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# FINITE POTENT GROUPS

JOHN POLAND

A group is potent if for any element of the group and any prescribed positive integer (dividing its order if this order is finite) there corresponds a finite homomorphic image of the group in which the element has the prescribed integer as its order. The finite potent groups form a finite variety that contains all finite nilpotent groups, all finite metabelian groups, and precisely one simple group, $A_5$ .

A finite group $G$ is called *potent* (following Lennox and Rhemtulla [2]) if and only if for any element $x$ of $G$ and divisor $n$ of the order of $x$ there is a normal subgroup $N$ of $G$ such that $xN$ has order precisely $n$ in $G/N$ .

The concept was developed originally in the context of residually finite groups: an arbitrary group $A$ is potent if and only if for every element $y$ of $A$ and every positive integer $m$ (dividing the order of $y$ if this order is finite), there is a normal subgroup $M$ of finite index in $A$ such that $yM$ has order (precisely) $m$ in $A/M$ . Finitely generated nilpotent groups are potent; as free groups have this property residually they too must be potent. Hence quotients of potent groups need not be potent, and Lennox and Rhemtulla illustrated that even metabelian groups need not be potent. The context of their discussion was primarily torsion-free groups.

In contrast we will show that the finite potent groups form a finite

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variety (subgroups, quotients and finite direct products of finite potent groups are potent), and all finite metabelian groups are potent. The implication for infinite groups is that periodic potent groups and torsion-free potent groups behave differently. And the classification of periodic potent groups will rest on results for finite potent groups: a periodic group is potent if it is residually a finite potent group because the finite potent groups form a finite variety.

The development of these results on finite potent groups is independent of the paper of Lennox and Rhemtulla [2], but I wish to thank Dr Rhemtulla for bringing to my attention the need for an investigation of these groups, and supplying a preprint of [2].

Let us begin with a simple test for potency.

**PROPOSITION.** *The finite group $G$ is not potent if and only if there exists an element $x$ of $G$ and a proper prime divisor $p$ of the order of $x$ such that $x$ lies in the normal closure of $x^p$ (that is, every normal subgroup containing $x^p$ also contains $x$).*

Proof. Suppose $G$ has an element $x$ lying in the normal closure of $x^p$ for some proper prime divisor $p$ of the order $n = pr$ of $x$, and suppose $xN$ has order $p$ for some normal subgroup $N$ of $G$. Then $N$ contains $x^p$ but not $x$. But if $N$ contains $x^p$ it must contain its normal closure and hence $x$, a contradiction. Thus $G$ is not potent.

Conversely, suppose $G$ is not potent so there is an element $y$ of $G$ of order $n$, and for some (proper) divisor $m$ of $n$, every normal subgroup $K$ of $G$ is such that $yK$ has order different from $m$. Pick $y$ and $m$ such that $m$ is minimal; thus, for every proper divisor $k$ of $m$ there is a corresponding normal subgroup $K$ such that $yK$ has order (precisely) $k$.

Suppose $m$ were a prime power, say $m = p^{t+1}$, $p$ prime. Put $x = y^{p^t}$ so $x$ has order $p(n/m)$ - thus $p$ is a proper divisor of the order of $x$. Let $N$ be any normal subgroup of $G$ and suppose $x^p$ lies in $N$. This means $N$ contains $y^m$ and so $yN$ has order dividing $m$.

By the conditions on $m$ and $y$, $yN$ must have order a proper divisor of $m = p^{t+1}$ and so $(yN)^{p^t} = N$. That is, $y^{p^t} = x$ must lie in $N$. This means that every normal subgroup containing $x^p$ must also contain $x$; as $p$ was a proper divisor of the order of $x$, we are done.

If $m$ were not a prime power, say $m = ab$ with $a$ and $b$ relatively prime proper divisors of $m$, then there would exist normal subgroups $K$ and $L$ with $yK$ of order $a$ and $yL$ of order $b$. The proof will be complete if we can just show that $y(K \cap L)$ would have as order the least common multiple of $a$ and $b$, namely $m$.

LEMMA. *If $N$ and $M$ are normal subgroups of the group $G$ then the order of $x(N \cap M)$ is the least common multiple of the orders of $xN$ and $xM$.*

Proof. If $n$ and $m$ are the orders of $xN$ and $xM$, and $k$ is the least common multiple of $n$ and $m$ then $x^k$ lies in $N$ and $M$ and hence in $N \cap M$. But $x^i$ lies in $N \cap M$ if and only if it lies in $N$ and in $M$ and hence $i$ is divisible by $n$ and $m$. Thus $k$ is the order of $x(N \cap M)$.

THEOREM. *(i) Subgroups of potent groups are potent.*

*(ii) Quotients of potent groups are potent.*

*(iii) Direct products of (finitely many) potent groups are potent.*

Proof. *(i)* If $H$ is a non-potent subgroup of the group $G$ then $H$ has an element $x$ that lies in $\langle x^p \rangle^H$ for some proper prime divisor $p$ of the order of $x$. Clearly $x$ has the same order in $G$, and $\langle x^p \rangle^G$ contains $\langle x^p \rangle^H$ and hence $x$, so $G$ cannot be potent, as we wished to show.

*(ii)* Suppose $G$ were a minimal counterexample: a potent group with a normal subgroup $N$ such that $G/N$ is not potent. Without loss of generality we can assume $N$ is a minimal normal subgroup of $G$ because for any non-trivial normal subgroup $M$ lying in $N$, $G/M$ cannot be potent (otherwise, it would be a smaller counterexample than $G$). Since $G/N$ is assumed not to be potent, there exists an $x$ in $G$ and a prime $p$

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properly dividing the order of $(xN)$ - and hence the order of $x$ too - such that for every normal subgroup $K$ of $G$, if $x^p$ lies in $KN$ then so does $x$. But $G$ is potent so $x$ does not lie in the normal closure $L$ of $x^p$ in $G$.

Now if $L$ contains $N$ then $xLN = xL$ with $xL \neq L$ but $(xL)^p = L$, a contradiction. Hence the minimal normal subgroup $N$ must avoid $L$ and $LN \simeq L \times N$. Since $(xLN)^p = x^pLN = LN$ then $x$ lies in $LN$ and so $x = ab$ for (commuting) elements $a$ of $L$ and $b$ of $N$. Consequently $x^p = (ab)^p = a^pb^p$ and we know $x^p$ (and $a^p$) lie in $L$, hence $b^p$ lies in $L \cap N = 1$. Therefore $b^p = 1$.

But now consider $a = xb^{-1}$. Since $xb = bx$ and $b^p = 1$ then $a^p = x^p$, so $L = \langle x^p \rangle^G$ and $L$ contains $a$. This will contradict the potency of $G$ if $p$ is a proper divisor of the order of $a$. But the latter is straightforward to verify: if $a^k = 1$ then $(xb^{-1})^k = 1$ so $x^k = b^k$ and $(xN)^k = N$ so the order of $xN$ divides the order of $a$ and $p$ must be a proper prime divisor of the order of $a$. This final contradiction establishes that quotients of potent groups are potent.

*(iii)* Let $G = A \times B$ be the direct product of potent groups $A$ and $B$ and let $x = ab$ for $a$ in $A$, $b$ in $B$. The order $n$ of $x$ is a least common multiple of the orders $h$ of $a$ and $k$ of $b$, so if $m$ is a divisor of $n$ then $m$ is a least common multiple of divisors $h'$ of $h$ and $k'$ of $k$. Now $A$ and $B$ have normal subgroups $A'$ and $B'$ respectively such that $aA'$ and $bB'$ have orders $h'$ and $k'$ respectively. Hence $xA'B$ and $xAB'$ have orders $h'$ and $k'$ so $x(A'B \cap AB')$ has order $m$ as required.

The following three corollaries can be easily deduced and we omit their proofs.

**COROLLARY.** *Every finite nilpotent group is potent.*

**COROLLARY.** *If $G$ is the smallest group in a quotient-closed class that is not potent, then $G$ is monolithic.*

**COROLLARY.** *The class of finite potent groups is a formation.*

**EXAMPLE.** The class of finite potent groups is not a Fitting class: the wreath product $G = S_3 \wr C_2$ is not a potent group (the base group $B$ is the normal closure of $x^3$ for each of its elements $x$ of order $6$ ) but $G = N_1N_2$ for proper normal potent subgroups $N_1$ and $N_2$ of $G$ (in fact, all proper subgroups of $G$ are potent).

**EXAMPLE.** The smallest group which is not potent is the binary tetrahedral group $G = \langle x, y \mid x^3 = y^3 = (xy)^2 = z, z^2 = 1\rangle$ of order $24$, a central extension of $A_4$ (equivalently, $G = \mathrm{SL}(2, 3)$ ). $G$ has a normal Sylow 2-subgroup which is quaternion; its compliments are not normal (and not maximal) in $G$ so every normal subgroup of $G$, and every maximal subgroup, contains $\langle z\rangle$. Clearly $\langle z\rangle = Z(G) = \Phi(G)$ is the unique minimal normal subgroup of $G$. For every normal subgroup $N$ of $G$ either $xN$ has order $6$, if $N = 1$, or $xN$ has order $3$ (or $1$ ) if $N$ contains $z = x^3$. Thus $G$ is not potent. The groups of smaller order are either $p$-groups for some prime $p$, or of order $pqr$ for primes $p$, $q$ and $r$. (It is easily checked that groups of order $pqr$ must be potent: a cyclic subgroup of order $pq$ is either normal or has a normal complement.) Notice that the binary tetrahedral group shows that $G/Z(G)$ and $G/\Phi(G)$ may be potent without $G$ being potent. Thus the formation of potent groups is not saturated.

The other two binary polyhedral groups, central extensions of $S_4$ and $A_5$, are also not potent, for similar reasons, even though $S_4$ and $A_5$ are potent.

**THEOREM.** $A_5$ *is the only non-abelian simple potent group.*

*Proof.* For a simple group $G$ to be potent, every element must have prime order, and so $G$ will be a $CN$-group with elementary abelian Sylow 2-subgroup. Suzuki [4] has shown that if $G$ is non-abelian then it must be isomorphic to the projective special linear group $\mathrm{PSL}(2, 2^n)$. It is will known that $\mathrm{PSL}(2, 2^n)$ has elements of orders $2^n + 1$ and $2^n - 1$, and for both of these to be primes we require $n = 2$. Thus $G \simeq \mathrm{PSL}(2, 4) \simeq A_5$.

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**COROLLARY.** *$A_5$ is the only non-abelian composition factor of a finite potent group.*

**EXAMPLE.** While every nilpotent group is potent, not every supersolvable group is. The smallest such group

$$G = \langle a,\ b,\ c,\ t \mid a^3 = b^3 = c^3 = [a,\ c] = [b,\ c] = 1,\ [a,\ b] = c,\ [a,\ t] = a,\ [b,\ t] = b,\ t^2 = 1\rangle$$

has centre $\langle c\rangle$ of order $3$, so $ct$ has order $6$ and $\langle t\rangle^G = G$ - hence $G$ is not potent - but $\langle c\rangle \subset \langle a,\ c\rangle \subset \langle a,\ b,\ c\rangle \subset G$ is a supersolvable series for $G$.

In order to establish that (finite) metabelian groups are potent, we collect together the basic conditions on a minimal counterexample, and then consider the intermediate step of metacyclic groups. This also has natural implications for Frobenius groups, which we discuss before completing the proof that metabelian groups are potent.

**LEMMA.** *If $G$ is a finite group containing an element $x$, in $K = \langle x^p\rangle^G$ for some proper prime divisor $p$ of $o(x) = pr$ - so $G$ is not a potent group - and if $N$ is an elementary abelian normal $q$-group with $G/N$ potent, then*

*(i)* $p = q$,

*(ii)* $\gcd(p,\ r) = 1$,

*(iii)* $o(xN) = r$ *with* $x^r$ *of order* $p$ *in* $N$,

*(iv)* $x = yz = zy$ *where* $o(y) = p$, $o(z) = r$ *and* $K = \langle z\rangle^G$,

*(v) the elements* $x^r$ *and* $y$ *lie in* $K \cap N$, *so* $K \cap N \neq 1$.

Proof. From the given conditions on $x$, every normal subgroup $M$ containing $N$ yields $o(xM) \neq p$ and since $G/N$ is potent, this means $p$ does not divide $o(xN)$. But $x^{pr} = 1$, and if $x^m \in N$ then $\left(x^m\right)^q = 1$. Hence $q = p$, $p$ does not divide $r$ and $o(xN) = r$, so $x^r$ is an element of order $p$ in $N$. Since there exist $s$ and $t$ such that $sp + rt = 1$ then putting $y = x^{rt}$ and $z = x^{ps}$ yields the results in

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*(iv).* Lastly, $x \in K$ and $x^r$ is a non-trivial element of $N$ so $K \cap N \neq 1$.

**THEOREM.** *Every finite metacyclic group is potent.*

Proof. Let $G$ be a minimal counterexample. Then $G$ has a unique minimal normal subgroup $N$, cyclic of prime order $p$, and $G/N$ is potent. Furthermore, there exist commuting elements $y$ and $z$ of orders $p$ and $r$, $r$ relatively prime to $p$, with $y$ in $N$ and $yz$ lying in $\langle z \rangle^G$.

As $G$ is metacyclic it is generated by elements $a$ and $b$ of orders $n$ and $m$, say, with $b^{-1}ab = a^s$ for some positive integer $s$. Because $\langle a \rangle$ is normal in $G$, and $G$ is monolithic, $n$ must be a power of $p$, say $n = p^k$, and we may assume $y = a^{p^{k-1}}$. For similar reasons, $\langle a \rangle$ must be the Fitting subgroup, so $C(a) = \langle a \rangle$.

Let $z = a^i b^j$ for some integers $i$ and $j$. Since $z^{-1}yz = y = a^{p^{k-1}}$ then $a^{s^j p^{k-1}}$. Hence $p^k$ (the order of $a$) divides $(s^j-1)p^{k-1}$ and $p$ divides $(s^j-1)$. Let us write $s^j - 1 = tp^h$ for some integers $h > 0$ and $t \geq 0$, where $t$ is prime to $p$ if $t > 0$.

Now should $h \geq k$ or $t = 0$ then $p^k$ divides $s^j - 1$ and so $az = za$. Then $C_G(a)$ would contain an element $z$ of order $r$ relatively prime to $p$, a contradiction. Hence $h < k$ and $t > 0$.

Since $z^r = 1$ then $a = z^{-r}az^r = a^{s^{rj}}$ so $p^k$ divides $(s^{rj}-1) = (1+tp^h)^r - 1 = rtp^h + p^{h+1}$ for some integer $g$. Because $0 < h < k$ this means $p$ divides $rt$ even though $r$ and $t$ were relatively prime to $p$. This final contradiction ensures that the minimal counterexample cannot exist.

**THEOREM.** *Every Frobenius group with metacyclic Frobenius complement is potent (this includes all Frobenius groups with non-abelian kernels), but there exist both solvable and non-solvable Frobenius groups that are not potent.*

Proof. A Frobenius group $G$ is the disjoint union of the Frobenius

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kernel $K$ and its Frobenius complements (which are all conjugate in $G$). If a Frobenius complement $H$ is metacyclic it is potent: if $x \in H$ and $m$ divides the order of $x$ there is a normal subgroup $N$ of $H$ so $xN$ has order $m$. But then $xNK$ has order $m$ and $NK \lhd G$. So we need only consider $y \in K$. The following lemma suffices to complete this case.

**LEMMA.** *If $K$ is nilpotent, $y \in K$, and $m$ divides the order of $y$ then there exists a characteristic subgroup $C$ of $K$ such that $yC$ has order $m$.*

Proof. It suffices to prove this in the case $K$ is a $p$-group. Then $K$ has a series $1 = C_0 < C_1 < C_2 < \ldots < C_k = K$ of characteristic subgroups $C_i$ such that $C_{i+1}/C_i$ is elementary abelian, of exponent $p$. If $y^{p^s} \in C_{i+1}$ then $y^{p^s} \in C_i$ or $y^{p^{s+1}} \in C_i$. Since $yC_k$ has order $p^0$ and $yC_0$ has order $p^n$ somewhere in between $yC_i$ has the required order $m$.

The group $\mathrm{PSL}(2, 31)$ contains copies of $A_4$, $S_4$, and $A_5$, so $\mathrm{SL}(2, 31)$ contains each of the binary polyhedral groups. As described in Huppert ([1], p. 500) each of these can serve as the (solvable or non-solvable) Frobenius complement by acting in the natural way as fixed-point-free automorphism groups of, say, $C_{31} \times C_{31}$. Since the binary polyhedral groups are not potent, these Frobenius groups cannot be potent either.

**THEOREM.** *Every metabelian group is potent.*

Proof. Let $G$ be a minimal counterexample. Since quotients of metabelian groups are metabelian, $G$ must have a unique minimal normal subgroup $N$ of prime order $p$, and $G$ must contain a $p'$-element $x$ that centralizes a generator $y$ of $N$ such that $\langle x\rangle^G$ contains $xy$ - that is, in no quotient does $xy$ have order $p$. We will call $x$ the witnessing element of $G$, witnessing that $G$ is not potent.

Since the derived group $G'$ of $G$ is abelian, any $p'$-elements it contains must form a Hall $p'$-subgroup characteristic in $G'$ and hence normal in $G$; because $G$ is monolithic we are forced to conclude $G'$ is a $p$-group, lying in a normal Sylow $p$-subgroup $P$ of $G$, and for similar reasons, $P$ must be the Fitting subgroup of $G$. This means $P$ cannot be cyclic, for otherwise $G/P$, as a group of automorphisms of $P$, would have to be cyclic and $G$ would be metacyclic - contradicting that $G$ is supposed not to be potent.

Let $H$ denote a Hall $p'$-subgroup of $G$ that contains the witnessing element $x$ and put $K = HG'$. Note that $H$ is abelian because it is embeddable in $G/G'$ and $K$ is normal in $G$. By a theorem of Zassenhaus (Huppert [1], p. 350), we then have

$$G' = [G', K] \times \big(G' \cap Z(K)\big) .$$

Since $G$ is monolithic one of these direct factors must be trivial. Now if $[G', K] = 1$ and $G' \leq Z(K)$ then $[G', H] = 1$ and so $[G', G] = [G', P]$, and the nilpotence of $P$ would force $G$ to be nilpotent, a contradiction since nilpotent groups are potent. Therefore $Z(K) = 1$ $\big($and hence $N \cap Z^G = 1$, meaning $Z(G) = 1\big)$, and $G' = [G', K] = [G', H]$.

With $Z(G) = 1$ it is now a straightforward application of Maschke's Theorem to show that $P$ cannot be abelian. For if $P$ were abelian, $G$ would have an elementary abelian normal subgroup $\Omega_1(P)$, properly containing $N$ as $P$ is not cyclic, with a nontrivial $p'$-group of automorphisms $G/C_G\big(\Omega_1(P)\big)$, so $\Omega_1(P) = N \times M$ with $M \lhd G$ - a contradiction.

Our aim at this point will be to show that $K$ is not a potent group. Since we have shown that $G'$ is a proper subgroup of $P$, $K$ is a proper subgroup of $G$ and this will contradict the minimality of $G$.

Notice that if $L$ is a Carter subgroup of $G$ then $G = LG'$ and $L \cap G' = 1$ (Schenkman [3], p. 227). Without loss of generality $L$ contains the witnessing element $x$ that centralized a generator $y$ of $N$ with $\langle x \rangle^G$ containing $xy$ (and hence demonstrated that $G$ was not potent). Recall that $x$ also lies in $H$ and in $K$. Now if $g$ is any element of $G = LG'$ then $g = uw$ for some $u$ in $L$, $w$ in $G'$. Since $L$ is isomorphic to $G/G'$, $L$ is abelian, and $x^g = x^{uw} = x^w$. In fact, $\langle x \rangle^G = \langle x \rangle^{G'} = \langle x \rangle^K$. Hence $K$ contains a $p'$-element $x$ and a

$p$-element $y$ with the property that $\langle x\rangle^K$ contains $xy$, so $K$ is not potent, yielding the desired contradiction.

The examples of non-potent groups presented earlier show the centre-by-metabelian, metabelian-by-cyclic, and abelian-by-nilpotent finite groups need not be potent.

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Department of Mathematics and Statistics,
Carleton University,
Ottawa,
Canada KIS 5B6.