1.4 and 0.1.5. Indeed, if $\theta : E_{|U} \to U \times \mathbb{C}^r$ is a trivialization of $E_{|U}$, and $(e_1, \dots, e_r)$ the corresponding local holomorphic frame of $E_{|U}$, the restriction of s to U can be uniquely written as

$$s = \sum_{j=1}^{r} \sigma_j \otimes e_j, \quad \sigma_j \in \mathcal{O}(U).$$

If D is the Chern connection of the Hermitian holomorphic vector bundle E, the operator D can be written as

$$Ds \simeq_\theta d\sigma + A \wedge \sigma,$$

where $A = (a_{jk})$ is the matrix of 1-forms representing the connection D in the trivialization θ. Since the coefficients a_{jk} of A are locally bounded (by constants depending implicitly on E), Lemma 0.4.1 and the expression of d in terms of D show that the radius of the coordinate ball on which Y can be straightened in a neighbourhood of a given point $y \in Y$ is bounded below by

$$C\rho(y) = C \frac{1}{\|Ds_y^{-1}\| \sup_\xi(\|D^2 s_\xi\| + \|Ds_\xi\|)},$$

the constant $C > 0$ depending only on E. This completes the proof of Theorem 0.1.5.

0.5. A Rauch-type comparison theorem

We still need to complete the proof of Theorem 0.1.4. Unlike its counterpart discussed in the previous section, Theorem 0.1.4 is set on a general Kähler manifold (X, ω). In order to get final estimates independent of the radii of local holomorphic coordinate balls of X, we will be working on the tangent space to X at a point instead of X itself. The exponential map locally identifies X to its tangent space. In order to estimate the deviation of the pull-back of ω to the tangent space from the standard Euclidian metric of this very tangent space, we need to establish a Riemannian geometric result related to the Rauch comparison theorem (see, for instance, [BC64, page 250]). The proof of this result will be a slight reshaping of the proof of Rauch's theorem and will use the Jacobi vector fields theory and an elementary Gronwall-type lemma.

Let (M,g) be a complete Riemannian manifold, $m \in M$ an arbitrary point, and $\exp_m : T_mM \to M$ the exponential map at the point m. Let $\mathrm{Id} := \mathrm{Id}_{T_mM}$ and, for an arbitrary point $x \in T_mM$, consider the tangent linear map (or the differential) $T_x \exp_m : T_mM \to T_{\exp_m(x)}M$ of $\exp_m$ at the point x. We can identify T_mM and $T_{\exp_m(x)}M$ via the isometry defined by parallel transport along the geodesic $\gamma(t)$ with initial conditions $\gamma(0) = m$ and $\gamma'(0) = x$. Our goal is to estimate

$$\|T_x \exp_m - \mathrm{Id}\,\|$$

in terms of $\|x\|$, when x ranges over the tangent space T_mM. Let $u \in T_mM$, $\|u\| = 1$, and γ_u the geodesic with $\gamma_u(0) = m$ and $\gamma_u'(0) = u$. We thus have

$$\gamma_u(t) = \exp_m(tu),$$

for all t in the interval of definition of γ_u. Recall that a vector field Y along the geodesic γ_u is said to be a *Jacobi field* if it satisfies the second order differential equation

$$Y'' + R(\gamma_u', Y)\gamma_u' = 0,$$

where R is the curvature tensor of (M,g) defined as $R(X,Y)Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z$. It is a well-known fact that the differential of the exponential map is given by a Jacobi field. More precisely, for any $u, v \in T_mM$, we have the relation

$$(T_{tu} \exp_m)(tv) = Y(t),$$

where Y is the unique Jacobi field along γ_u such that $Y(0) = 0$ and $Y'(0) = v$.

Assume now the sectional curvature of (M,g) to be bounded, namely that there exists a constant $k > 0$ such that

$$-k \le K(p,P) \le k,$$

for every point $p \in M$ and every plane $P \subset T_pM$, where $K(p,P)$ stands for the sectional curvature of the plane P. To estimate $\|T_x \exp_m - \mathrm{Id}\,\|$ we need estimate

$$\|(T_{tu} \exp_m)(tv) - \mathrm{Id}(tv)\| = \|Y(t) - Y'(0)t\|,$$

when t ranges over $\mathbb{R}$. We need therefore an estimate for Y given that Y satisfies a second order linear differential equation. The following elementary lemma, of Gronwall-type, provides the necessary estimation.

LEMMA 0.5.1. *Let* $v : [0,T] \to \mathbb{R}$ *be a* C^2 *function,* $v \geq 0$*, such that* $v(0) = 0$*,* $v'(0) = A$ *and*

$$-kv \leq v'' \leq kv, \quad \text{on } [0,T], \quad \text{where } k > 0 \text{ is a constant.}$$

Then,

$$A\frac{1}{\sqrt{k}}\sin(\sqrt{k}t) \leq v(t) \leq A\frac{1}{\sqrt{k}}\sinh(\sqrt{k}t), \quad \text{for all } t \in [0,T].$$

Proof. Let us first prove the right-hand inequality. Let u be the solution to the Cauchy problem $u'' = ku$ with initial conditions $u(0) = 0$ and $u'(0) = 1$. Then, $u(t) = \frac{1}{\sqrt{k}}\sinh(\sqrt{k}t)$. In particular, $u \geq 0$, and $u(t) = 0$ if and only if $t = 0$. The hypothesis shows that

$$\frac{v''}{v} \leq k = \frac{u''}{u} \iff (v'u - vu')' \leq 0 \implies v'u - vu' \leq 0,$$

on $[0,T]$. This implies

$$\left(\frac{v}{u}\right)' \leq 0 \implies \frac{v(t)}{u(t)} \leq \frac{v}{u}(0_+),$$

for all $t \in [0,T]$. Therefore,

$$v(t) \leq \frac{v}{u}(0_+)\,\frac{1}{\sqrt{k}}\sinh(\sqrt{k}t),$$

for all $t \in [0,T]$. On the other hand, we see that

$$\frac{v}{u}(0_+) = \lim_{t\to 0}\frac{v(t)}{u(t)} = \lim_{t\to 0}\frac{v'(t)}{u'(t)} = \frac{v'(0)}{u'(0)} = A,$$

which proves the right-hand inequality. Let us now prove the left-hand inequality.

Let u be the solution to the Cauchy problem $u'' = -ku$, with initial conditions $u(0) = 0$ and $u'(0) = 1$. Then, $u(t) = \frac{1}{\sqrt{k}}\sin(\sqrt{k}t)$. In particular, $u \geq 0$, and $u(t) = 0$ if and only if $t = 0$. By hypothesis, we see that

$$\frac{v''}{v} \geq -k = \frac{u''}{u} \iff (v'u - vu')' \geq 0 \implies v'u - vu' \geq 0,$$

on $[0,T]$. This implies

$$\left(\frac{v}{u}\right)' \geq 0 \implies \frac{v(t)}{u(t)} \geq \frac{v}{u}(0_+),$$

for all $t \in [0, T]$. Consequently,

$$v(t) \geq \frac{v}{u}(0_+) \, \frac{1}{\sqrt{k}} \sin(\sqrt{k}t),$$

for all $t \in [0, T]$. As before, $\frac{v}{u}(0_+) = \frac{v'(0)}{u'(0)} = A$, which proves the left-hand inequality. ∎

We shall now apply this lemma to the components Y_j of the Jacobi field $Y = (Y_1, \ldots, Y_{2n})$ which are real functions satisfying $Y_j(0) = 0$, $Y_j'(0) = v_j$, and $-kY_j \leq Y_j'' \leq kY_j$, for all $j = 1, \ldots, 2n$, where $2n$ is the real dimension of the manifold M, and $v = (v_1, \ldots, v_{2n})$ are the components of $v \in T_m M \simeq \mathbb{R}^{2n}$. We get

$$|Y_j(t) - Y_j'(0)t|^2 \leq \left| \frac{\sinh(\sqrt{k}t)}{\sqrt{k}} - t \right|^2 |v_j|^2, \quad \text{for } j = 1, \ldots, 2n,$$

if we take into account that $\sin x \leq x \leq \sinh x$, for $x \geq 0$. A summation over $j = 1, \ldots, 2n$ gives

$$\|Y(t) - Y'(0)t\| \leq \left| \frac{\sinh(\sqrt{k}t)}{\sqrt{k}} - t \right| \|v\|,$$

for all t, v, u. From this we get, after dividing out by t, that

$$\|(T_{tu} \exp_m)(v) - \mathrm{Id}(v)\| \leq \left| \frac{\sinh(\sqrt{k}t)}{\sqrt{k}t} - 1 \right| \|v\|,$$

$$\|T_{tu} \exp_m - \mathrm{Id}\,\| \leq \left| \frac{\sinh(\sqrt{k}t)}{\sqrt{k}t} - 1 \right|,$$

for all t, u. If we set $x = tu$, we find

$$\|T_x \exp_m - \mathrm{Id}\,\| \leq \left| \frac{\sinh(\sqrt{k}\|x\|)}{\sqrt{k}\|x\|} - 1 \right|, \quad \text{for all } x \in T_m M.$$

Since $\sinh x \geq x$, for all $x \geq 0$, the absolute value is superfluous in the right-hand term. We have thus proved the following.

PROPOSITION 0.5.2. *If there exists a constant $k > 0$ such that*

$$-k \leq K(p, P) \leq k,$$

for every point $p \in M$ *and every plane* $P \subset T_pM$*, then*

$$\|T_x \exp_m - \mathrm{Id}\,\| \leq \frac{\sinh(\sqrt{k}\|x\|)}{\sqrt{k}\|x\|} - 1, \quad \textit{for all } x \in T_mM.$$

Remark. The Rauch comparison theorem estimates $\|T_x \exp_m\|$. The above proposition estimates the distance between $T_x \exp_m$ and $T_0 \exp_m = \mathrm{Id}$. The latter is therefore slightly more general.

0.6. Final estimate

We will now complete the proof of Theorem 0.1.4. We only have to get a uniform control of

$$\int_{Y_c} \frac{|\nabla^k(J^kF_{k-1})|^2}{|\Lambda^r(ds)|^{2\frac{r+k}{r}}}\, dV_{Y,\omega}$$

(see the end of Section 0.3).

Fix a point $y_0 \in Y \subset X$, and let $\Phi := \exp_{y_0} : T_{y_0}X \to X$ be the exponential map. The Kähler metric ω on the weakly pseudoconvex manifold X can be made complete by a standard well-known procedure. We may therefore assume, without loss of generality, that the exponential map is defined on the whole tangent space. Let ω_0 be the standard Kähler metric on the Euclidian space $T_{y_0}X \simeq \mathbb{C}^n$. Our first goal in this section is to find an explicit formula for the radius of the ball in the tangent space $T_{y_0}X$ on which the two metrics, $\Phi^\star\omega$ and ω_0, can be compared. Let us set

$$r(y_0) := \sup\Big\{r > 0\,;\ \sup_{\substack{x\in B(y_0,\,r)\\ 0\le l\le m}} r^{2+l}\,\|\nabla^l\Theta(T_X)(x)\| < 10^{-2a}\Big\}, \tag{0.6.1}$$

where $a > 0$ is a constant to be specified later, and $\nabla^l\Theta(T_X)$ stands for the l^{th} order derivative of the curvature tensor $\Theta(T_X)$ viewed as a section of the C^∞ bundle $\Lambda^{1,1}T_X^\star \otimes \mathrm{Hom}(T_X, T_X)$. Locally, this boils down to deriving the coefficients of $\Theta(T_X)$. In particular, we get

$$\sup_{x\in B(y_0,\,r(y_0))} \|\Theta(T_X)\| \leq \frac{10^{-2a}}{r(y_0)^2} := k,$$

and hence the sectional curvature of X satisfies:

$$-k \leq K(p, P) \leq k,$$

for all $p \in B(y_0, r(y_0))$, and all planes $P \subset T_{y_0}X$ in the tangent space at y_0 to X.

This shows that the hypothesis of Proposition 0.5.2 is fulfilled in the ball $B(y_0, r(y_0))$. Then we get

$$(\star) \qquad \|T_v \exp_{y_0} - \mathrm{Id}\,\| \leq \frac{\sinh(\sqrt{k}\|v\|)}{\sqrt{k}\|v\|} - 1,$$

for all $v \in T_{y_0}X$, such that $\|v\| < r(y_0)$. If $\|T_v \exp_{y_0} - \mathrm{Id}\,\| < 1$, the map $T_v \exp_{y_0}$ is invertible. Consequently, $\exp_{y_0}$ is an immersion on $B(0, r(y_0)) \subset T_{y_0}X$, if $\frac{\sinh(\sqrt{k}\|v\|)}{\sqrt{k}\|v\|} < 2$ for all v such that $\|v\| < r(y_0)$. To achieve this, it is enough to have

$$(1) \qquad \frac{\sinh(10^{-a})}{10^{-a}} < 2.$$

On the other hand, we need a value of the constant a such that we may have the bounds

$$(\star\star) \qquad \frac{1}{2}\omega_0 \leq \exp_{y_0}^{\star} \omega \leq 2\omega_0, \quad \text{on the ball } B(0, r(y_0)) \text{ in } T_{y_0}X.$$

In order to have these bounds, it is enough to have

$$\frac{1}{2} \leq \|T_v \exp_{y_0}\| \leq 2,$$

for all $v \in T_{y_0}X$ such that $\|v\| < r(y_0)$. We thus infer from $(\star)$ that

$$2 - \frac{\sinh(\sqrt{k}\|v\|)}{\sqrt{k}\|v\|} \leq \|T_v \exp_{y_0}\| \leq \frac{\sinh(\sqrt{k}\|v\|)}{\sqrt{k}\|v\|},$$

for all $v \in T_{y_0}X$, $\|v\| < r(y_0)$. This shows that it is enough to have $\frac{\sinh(\sqrt{k}\|v\|)}{\sqrt{k}\|v\|} \leq \frac{3}{2}$, for all v such that $\|v\| < r(y_0) = \frac{10^{-a}}{\sqrt{k}}$. The bounds $(\star\star)$ are therefore guaranteed as soon as the constant a satisfies the inequality

$$(2) \qquad \frac{\sinh(10^{-a})}{10^{-a}} \leq \frac{3}{2}.$$

In short, we have proved the following.

Lemma 0.6.1. *For a choice of the constant $a > 0$ satisfying inequality* (2), *and for $r(y_0)$ defined by relation* (0.6.1), *the exponential map $\Phi = \exp_{y_0}$ is an immersion and the bounds $(\star\star)$ hold on the ball $B(0, r(y_0))$ in the tangent space $T_{y_0}X$.*

Lemma 0.4.1 shows that there exist local holomorphic coordinates $\zeta = (\zeta', \zeta'')$, $\zeta' = (\zeta_1, \dots, \zeta_r)$, $\zeta'' = (\zeta_{r+1}, \dots, \zeta_n)$ on the ball $B(0,\, r) \subset T_{y_0}X$ such that the subvariety $\Phi^{-1}(Y \cap B(y_0,\, r)) \subset B(0,\, r)$ is defined by the equations $\zeta' = 0$, for the following radius

$$r = \rho(y_0) = \frac{1}{6\,\|Ds_{y_0}^{-1}\|_{\omega_0} \sup_\xi(\|(D^2 s_\xi\|_{\omega_0} + \|Ds_\xi\|_{\omega_0})}.$$

Moreover, the bounds $(\star\star)$ imply

$$r \geq \frac{1}{24\,\|Ds_{y_0}^{-1}\|_{\omega} \sup_\xi(\|D^2 s_\xi\|_{\omega} + \|Ds_\xi\|_{\omega})} := r_0(y_0).$$

In the above expressions all $\sup_\xi$ are computed for $\xi \in B(y_0,\, r(y_0))$. Let us set from now on:

$$r_1(y_0) = \min(r(y_0),\, r_0(y_0)). \tag{0.6.2}$$

Recall that $F_{k-1} \in H^0(X,\, \Lambda^n T_X^\star \otimes L)$ is the $(k-1)$-order extension of the jet $f \in H^0(X,\, \Lambda^n T_X^\star \otimes L \otimes \mathcal{O}_X/\mathcal{J}_Y^{k+1})$ given by the induction hypothesis we have set up to prove Theorem 0.1.4 (see the beginning of Section 0.3). The holomorphic line bundle $L' := \Lambda^n T_X^\star \otimes L$ is equipped with a C^∞ Hermitian metric h. Let us consider the C^∞ line bundle $\Phi^\star L'$ equipped with the metric $\phi^\star h$, and the section $\Phi^\star F_{k-1} \in C^\infty(T_{y_0}X,\, \Phi^\star L')$.

Let $J_X \in \mathrm{End}(T_X)$ be the complex structure of the manifold X, and $J := \Phi^\star J_X$ the almost complex structure induced on $T_{y_0}X$. If J_0 is the canonical complex structure of $T_{y_0} \simeq \mathbb{C}^n$, the map Φ is not (J_0, J_X)-holomorphic, but it certainly is (J, J_X)-holomorphic. If $i\Theta(L')$ is the curvature form (of type $(1,1)$) of (L', h), $\Phi^\star(i\Theta(L'))$ is a type $(1,1)$-form for J on $T_{y_0}X$.

LEMMA 0.6.2. *There exists a real function $\tilde\varphi \in C^\infty$ on the ball $B = B(0,\, r_1(y_0))$ in the tangent space $T_{y_0}X$ such that $i\partial_J\bar\partial_J\tilde\varphi = \Phi^\star(i\Theta(L'))$ and*

$$\sup_B |\tilde\varphi| \leq C \sup_B \|\Phi^\star(i\Theta(L'))\|,$$

where $C > 0$ is a constant depending only on $r_1(y_0)$.

Proof. With respect to real coordinates $x_1, \dots, x_{2n}$ on B, the real d-closed 2-form $\Phi^\star(i\Theta(L'))$ can be written as $\Phi^\star(i\Theta(L')) = \sum_{i<j} v_{ij}\, dx_i \wedge$

dx_j, with functions $v_{ij} \in C^\infty(B)$. The Poincaré lemma gives the explicit formula:

$$U(x) = \sum_{i<j} \left(\int_0^1 t\, v_{ij}(tx)\, dt \right) (x_i\, dx_j - x_j\, dx_i),$$

for a C^∞ solution to the equation $dU = \Phi^\star(i\Theta(L'))$ on B. We see then that

$$\|U\|_{L^\infty(B)} \leq C_1\, \|\Phi^\star(i\Theta(L'))\|_{L^\infty(B)},$$

with a constant $C_1 > 0$ depending only on the radius of B. With respect to the almost complex structure J, the real 1-form U decomposes as $U = U^{1,0} + U^{0,1}$, with $U^{0,1} = \overline{U^{1,0}}$. Then $dU = \partial_J U^{0,1} + \overline{\partial_J U^{0,1}}$, since dU is of type $(1,1)$ for J. The almost complex structure J is integrable as the inverse image of an integrable almost complex structure. Let $(z_1, \ldots, z_n)$ be J-holomorphic complex coordinates centred at 0 on a neighbourhood of the ball $B \subset T_{y_0}X$. We thus have $\bar\partial_J U^{0,1} = 0$ on B. The bounds $(\star\star)$, relating the metrics ω and ω_0, allow us to assume that the ball B is J-pseudoconvex (if not so, we multiply the radius $r_1(y_0)$ by a fixed constant). Since for an integrable almost complex structure we have the same formalism as for a complex analytic structure, a classical result on the solvability of the $\bar\partial$ operator on bounded strictly pseudoconvex domains with a C^2 boundary in $\mathbb{C}^n$ (see, for instance, [HL84, Theorem 2.3.5]), yields the existence of a constant $C_2 > 0$ depending only on the radius of the ball B, and of a solution to the equation $\bar\partial_J v = U^{0,1}$ on B obtained by an explicit integral formula, such that

$$\|v\|_{L^\infty(B)} \leq C_2\, \|U^{0,1}\|_{L^\infty(B)} \leq 2\, C_2\, \|U\|_{L^\infty(B)}.$$

Then $\tilde\varphi := i(\bar v - v)$ is the function we were looking for. □

Since ϕ is an immersion on $B(0, r_1(y_0))$, there exists a neighbourhood $V \subset B(0, r_1(y_0))$ of 0 such that ϕ is a diffeomorphism of V onto a neighbourhood U of y_0 in X. Let $\psi : U \to V$ be the inverse diffeomorphism. In a local trivialization of L' in a neighbourhood of y_0, the section F_{k-1} can be written as $F_{k-1} = u \otimes e$, with respect to a local holomorphic frame e. The function $v = u \circ \Phi$ is then C^∞ on V, and u being holomorphic implies $\bar\partial(v \circ \psi) = 0$. If $z = (z_1, \ldots, z_n)$ is a system of local holomorphic coordinates on U, this means that v is a solution to the following elliptic system

$$(\star\star\star) \qquad \sum_j \frac{\partial v}{\partial \zeta_j} \circ \psi\, \frac{\partial \psi_j}{\partial \bar z_k} + \sum_j \frac{\partial v}{\partial \bar\zeta_j} \circ \psi\, \frac{\partial \bar\psi_j}{\partial \bar z_k} = 0, \quad k = 1, \ldots, n.$$

Let us remind now a standard differential operator theory result. Gårding's lemma controls the growth of the derivatives of a solution to an elliptic equation in terms of the growth of this very solution. This lemma plays the role of Cauchy's inequalities in the nonholomorphic case. Let H_j^{loc} be the Sobolev space of locally L^2 functions whose all derivatives in the sense of distributions up to order j are still locally L^2, and let $\| \ \|_j$ be its Sobolev norm. We refer for the details to [Agm65] (Lemma 6.1 and Theorems 6.2–6.7, pages 53–67).

Theorem 0.6.3. (Theorem 6.5 in [Agm65]) *Let* Ω *be an open subset of* $\mathbb{R}^n$*, and* $A_1(x,D),\ldots,A_N(x,D)$ *differential operators of respective orders* $m_1,\ldots,m_N$*, with coefficients* $a_\alpha^i \in C^\infty$*, which make up an elliptic system in* Ω*. Let* $u \in L^2_{\text{loc}}(\Omega)$ *such that* $A_i^\star u \in H_{k_i}^{\text{loc}}(\Omega)$*, for all* $i = 1,\ldots,N$*.*

If $j := \min(m_1+k_1,\ldots,m_N+k_N)$*, then* $u \in H_j^{\text{loc}}(\Omega)$*. In addition, for all* $\Omega' \subset\subset \Omega$*, there exists* $\gamma = \gamma(A_i,\Omega',\Omega)$ *such that*

$$\|u\|_{j,\Omega'} \leq \gamma\left(\sum_{i=1}^N |A_i^\star u|_{k_i,\Omega} + \|u\|_{0,\Omega}\right),$$

where $\gamma = \text{Const}\cdot p\cdot N\cdot K\cdot M$*,* Const *is a universal constant,* $p = p(n,l) = \text{card}\{\alpha \in \mathbb{N}^n \mid |\alpha| = l\}$*,* $K = \sup_{\xi\in\Omega',\,|\alpha|\leq l,i} |da_\alpha^i(\xi)|$*,* $M = \sup_{x\in\Omega',\,|\alpha|\leq l,i} |a_\alpha^i(x)|$*.*

The actual dependence of the constant γ on the data is not explicit in [Agm65], but it can be easily inferred from the proofs given there to Theorems 6.2–6.7. Likewise, the statement given there is slightly more general as the coefficients of the operators $A_i(x,D)$ are only assumed to be "s-smooth".

Since v is a solution to the elliptic system $(\star\star\star)$, the previous theorem shows that we have the estimate

$$\sup_{\|\zeta''\|\leq\frac{1}{2}r_1(y_0)} \sum_{|\alpha|=k} \left|\frac{\partial^\alpha v}{\partial\zeta'^\alpha}(0,\zeta'')\right|^2 \leq \gamma_k \int_{B(0,\,r_1(y_0))} |v(\zeta',\zeta'')|^2\, d\lambda(\zeta',\zeta''),$$

where $\gamma_k = \mathit{Const}\cdot p_k\cdot \max\bigl(\sup_{\xi\in U}\|d_\xi\psi\|, \sup_{\xi\in U}\|d_\xi^2\psi\|\bigr)$, $p_k = \text{card}\{\alpha \mid |\alpha| = k\}$, and *Const* is a universal constant. For the following norms computed in the Hermitian vector bundle $(\Phi^\star L', \Phi^\star h)$, equipped with the local weight $\tilde{\varphi}$,

$$\left\| \frac{\partial^\alpha v}{\partial \zeta'^\alpha}(0,\zeta'') \right\|^2 = \left| \frac{\partial^\alpha v}{\partial \zeta'^\alpha}(0,\zeta'') \right|^2 e^{-2\tilde{\varphi}(0,\zeta'')},$$
$$\|v(\zeta',\zeta'')\|^2 = |v(\zeta',\zeta'')|^2 \, e^{-2\tilde{\varphi}(\zeta',\zeta'')},$$

we get the estimate

$$\int_{\|\zeta''\| \leq \frac{1}{2} r_1(y_0)} \sum_{|\alpha|=k} \left\| \frac{\partial^\alpha v}{\partial \zeta'^\alpha}(0,\zeta'') \right\|^2 d\zeta''$$
$$\leq \gamma_k \int_{B(0,\, r_1(y_0))} \|v(\zeta',\zeta'')\|^2 \, e^{2(\tilde{\varphi}(\zeta',\zeta'') - \tilde{\varphi}(0,\zeta''))} \, d\lambda(\zeta',\zeta''),$$

and also, thanks to Lemma 0.6.2,

$$\int_{\|\zeta''\| \leq \frac{1}{2} r_1(y_0)} \sum_{|\alpha|=k} \left\| \frac{\partial^\alpha v}{\partial \zeta'^\alpha}(0,\zeta'') \right\|^2 d\,\zeta'' \tag{3}$$
$$\leq \gamma_k \, C_{L'} \int_{B(0,\, r_1(y_0))} \|v(\zeta',\zeta'')\|^2 \, d\lambda(\zeta',\zeta''),$$

where the constant $C_{L'} := e^{2C \sup_U \|i\Theta(L')\|}$ depends only on the growth of the curvature of L'.

It remains to infer from the estimate (3) for v an analogous estimate for u. If z is the variable on $U \subset X$, and ζ is the variable on $V \subset T_{y_0}X$, the change of variable $\zeta = \psi(z)$ implies the following estimate for u

$$\|u\|^2_{k, U' \cap Y} \leq \tilde{\gamma}_k \, C_{L'} \, \|u\|^2_{0,U}, \quad U' \subset\subset U, \tag{4}$$

where $\tilde{\gamma}_k = Const \cdot p_k \cdot \sup_{\substack{1 \leq l \leq k \\ \xi \in U}} \|d^l_\xi \psi\|$, *Const* being a universal constant.

Proposition 0.5.2 has already given an estimate for the norm of the differential of the exponential map ϕ, and implicitly for the differential map of ψ. The formula for $\tilde{\gamma}_k$ would also require an estimation of the growth of the differentials of order $\leq k$ of ψ. It is clear that $\sup_{\substack{1 \leq l \leq k \\ \xi \in U}} \|d^l_\xi \psi\|$ is bounded above by a constant depending only on the radius $r_1(y_0)$ of the ball on which we are working. These are standard calculations that can well be left to the reader.

We are now in a position to conclude that the constant $C_r^{(k)}$ in the statement of Theorem 0.1.4 depends only on r, on k, on E, and on $\sup_\Omega \|i\Theta(L)\|$.

A standard argument (see [Dem00, 4.8, p. 12]) shows that the restriction imposed at the beginning of Section 0.3 on the singular set $\Sigma = \{s = 0, \Lambda^r(ds) = 0\}$ of Y to be empty, is superfluous.

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