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ON THE VECTOR SUM OF TWO CONVEX SETS IN SPACE

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0. **Introduction.** In the paper [KIS2], C. Kiselman studied the boundary smoothness of the vector sum of two smoothly bounded convex sets A and B in $\mathbb{R}^2$. He discovered the startling fact that even when A and B have real analytic boundary the set $A + B$ need not have boundary smoothness exceeding $C^{20/3}$ (this result is sharp). When A and B have C^∞ boundaries, then the smoothness of the sum set breaks down at the level C^5 (see [KIS2] for the various pathologies that arise).

Kiselman considers sets A and B that are supergraphs of functions f and g; the sum of the two sets corresponds to an infimal convolution of f and g. Kiselman's theorems reduce to questions about the infimal convolution calculus. The methods work only in $\mathbb{R}^2$.

The purpose of the present note is three-fold. First, we discover what positive results hold in $\mathbb{R}^N$ for any N. We find that boundary smoothness of the sum depends only on boundary smoothness of one of the sets. Secondly, we discover how the smoothness of the sum depends in an explicit geometric fashion on the smoothness of the boundary of just one of the domains. Thirdly, we isolate a collection of domains for which the sum is always C^∞. It should be noted that, in contrast to Kiselman's work, many arguments of this paper apply only in case one of the domains is bounded. The reason is that we use repeatedly the relation $\partial(A + B) \subseteq \partial A + \partial B$, which is not in general true if both A and B are unbounded.

Kiselman makes special note that, as a result of his infimal calculus, his results apply to sums of finitely many domains. The same is true of results in the present paper, for a more direct reason: In our theorems about two domains, one of the domains needs only to be convex (no extra smoothness or non-degeneracy is required); therefore, by the associativity of addition, our results will apply automatically to a sum of finitely many domains.

In recent years, questions about how regularity of partial differential equations depends on convexity conditions of the boundary have played a central role in several complex variables (see, for example, [FOK], [FEK]). More recently Baouendi [BAR] and D'Angelo [JPD] have isolated convexity conditions of "finite type" which arise in

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the study of other kinds of partial differential equations. The present paper is a contribution to the study of this new geometric phenomenon.

Throughout this paper, the symbols c, C, c', etc. will be used to denote constants that are independent of the relevant parameters in any given inequality. The specific values of these constants may change from occurrence to occurrence.

Finally, a note about the form of this paper. There exist a number of algebraic and analytic structures which, in principle, shed light on the questions considered here. These include the infimal convolution calculus, the generalized matrix inverses of Moore and Penrose, and the parallel addition of matrices introduced by Anderson and Duffin. It is our feeling that the use of this machinery would serve to obscure the essential geometry of the problems we address. Therefore we use strictly geometric methods (in the spirit of the regularity theory of Almgren—see [ALA]) and, with no gain in length, are able to present the results in a self-contained fashion.

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1. **A class of domains in $\mathbb{R}^{N+1}$ for which the sum is always smooth.** Let α and β be $N \times N$ symmetric real matrices and assume that α is positive definite and that β is positive semi-definite. Define

$$A = \{ (x_1, x_2, \ldots, x_N, y) \in \mathbb{R}^{N+1} : y \geq x\alpha\,{}^t x\}$$

and

$$B = \{ (x_1, x_2, \ldots, x_N, y) \in \mathbb{R}^{N+1} : y \geq x\beta\,{}^t x\}.$$

Here if M is any matrix then ${}^tM$ denotes its transpose. We have:

THEOREM 1. *The sum $A + B \equiv \{a + b : a \in A, b \in B\}$ has real analytic boundary.*

PROOF. We will use the notation $X = (x; y)$ with $x \in \mathbb{R}^N$ and $y \in \mathbb{R}$ to denote a point of $\mathbb{R}^{N+1}$. If $X \in \partial(A + B)$, $X = X_1 + X_2$ with $X_1 \in A$, $X_2 \in B$, then it follows that $X_1 \in \partial A$ and $X_2 \in \partial B$ (see Remark 3). Writing $X_1 = (x^1; y^1)$ and $X_2 = (x^2; y^2)$ we have the following equations:

$$y^1 = x^1 \alpha\,{}^t(x^1) \tag{1}$$

$$y^2 = x^2 \beta\,{}^t(x^2) \tag{2}$$

$$y = y^1 + y^2 \tag{3}$$

$$x = x^1 + x^2 \tag{4}$$

$$\alpha\,{}^t(x^1) = \beta\,{}^t(x^2). \tag{5}$$

The only one of equations (1) – (5) that is not obvious is (5): it expresses the fact that the tangent plane to ∂A at x^1 is parallel to the tangent plane to ∂B at x^2. An independent proof of this fact is given in Lemma 1 of the next section.

Now since α is non-singular we may solve equations (1) – (5) as follows: from (5),

$$ {}^t(x^1) = \alpha^{-1}\beta\,{}^t(x^2) \tag{6} $$

hence, by (4),

$$ {}^t x = (I + \alpha^{-1}\beta)\,{}^t(x^2). \tag{7} $$

Now $(1 + \alpha^{-1}\beta)$ is invertible by the definiteness hypotheses on α and β so that the last line gives

$$ {}^t(x^2) = (I + \alpha^{-1}\beta)^{-1}\,{}^t x. \tag{8} $$

We plug this into (6) to obtain

$$ {}^t(x^1) = \alpha^{-1}\beta(I + \alpha^{-1}\beta)^{-1}\,{}^t x. \tag{9} $$

We substitute (8) into (2) and note that α and β are self-adjoint to conclude that

$$ {}^t(y^2) = x(I + \alpha^{-1}\beta)^{-1}\beta(I + \alpha^{-1}\beta)^{-1}\,{}^t x. \tag{10} $$

We also have

$$ \alpha^{-1}\beta(I + \alpha^{-1}\beta)^{-1} = I - (I + \alpha^{-1}\beta)^{-1}. \tag{11} $$

We substitute (9) into (1) and utilize (11) to get

$$ \begin{aligned} {}^t(y^1) &= x(I + \alpha^{-1}\beta)^{-1}\beta\alpha^{-1}\beta(I + \alpha^{-1}\beta)^{-1}]\,{}^t x \\ &= x(I + \alpha^{-1}\beta)^{-1}\beta[I - (I + \alpha^{-1}\beta)^{-1}]\,{}^t x. \end{aligned} \tag{12} $$

Adding (10) and (12), we have

$$ y = x\beta(I + \alpha^{-1}\beta)^{-1}({}^t\beta)x. $$

We conclude that y is a real analytic function of x, hence that $A + B$ has real analytic boundary. ∎

REMARKS.

(1) if A is a convex domain with C^∞ boundary then, near a point $P \in \partial\Omega$ of strong convexity, the boundary can be written in local coordinates so that P is the origin of coordinates and

$$ y^1 = \mathcal{A}(x) = x^1\alpha x^1 + O(|x^1|^3) $$

with α a symmetric, positive definite $N \times N$ matrix. Likewise, near a point $Q \in \partial B$ of convexity the boundary of B can be described in local coordinates so that Q is the origin of coordinates, the tangent hyperplane at Q is parallel to the tangent hyperplane to ∂A at P, and

$$ y^2 = \mathcal{B}(x), $$

where $\mathcal{B}$ is a function of N real variables with positive semi-definite Jacobian. Then all of the steps of the preceding proof can be imitated to determine that $\partial A + \partial B$ can be described by

$$y = x(I + \alpha^{-1}\,\mathrm{Jac}(\mathcal{B}))^{-1}\mathcal{B}^t x + O(|x|)^3,$$

and the right side of this equation is a smooth function of x. It follows under these circumstances that $A + B$ has smooth boundary near $P + Q$.

(2) In the proof of Theorem 1, we used the positive definiteness of the matrices only to verify that $(I + \alpha^{-1}\beta)$is invertible. Indeed this invertibility holds if all the eigenvalue of α exceed 1 in absolute value and all those of β are less than one in absolute value. Therefore Theorem 1, and Remark (1), would hold in these circumstances. We do not understand the geometric significance of this observation.

(3) We mentioned in the introduction that the relation $\partial(A + B) \subseteq \partial A + \partial B$ does not generally hold for unbounded sets. A simple counterexample is

$$A = \{(x, y) : xy \geq 1, y > 0\},$$
$$B = \{(x, y) : xy \leq -1, y > 0\}.$$

Then $A + B = \{(x, y) : y > 0\}$ and clearly

$$(*) \qquad \partial(A + B) \not\subseteq \partial A + \partial B.$$

However we can show that $(*)$ *does hold* for the domains of the type considered in this section. For if $(x, y) = X \in \partial(A+B)$ then there must be sequences $(x^{1,i}, y^{1,i}) = X_{1,i} \in A$ and $(x^{2,i}, y^{2,i}) = X_{2,i} \in B$ such that

$$(X_{1,i} + X_{2,i}) \longrightarrow X.$$

By the positive definiteness of α we have that $y^{1,i} \geq c|x^{1,i}|$ while the semi-definiteness of β gives us $y^{2,i} \geq 0$. Since $y^{1,i} + y^{2,i} \longrightarrow y$, we conclude that $\{y^{1,i}\}$ and $\{y^{2,i}\}$ are bounded sequences. It then follows that $\{x^{1,i}\}$ and $\{x^{2,i}\}$ are bounded sequences. By extracting convergent subsequences, we see that $X \in A+B$ and, of course, it then follows that $X \in \partial A + \partial B$.

(4) The domains treated in Theorem 1 may be handled rather naturally with the algebraic machinery of the Moore-Penrose generalized inverse for a matrix (see [PEN]) and the matrix "parallel sum" of Anderson and Duffin [AD]. However the introduction and utilization of that language would, in the present context, serve only to obscure matters and would not shorten the proof. We therefore merely provide references for the interested reader.

2. **Sufficient conditions for the boundary of the sum to be C^1.** The principle result of this section is the following:

THEOREM 2. *Assume that either the set A is bounded or the set B is bounded. If A is a convex set with C^1 boundary and B is any convex set then $A + B$ has C^1 boundary.*

Christer Kiselman has informed us that this result is known (private communication); however parts of the proof will be needed later and we present the details. The proof hinges on the following lemma:

LEMMA. *Let A and B be as in Theorem 2. Then each point in the boundary of $A + B$ has a unique supporting hyperplane.*

PROOF. Let $z \in \partial(A + B)$. Write $z = a + b$ with $a \in \partial A$ and $b \in \partial B$. Let v^1 be the inward unit normal to ∂A at a. Then for each $0 < \tau \leq 1$ we can find $0 < \rho < \infty$ such that

$$\{ u : |u - a| \leq \rho \text{ and } \tau|u - a| \leq v^1 \cdot (u - a)\} \subseteq A.$$

Here "$\cdot$" is the usual Euclidean dot product. Therefore, by addition,

$$S_\tau \equiv \{ u : |u - z| \leq \rho \text{ and } \tau|u - z| \leq v^1 \cdot (u - z)\} \subseteq A + B.$$

Now the union over τ of S_τ determines a unique hyperplane at a, and this is the supporting plane that we seek. ∎

The proof also yields the following:

COROLLARY. *With A, B as in the theorem, $z \in \partial(A + B), z = a + b$ with $a \in \partial A$ and $b \in \partial B$, it holds that the supporting hyperplane to $A + B$ at z is parallel to the supporting hyperplane to A at x.*

PROOF OF THE THEOREM. Fix $z \in \partial(A + B)$. Let $z^j \in \partial(A + B)$ converge to z. We may choose $a^j \in \partial A$ and $b^j \in \partial B$ such that $a^j + b^j = z^j$ and we may assume that the a^j converge to some a^0 in ∂A and the b^j converge to some b^0 in ∂B. By the continuity of addition, $z = a^0 + b^0$. Of course the supporting hyperplane to ∂A at a^j converges to the supporting hyperplane at a^0. So the lemma guarantees that the supporting hyperplane to $\partial(A + B)$ at $a^j + b^j$ converges to the supporting hyperplane at $z = a^0 + b^0$ (because these are the very same hyperplanes as in the preceding sentence). But this means that the boundary of $A + B$ is continuously differentiable. ∎

REMARK. It seems to be a standard fact, at least in folklore, that a convex set has C^1 boundary if and only if the tangent plane at each boundary point is uniquely determined. A part of the above proof formalizes this idea. ∎

Recall (see [FED1], [FED2, 3.2.36]) that a closed set $S \subseteq \mathbb{R}^N$ is of *positive reach* ϵ if each point x of distance less than ϵ from S has a unique nearest point in S. For example, any closed C^2 hypersurface in $\mathbb{R}^N$ is a set of positive reach. In fact it is enough for the hypersurface to be C^1 with Lipschitz turning tangent planes (see [KRP]). However, it is also shown in [KRP] that this boundary smoothness cannot be essentially weakened further. It is important in geometric analysis to be able to recognize sets of positive reach. To this end we have

PROPOSITION 1. *Assume that either the set A is bounded or the set B is bounded. If A and B are closed convex sets with C^1 boundaries and if ∂A has positive reach r_A and ∂B has positive reach r_B then $\partial(A+B)$ has positive reach at least $r = r_A + r_B$.*

PROOF. Since A and B are convex, it is only necessary to check positive reach for points in the interior of $A+B$. Let z be an interior point of $A+B$ and suppose that

$$\operatorname{dist}\left[z, \partial(A+B)\right] < r.$$

Let $z^* \in \partial(A+B)$ be some nearest point to z, so that

$$|z - z^*| = \operatorname{dist}(z, \partial(A+B)).$$

Write $z^* = x^* + y^*$ for some $x^* \in \partial A$ and some $y^* \in \partial B$. Let ν^* be the inward unit normal to ∂A at x^*. By the Corollary just proved it follows that ν^* is also the inward unit normal to ∂B at y^*. Choose $s_A < r_A$ and $s_B < r_B$ such that

$$s_A + s_B = |z - z^*|.$$

Then $x \equiv x^* + s_A \nu^* \in A$ and $y \equiv y^* + s_B \nu^* \in B$. Also

$$\operatorname{dist}(x, \partial A) = s_A \text{ and } \operatorname{dist}(y, \partial B) = s_B.$$

Now suppose that z^{**} is a point of $\partial(A+B)$ such that $|z^{**} - z| = |z^* - z|$ and that z^{**} is different from z^*. We will show that z^{**} cannot lie in $\partial(A+B)$. For set

$$\nu = \frac{z^{**} - z}{|z^{**} - z|}.$$

It follows that $x + s_A \nu \in \operatorname{Int} A$ and $y + s_B \nu \in \operatorname{Int} B$, whence

$$z^{**} = (x + s_A \nu) + (y + s_B \nu) \in \operatorname{Int}(A+B).$$

This is the desired result. ∎

Notice that in fact the Proposition holds if only one of the two sets has positive reach.

3. **Sufficient conditions for the sum to be $C^{1,\alpha}$.** In this section we develop a geometric characterization of $C^{1,\alpha}$ boundary for a domain in Euclidean space. This result should prove useful in other contexts of geometric analysis. For us, the interest is in its application to the vector sum of convex sets.

Throughout this section, let C be a compact, convex subset of $\mathbb{R}^N$ with C^1 boundary ∂C. If $x \in \partial C$ then let ν_x be the unit inward normal at x. Define

$$\Pi_x(y) = y - (y \cdot \nu_x)\nu_x,$$

the orthogonal projection into the tangent plane to ∂C at x. Write

$$\nu_x(y) \equiv y - \Pi_x(y).$$

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LEMMA. *Let C be a compact convex domain in $\mathbb{R}^N$ with C^1 boundary. Let notation be as above. Fix $0 < \alpha \leq 1$. Then ∂C is $C^{1,\alpha}$ if and only if there exists a positive constant c such that for each $x \in \partial C$ it holds that*

$$(*) \qquad \{y : c|\Pi_x(y) - x|^{1+\alpha} \leq v_x \cdot y - x \leq c^{-1}\} \subseteq A$$

PROOF. If ∂C is $C^{1,\alpha}$ then it is a straightforward exercise to see that $(*)$ holds.

For the converse, we begin with some definitions. Because of the smoothness of ∂C there are positive constants γ_1, γ_2 such that for each $x \in \partial C$ it holds that

$$\partial C \cap \{y : |\Pi_x(y) - x| < \gamma_1,\ |v_x(y) - x| < \gamma_2\}$$

is the graph of a C^1 function. We denote that function by f_x. Notice that $Df_x(x) = 0$. Thus given an $\epsilon > 0$ there is a $\delta > 0$ such that for $\xi \in \mathbb{R}^{N-1}$ with $|\xi| < \delta$ we have $|Df_x(\xi)| < \epsilon$.

Now fix α and c for which $(*)$ is assumed to be true. Fix also a point x_0 of ∂C. We show that, in a neighbourhood of x_0, ∂C is the graph of a $C^{1,\alpha}$ function; the size of the neighbourhood and the $C^{1,\alpha}$ norm of the function will not depend on x_0.

There is no loss of generality to suppose that $x_0 = 0$ and that the tangent plane to ∂C at x_0 is

$$\Pi_0 \equiv \{y : y \cdot e_N = 0\}.$$

Here $e_N = (0, 0, \cdots, 0, 1)$. We also will write points $y \in \mathbb{R}^N$ in the form $y = (y', y_N)$ with $y' \in \mathbb{R}^{N-1}$ and $y_N \in \mathbb{R}$.

We choose $\delta > 0$ such that for any $x \in \partial C$ and any $\xi \in \mathbb{R}^{N-1}$ with $|\xi| < \delta$ it holds that

$$|Df_x(\xi)| < \delta \cdot c^{-1}.$$

Let y be a point of ∂C with

$$|x_0 - y| < \min\{1, \delta, \gamma_1, \gamma_2\}.$$

Let Π_y be the tangent plane to ∂C at y. Since $(*)$ holds there exists a positive constant d such that

$$\Xi_d \equiv \{(y', d|y'|^{1+\alpha}) : y \in \mathbb{R}^{N-1}\}$$

is tangent to Π_0 at some point (s', s).

Let us examine this situation more closely. The slope of Ξ_y at the point y in the radial direction (from $x_0 = 0$) is, by calculus,

$$(**) \qquad (1 + \alpha) \cdot d \cdot |s'|^{\alpha}.$$

This is the slope of the tangent plane Π_y also, so we can estimate how close the set $\Pi_y \cap \Pi_0$ is to the origin. The radial symmetry of the situation reduces this to a problem in plane geometry, and we find that

$$\operatorname{dist}\left[0, \Pi_y \cap \Pi_0\right] = \alpha(1 + \alpha)^{-1}|s'|.$$

Here "dist" denotes the usual Euclidean distance of sets. We also have that

$$|y'| \geq \text{dist}\left[0, \Pi_y \cap \Pi_0\right].$$

Combining these two estimates gives

$$(***) \qquad |s'| \leq (1+\alpha)\alpha^{-1}|s|.$$

We claim that $d \leq c$. This would imply that the slope of the tangent plane Π_y is bounded by

$$(1+\alpha)^{(1+\alpha)}\alpha^{-\alpha}c|y'|^{\alpha},$$

proving the lemma.

In order to prove the claim, we assume the contrary. Then $s > c^{-1}$ so that

$$d|s'|^{1+\alpha} = s > c^{-1}.$$

It follows that

$$\begin{aligned}(1+\alpha)d|s'|^{\alpha} &> (1+\alpha)c^{-1}|s'|^{-1}\\ &\geq \alpha c^{-1}|y'|^{-1}\\ &\geq \alpha \cdot c^{-1}.\end{aligned}$$

On the other hand,

$$(1+\alpha)d|s'|^{\alpha} = |Df_x(y')| \leq \alpha c^{-1}.$$

The last two strings of inequalities taken together provide the desired contradiction. ∎

We use the lemma to obtain the following result.

THEOREM 3. *For $0 < \alpha \leq 1$ let A be bounded, convex, and have $C^{1,\alpha}$ boundary. Let B be any convex domain. Then the vector sum $A + B$ has $C^{1,\alpha}$ boundary.*

PROOF. For simplicity we give the proof for $\alpha = 1$ and comment on the general case afterward. If ∂A is $C^{1,1}$ then, by the work in [KRP] there is a number $r > 0$ so that for each $x \in \partial A$ there is a Euclidean ball B_x of radius r that is internally tangent to ∂A at x. The same claim cannot be made for B; however there is an $\epsilon > 0$ such that at each point y of ∂B there is the inward normal segment S_y^ϵ of length ϵ that is contained in B. But then each point z of $\partial(A + B)$, coming as it does from a point $x \in \partial A$ and a point $y \in \partial B$, has the internally tangent region given by $B_x + S_y^\epsilon$. This region is convex and has C^2 boundary—indeed the boundary is spherical—near $z = x + y$. It follows immediately that the hypotheses of the lemma are satisfied for the set $A + B$ with $\alpha = 1$. Therefore $A + B$ has $C^{1,1}$ boundary. ∎

To treat the case of Theorem 3 when $\alpha < 1$ we replace internally tangent balls by homothetes of regions of the form $\{(x', x_N) : x_N > c \cdot |x'|^{1+\alpha}\}$ for some fixed c. The rest of the proof is the same.

We refer to [KIS2] and [BOM] for examples indicating the limits of what is possible for the smoothness of vector sums of convex sets. If no hypotheses are made about the

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flatness of the boundary, then the boundary of the vector sum of two sets with C^k boundaries is C^k for $k = 1, 2, 3$, and 4—*provided* that the dimension is two. This assertion fails for $k = 5$ in dimension 2. A recent startling result of Jan Boman [BOM] exhibits in all dimensions greater than three a pair of bounded real analytic convex domains whose sum domain is not C^2 (by our Theorem 3, the sum is $C^{1,1}$). It remains an open problem to see whether there exist such examples in $\mathbb{R}^3$. The only general positive results in $\mathbb{R}^N$ for $N \geq 3$ are those presented in this paper.

For some purposes, it is natural to view the results of this paper in the context of *projections* of convex sets. For a consideration of this point of view, we refer the reader to [KIS1].

NOTE ADDED IN PROOF. The recent reprint *Regularity classes for operations in convexity theory* by C. O. Kiselman sheds new light on the topics considered here.

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