PROOF. Let the circles AEC and ABD cut AO in X and Y respectively. Then each of the angles XEC, XCE, YDB, YBD is in the same segment of a circle as OAB or OAC, and is equal to $\frac{1}{2}A$. Consequently the triangles XEC, YDB are congruent, and the four lines XE, XC, YD, YB are equal. It also follows that XE is a tangent to the circle EAO, and YD to DAO, so that, $XO.XA = XE^2 = YD^2 = YO.YA$. Accordingly X and Y must coincide, at the centre of a circle $BCDE$. In this circle the equal chords BD and CE subtend equal angles at the circumference. Therefore angles B and C are equal (or supplementary: this is impossible), and the triangle is isosceles.

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A note on equilateral polygons

By A. D. RUSSELL.

Theorem. If a circle cut all the sides (produced if necessary) of an equilateral polygon, the algebraic sum of the intercepts between the vertices and the circle is zero; *i.e.*, if any side AB of the polygon be cut by the circle in P and Q, then $\Sigma(AP + BQ) = 0$, the intercepts being signed by fixing a positive direction round the contour of the polygon.

This was proved in a previous note,[1] using Euclid III. 35, for the special case of a *regular* polygon; but the same proof applies to any *equilateral* polygon, which may be re-entrant or self-crossing.

Corollary. If n chords P_iOQ_i $(i = 1, \ldots, n)$ of a circle are drawn through a point O parallel to the sides of an equilateral n-gon, the

[1] "Theorem regarding a regular polygon and a circle cutting its sides," *Mathematical Notes*, No, 22, 1924.

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algebraic sum of their $2n$ segments, $\Sigma(OP_i + OQ_i)$, is zero. (As before, the segments are signed according to a positive direction round the contour of the polygon.)

This is a limiting case of the theorem, deduced by considering a variable polygon whose sides tend to zero while remaining parallel to those of the given polygon.

Trigonometrical Interpretation. Let the exterior angles of the equilateral n-gon (*i.e.*, the angles between the positive directions of consecutive sides) be $a_1, a_2, \ldots, a_n$. These are also the angles between the consecutive chords $P_1OQ_1, P_2OQ_2, \ldots, P_nOQ_n$. If the first of these chords makes θ with OC, where C is the centre, then, paying due attention to sign,

$$OC\cos\theta = \tfrac{1}{2}(OP_1 + OQ_1), \qquad OC\cos(\theta + a_1) = \tfrac{1}{2}(OP_2 + OQ_2), \quad \text{etc.};$$

and the Corollary gives

$$\cos\theta + \cos(\theta + a_1) + \cos(\theta + a_1 + a_2) + \ldots + \cos(\theta + a_1 + a_2 + \ldots + a_{n-1}) = 0.$$

Since θ depends on the position of O, which is arbitrary, this is true for all values of θ. Replacing θ by $\theta - \frac{1}{2}\pi$, it follows that

$$\sin\theta + \sin(\theta + a_1) + \sin(\theta + a_1 + a_2) + \ldots + \sin(\theta + a_1 + a_2 + \ldots + a_{n-1}) = 0.$$

Formulae involving the interior angles $A_1 = \pi - a_1$, etc., may be deduced. As a verification, taking $\theta = 0$ in the case of a rhombus ($n = 4$), we have the obviously correct results

$$1 - \cos A_1 + \cos(A_1 + A_2) - \cos(A_1 + A_2 + A_3) = 0,$$
$$\sin A_1 - \sin(A_1 + A_2) + \sin(A_1 + A_2 + A_3) = 0.$$

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