ON THE CORRELATION BETWEEN THE VOLUMES OF THE TYPICAL POISSON–VORONOI CELL AND THE TYPICAL STIENEN SPHERE

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Abstract

In this paper we consider a tessellation $\mathcal{V}$ generated by a homogeneous Poisson process Φ in $\mathbb{R}^d$ and, furthermore, the random set of spheres with centres being the points in Φ and having radii equal to half the distance to their closest neighbouring point in Φ. In $\mathbb{R}^d$ we give an integral formula for the correlation between the volume of the typical cell and the volume of the sphere in the typical cell, and we also show that this correlation is strictly positive. Furthermore, on the real line we give an analytical expression for the correlation, and in the plane and in space we give simplified integral formulae. Numerical values for the correlation for $d = 2, \ldots, 7$ are also given.

Keywords: Correlation; Poisson process; Robbins' formula; Stienen model; typical cell; Voronoi tessellation

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1. Introduction

Let Φ be a homogeneous Poisson process in $\mathbb{R}^d$ with intensity $\lambda \in (0, \infty)$ and let $\mathcal{V}$ denote the tessellation generated by Φ. This means that, for each point $\boldsymbol{x} \in \Phi$, we let $\mathcal{V}_{\boldsymbol{x}}$ be the cell that has the point $\boldsymbol{x}$ as its nucleus (or generator), i.e. $\mathcal{V}_{\boldsymbol{x}} = \{\boldsymbol{y} \in \mathbb{R}^d \colon \|\boldsymbol{y} - \boldsymbol{x}\| \leq \|\boldsymbol{y} - \boldsymbol{z}\|, \boldsymbol{z} \in \Phi\}$, where $\|\cdot\|$ denotes the Euclidean norm, and then $\mathcal{V} = \{\mathcal{V}_{\boldsymbol{x}} \colon \boldsymbol{x} \in \Phi\}$. The tessellation $\mathcal{V}$ is known as the *Poisson–Voronoi tessellation* and was introduced by Meijering [5]. For a more in-depth view of Poisson–Voronoi tessellations, the reader is referred to [6] and [7]. Calka [1], [2] investigated the size and the form of Voronoi cells in the plane. Hug *et al.* [4] investigated the shape of large Voronoi cells.

In order to describe the statistical properties of $\mathcal{V}$ it is useful to introduce the *typical cell*. Following the approach introduced in [6], we let Γ denote the set of polytopes in $\mathbb{R}^d$ equipped with a 'suitable' σ-field $\mathcal{A}$. For a set $A \in \mathcal{A}$, we define the following distribution:

$$Q(A) = \frac{1}{\lambda v_d(B)} \mathrm{E}\left[\sum_{\boldsymbol{x} \in \Phi \cap B} \mathbf{1}(\mathcal{V}_{\boldsymbol{x}} - \boldsymbol{x} \in A) \right]$$

for an arbitrary Borel set B such that $0 < v_d(B) < \infty$. Here, v_d denotes the Lebesgue measure. The typical Voronoi cell is defined as the random polytope $\mathcal{V}_\mathrm{t}$ on $(\Gamma, \mathcal{A})$ having distribution Q. It is known (see, e.g. [6]) that $\mathcal{V}_\mathrm{t}$ has the same distribution as $\mathcal{V}_\mathrm{o}$, where $\mathcal{V}_\mathrm{o}$ denotes the cell whose nucleus is the origin. This means that the distribution P of Φ is given by the Palm distribution at the origin. To emphasize this we use the notation Φ_o. On the real line $v_1(\mathcal{V}_\mathrm{o}) \sim \Gamma(2, 2\lambda)$
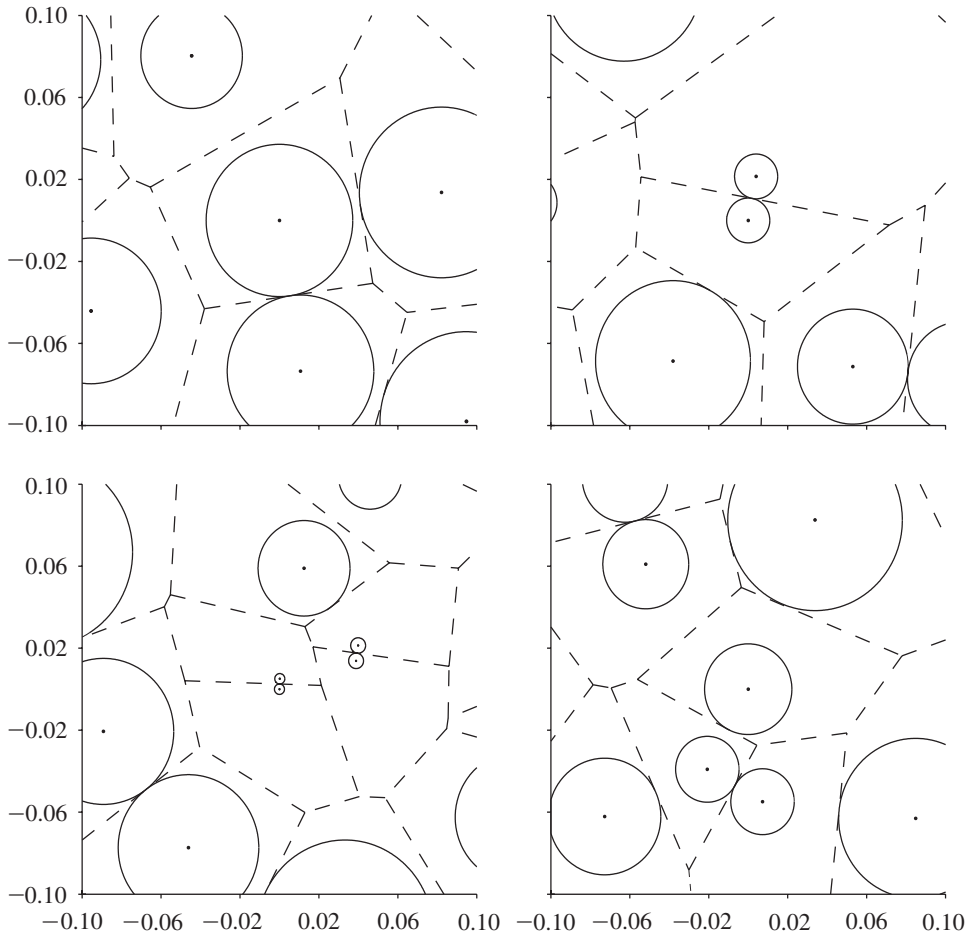
FIGURE 1: Examples of realizations of $\mathcal{V}_o$ and $\mathcal{S}_o$.

and, for $\mathbb{R}^2$, Calka [2] gave an expression for both the density of the number of neighbours and the conditional distribution function of the area of the typical cell, conditioned on the number of neighbours. For $d \geq 3$, the distribution of the volume of the typical cell is unknown.

Around each point $x \in \Phi$ we place a sphere with diameter equal to the distance to the closest neighbour of x. The random closed set made up of the union of these spheres is known as the Stienen model and was introduced by Stienen [11], motivated by an application in materials science. Conditioning on Φ with a point at the origin, we call the sphere having the origin as its centre the *typical Stienen sphere* (or the typical sphere) and denote it by $\mathcal{S}_o$. Distributional properties of the Stienen model were studied by Schlater and Stoyan [10]. Although the distributions of the volumes of the typical sphere and the typical cell (at least for $d = 2$) are known, the correlation between them is not available in the literature. At first it seems obvious that this correlation must be positive, but examining the selected realizations in Figure 1 suggests that the answer is not this obvious as, for example, two neighbouring points lying close together often produce small spheres and large Voronoi cells.

2. Main results

In this section we state and discuss the main results while the proofs are postponed to Section 3. In Proposition 2.1, below, we give a closed-form expression for the correlation between $v_1(\mathcal{V}_\mathrm{o})$ and $v_1(\mathcal{S}_\mathrm{o})$. In Theorem 2.1, below, we give an integral formula for $\mathrm{corr}(v_d(\mathcal{V}_\mathrm{o}), v_d(\mathcal{S}_\mathrm{o}))$ in arbitrary dimensions, $d \geq 2$.

Proposition 2.1. *On the real line it holds that*

$$\mathrm{corr}(v_1(\mathcal{V}_\mathrm{o}), v_1(\mathcal{S}_\mathrm{o})) = \frac{1}{\sqrt{2}}.$$

The real line is the only case where we have been able to obtain an analytical expression for $\mathrm{corr}(v_d(\mathcal{V}_\mathrm{o}), v_d(\mathcal{S}_\mathrm{o}))$. An integral formula for $\mathrm{corr}(v_d(\mathcal{V}_\mathrm{o}), v_d(\mathcal{S}_\mathrm{o}))$ in arbitrary dimensions, $d \geq 2$, is given in Theorem 2.1, below. Lemma 3.1 in Section 3 gives an integral formula that holds for any $d \geq 1$.

Theorem 2.1. *For $d \geq 2$, the correlation between the volume of $\mathcal{V}_\mathrm{o}$ and the volume of $\mathcal{S}_\mathrm{o}$ is given by*

$$\mathrm{corr}(v_d(\mathcal{V}_\mathrm{o}), v_d(\mathcal{S}_\mathrm{o}))$$

$$= \frac{\sigma_d^2 \int_0^\infty \int_0^{2v} u^{d-1} v^{d-1} \mathrm{e}^{-b_{d-1} W_d(u,v)} \, \mathrm{d}u \, \mathrm{d}v - (2^d - 1)/2^d}{\sqrt{(d-1)(b_d/b_{d-1}) \int_0^\pi \int_0^\infty (\sin^{d-2}(\alpha) R^{d-1}/S_d(R,\alpha)^2) \, \mathrm{d}R \, \mathrm{d}\alpha - 1}}, \qquad (2.1)$$

where

$$W_d(u, v) = u^d \int_{\arccos(u/2v)}^\pi \sin^d t \, \mathrm{d}t + v^d \int_{\arccos(2v^2-u^2/2v^2)}^\pi \sin^d t \, \mathrm{d}t \qquad (2.2)$$

and

$$S_d(R, \alpha) = R^d \int_\alpha^\pi \sin^d t \, \mathrm{d}t + (R^2 + 1 - 2R\cos\alpha)^{d/2} \int_{T(R,\alpha)}^\pi \sin^d t \, \mathrm{d}t, \qquad (2.3)$$

with

$$T(R, \alpha) = \arccos\left(\frac{1 - R\cos\alpha}{\sqrt{R^2 + 1 - 2R\cos\alpha}}\right).$$

Here, σ_d and b_d denote the surface area and the volume of the d-dimensional unit sphere, respectively.

The function $b_{d-1} W_d(u, v)$ can be interpreted as the volume of the union of two d-dimensional spheres with radii u and v and their centres a distance v apart. Furthermore, $b_{d-1} S_d(R, \alpha)$ can be interpreted as the volume of two spheres, with their centres a unit distance apart, where one has radius R and the other has radius $\sqrt{R^2 + 1 - 2R\cos\alpha}$. For any $d \geq 2$, the integrals defining $S_d(R, \alpha)$ and $W_d(u, v)$ can be evaluated in terms of elementary functions. This means that, for example, in the plane and in space it is straightforward to use (2.1) to obtain numerical approximations for the correlation between $v_d(\mathcal{V}_\mathrm{o})$ and $v_d(\mathcal{S}_\mathrm{o})$.

Corollary 2.1. *In the plane and in space the following two assertions hold.*

(i)

$$\mathrm{corr}(v_2(\mathcal{V}_\mathrm{o}), v_2(\mathcal{S}_\mathrm{o})) = \frac{4\pi^2 \mathcal{I}_2 - 3/4}{\sqrt{(\pi/2)\mathcal{I}_1 - 1}} \approx 0.7051,$$

where

$$\mathcal{I}_1 = \int_0^\pi \int_0^\infty \frac{R}{S_2(R,\alpha)^2}\, dR\, d\alpha$$

and

$$\mathcal{I}_2 = \int_0^\infty \int_0^{2v} uv e^{-2W_2(u,v)}\, du\, dv,$$

with

$$S_2(R,\alpha) = \frac{1}{2}\left\{(1 + R^2 - 2R\cos\alpha)\left(\pi - \arccos\left(\frac{1 - R\cos\alpha}{\sqrt{1 + R^2 - 2R\cos\alpha}}\right)\right) \right.$$
$$\left. + R^2(\pi - \alpha) + R\sin\alpha\right\}$$

and

$$W_2(u,v) = \frac{u^2}{2}\left(\pi - \arccos\frac{u}{2v}\right) + \frac{v^2}{2}\left(\pi - \arccos\left(1 - \frac{u^2}{2v^2}\right)\right) + \frac{u}{4}\sqrt{4v^2 - u^2}.$$

(ii)

$$\mathrm{corr}(\nu_3(\mathcal{V}_o), \nu_3(\mathcal{S}_o)) = \frac{16\pi^2 \mathcal{J}_2 - 7/8}{\sqrt{(8/3)\mathcal{J}_1 - 1}} \approx 0.6778,$$

where

$$\mathcal{J}_1 = \int_0^\pi \int_0^\infty \frac{R^2 \sin\alpha}{S_3(R,\alpha)^2}\, dR\, d\alpha$$

and

$$\mathcal{J}_2 = \int_0^\infty \int_0^{2v} u^2 v^2 \exp\left(-\frac{\pi(3u^4 + 8u^3 v + 16v^4)}{12v}\right) du\, dv,$$

with

$$S_3(R,\alpha) = \tfrac{1}{6}(4 + 9R^2 + 4R^3 - 12R\cos\alpha + 4(1 + R^2 - 2R\cos\alpha)^{3/2} + 3R^2\cos 2\alpha).$$

Table 1 displays numerical estimates of $\mathrm{corr}(\nu_d(\mathcal{V}_o), \nu_d(\mathcal{S}_o))$ for $d = 1, \ldots, 7$. The numerical results were obtained using the NIntegrate routine in MATHEMATICA®. As can be seen from Table 1, the correlation seems to decrease as the dimension increases. Using the bounds in (2.4), below, it follows that $\mathrm{corr}(\nu_d(\mathcal{V}_o), \nu_d(\mathcal{S}_o)) \to 0$ as $d \to \infty$.

TABLE 1: Numerical estimates of $\mathrm{corr}(\nu_d(\mathcal{V}_o), \nu_d(\mathcal{S}_o))$ for $d = 1, \ldots, 7$.

d	$\mathrm{corr}(\nu_d(\mathcal{V}_o), \nu_d(\mathcal{S}_o))$
1	0.707 107
2	0.705 143
3	0.677 790
4	0.649 534
5	0.623 393
6	0.599 667
7	0.578 145

It was questioned above whether $\mathrm{corr}(v_d(\mathcal{V}_\mathrm{o}), v_d(\mathcal{S}_\mathrm{o}))$ is positive or not. As seen, we cannot find a closed form for the correlation between $v_d(\mathcal{V}_\mathrm{o})$ and $v_d(\mathcal{S}_\mathrm{o})$ in arbitrary dimensions, but we can give a lower bound which shows that it is strictly positive, a result which is in accordance with intuition. Furthermore, we give bounds for the covariance between $v_d(\mathcal{V}_\mathrm{o})$ and $v_d(\mathcal{S}_\mathrm{o})$.

Proposition 2.2. *For any dimension, it holds that*

(i)

$$\frac{1}{2^{2d}(1+2^d)\lambda^2} \leq \mathrm{cov}(v_d(\mathcal{V}_\mathrm{o}), v_d(\mathcal{S}_\mathrm{o})) \leq \frac{1}{2^d\lambda^2}, \tag{2.4}$$

(ii)

$$\mathrm{corr}(v_d(\mathcal{V}_\mathrm{o}), v_d(\mathcal{S}_\mathrm{o})) \geq \frac{1}{2^d(1+2^d)}.$$

A result which lies a little outside the main focus of this article is the following proposition, which gives bounds on the second moment of the volume of the typical cell in arbitrary dimensions.

Proposition 2.3. *It holds that*

$$\frac{1}{\lambda^2} \leq \mathrm{E}[v_d(\mathcal{V}_\mathrm{o})^2] \leq \frac{2}{\lambda^2}.$$

Remark 2.1. The lower bound in Proposition 2.3 is elementary since $\mathrm{E}[v_d(\mathcal{V}_\mathrm{o})] = 1/\lambda$; see (3.3), below. Also note that, for $d=1$, we have $\mathrm{E}[v_1(\mathcal{V}_\mathrm{o})^2] = 3/2\lambda^2$.

3. Proofs

A result, given in [8] and [9], that will be central in the proofs states that

$$\mathrm{E}[v_d(X)^n] = \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathrm{P}(x_1, \ldots, x_n \in X)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n \tag{3.1}$$

for any random closed set X.

3.1. Moments of the typical cell

Equation (3.1) implies that

$$\begin{aligned}
\mathrm{E}[v_d(\mathcal{V}_\mathrm{o})^n] &= \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathrm{P}(x_1, \ldots, x_n \in \mathcal{V}_\mathrm{o})\, \mathrm{d}x_1 \cdots \mathrm{d}x_n \\
&= \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathrm{e}^{-\lambda U_d(x_1,\ldots,x_n)}\, \mathrm{d}x_1 \cdots \mathrm{d}x_n,
\end{aligned} \tag{3.2}$$

where $U_d(x_1, \ldots, x_n)$ is the volume of the union of n spheres in $\mathbb{R}^d$, centred at $x_1, \ldots, x_n$ and having radii $\|x_1\|, \ldots, \|x_n\|$, respectively. The property that

$$\mathrm{P}(x_1, \ldots, x_n \in \mathcal{V}_\mathrm{o}) = \mathrm{e}^{-\lambda U(x_1,\ldots,x_n)}$$

follows from the fact that the points $x_1, \ldots, x_n$ lie inside the typical cell if and only if

$$\Phi_\mathrm{o}^! \cap \bigcup_{i=1,\ldots,n} B_d(x_i, \|x_i\|) = \varnothing,$$

where $\Phi_o^!$ denotes $\Phi_o \setminus \{o\}$ and $B_d(\boldsymbol{x}, r) = \{\boldsymbol{y} \in \mathbb{R}^d : \|\boldsymbol{x} - \boldsymbol{y}\| \leq r\}$. It is well known that

$$\mathrm{E}[\nu_d(\mathcal{V}_o)] = \frac{1}{\lambda}, \tag{3.3}$$

which is straightforward to obtain from (3.2). This result was initially given in [3]. Unfortunately it is the only moment known in analytical form, except for the special case in which $d = 1$, where all moments are known. On the real line, $\mathcal{V}_o$ is an interval. The distance from the origin to the left-hand endpoint of $\mathcal{V}_o$ and the distance from the origin to the right-hand endpoint of $\mathcal{V}_o$ are two independent $\mathrm{Exp}(2\lambda)$-distributed random variables. This means that $\nu_1(\mathcal{V}_o) \sim \Gamma(2, 2\lambda)$, which implies that

$$\mathrm{E}[\nu_1(\mathcal{V}_o)^n] = \frac{(n+1)!}{(2\lambda)^n}. \tag{3.4}$$

3.2. Distribution of the typical sphere

Let R_o denote the radius of a typical sphere in the Stienen model, i.e. the sphere that has the origin as its centre. We have $\mathrm{P}(R_o > r) = \mathrm{e}^{-\lambda 2^d b_d r^d}$, as R_o is greater than r if and only if $\Phi_o^! \cap B_d(o, 2r) = \varnothing$. This means that

$$\mathrm{P}(\nu_d(S_o) > s) = \mathrm{P}(b_d R_o^d > s) = \mathrm{P}\left(R_o > \left(\frac{s}{b_d}\right)^{1/d}\right) = \mathrm{e}^{-\lambda 2^d s},$$

i.e. $\nu_d(S_o) \sim \mathrm{Exp}(2^d \lambda)$; hence,

$$\mathrm{E}[\nu_d(S_o)^n] = \frac{n!}{(2^d \lambda)^n}. \tag{3.5}$$

3.3. Proofs of Proposition 2.1 and Theorem 2.1

In this section we give the proofs of Theorem 2.1 and Proposition 2.1. To prove these results we will use the following lemma, which gives an integral formula for the correlation between $\nu_d(\mathcal{V}_o)$ and $\nu_d(S_o)$ for any d.

Lemma 3.1. *The correlation between the volume of $\mathcal{V}_o$ and the volume of S_o is given by*

$$\mathrm{corr}(\nu_d(\mathcal{V}_o), \nu_d(S_o)) = \frac{(1 - 2^d)/2^d + \iint_{\|\boldsymbol{x}_1\| \leq 2\|\boldsymbol{x}_2\|} \mathrm{e}^{-V_d(\boldsymbol{x}_1, \boldsymbol{x}_2)} \, \mathrm{d}\boldsymbol{x}_1 \, \mathrm{d}\boldsymbol{x}_2}{\sqrt{\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathrm{e}^{-U_d(\boldsymbol{x}_1, \boldsymbol{x}_2)} \, \mathrm{d}\boldsymbol{x}_1 \, \mathrm{d}\boldsymbol{x}_2 - 1}}, \tag{3.6}$$

where $V_d(\boldsymbol{x}_1, \boldsymbol{x}_2)$ is the volume of the union of two spheres in $\mathbb{R}^d$, centred at the origin and $\boldsymbol{x}_2$, with radii $\|\boldsymbol{x}_1\|$ and $\|\boldsymbol{x}_2\|$, respectively.

Proof. Define $\tilde{\mathcal{V}}_o$ as the part of $\mathcal{V}_o$ that lies outside S_o, i.e. $\tilde{\mathcal{V}}_o = \mathcal{V}_o \setminus S_o$. Now, to obtain the correlation between $\nu_d(\mathcal{V}_o)$ and $\nu_d(S_o)$ we first compute the covariance. We have

$$\begin{aligned}
\mathrm{var}(\nu_d(\tilde{\mathcal{V}}_o)) &= \mathrm{var}(\nu_d(\mathcal{V}_o) - \nu_d(S_o)) \\
&= \mathrm{var}(\nu_d(\mathcal{V}_o)) + \mathrm{var}(\nu_d(S_o)) - 2\,\mathrm{cov}(\nu_d(\mathcal{V}_o), \nu_d(S_o)),
\end{aligned}$$

which implies that

$$\mathrm{cov}(\nu_d(\mathcal{V}_o), \nu_d(S_o)) = \tfrac{1}{2}(\mathrm{var}(\nu_d(\mathcal{V}_o)) + \mathrm{var}(\nu_d(S_o)) - \mathrm{var}(\nu_d(\tilde{\mathcal{V}}_o))). \tag{3.7}$$

Now, from (3.2), (3.3), and (3.5) it follows that

$$\mathrm{var}(\nu_d(\mathcal{V}_{\mathrm{o}})) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-\lambda U_d(\mathbf{x}_1, \mathbf{x}_2)} \, d\mathbf{x}_1 \, d\mathbf{x}_2 - \frac{1}{\lambda^2}$$

and

$$\mathrm{var}(\nu_d(\mathcal{S}_{\mathrm{o}})) = \frac{1}{(2^d \lambda)^2}. \tag{3.8}$$

It remains to find $\mathrm{var}(\nu_d(\tilde{\mathcal{V}}_{\mathrm{o}}))$. The expectation is given by

$$\mathrm{E}[\nu_d(\tilde{\mathcal{V}}_{\mathrm{o}})] = \mathrm{E}[\nu_d(\mathcal{V}_{\mathrm{o}}) - \nu_d(\mathcal{S}_{\mathrm{o}})] = \frac{2^d - 1}{2^d \lambda}. \tag{3.9}$$

This is what we would expect since the volume fraction of the Stienen model is equal to 2^{-d}. We will use (3.1) to find the second moment of $\nu_d(\tilde{\mathcal{V}}_{\mathrm{o}})$; then we will need to compute $\mathrm{P}(\mathbf{x}_1, \mathbf{x}_2 \in \tilde{\mathcal{V}}_{\mathrm{o}})$. Now, for $\|\mathbf{x}_1\| \leq \|\mathbf{x}_2\|$, we have

$$\mathrm{P}(\mathbf{x}_1, \mathbf{x}_2 \in \tilde{\mathcal{V}}_{\mathrm{o}}) = \mathrm{P}(R_{\mathrm{o}} < \|\mathbf{x}_1\|, \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{V}_{\mathrm{o}})$$

and

$$\mathrm{P}(\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{V}_{\mathrm{o}}) = \mathrm{P}(R_{\mathrm{o}} < \|\mathbf{x}_1\|, \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{V}_{\mathrm{o}}) + \mathrm{P}(R_{\mathrm{o}} \geq \|\mathbf{x}_1\|, \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{V}_{\mathrm{o}}).$$

Furthermore, $\mathbf{x}_1$ and $\mathbf{x}_2$ lie in $\mathcal{V}_{\mathrm{o}}$ and $R_{\mathrm{o}} \geq \|\mathbf{x}_1\|$ if and only if there are no points of $\Phi_{\mathrm{o}}^!$ in $B_d(\mathrm{o}, 2\|\mathbf{x}_1\|) \cup B_d(\mathbf{x}_2, \|\mathbf{x}_2\|)$ and, therefore,

$$\mathrm{P}(R_{\mathrm{o}} \geq \|\mathbf{x}_1\|, \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{V}_{\mathrm{o}}) = e^{-\lambda V_d(\mathbf{x}_1, \mathbf{x}_2, 2)},$$

where

$$V_d(\mathbf{x}_1, \mathbf{x}_2, 2) = \nu_d(B_d(\mathrm{o}, 2\|\mathbf{x}_1\|) \cup B_d(\mathbf{x}_2, \|\mathbf{x}_2\|)).$$

Using (3.1), the above means that

$$\mathrm{E}[\nu_d(\tilde{\mathcal{V}}_{\mathrm{o}})^2] = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-\lambda U_d(\mathbf{x}_1, \mathbf{x}_2)} \, d\mathbf{x}_1 \, d\mathbf{x}_2 - 2 \iint_{\|\mathbf{x}_1\| \leq \|\mathbf{x}_2\|} e^{-\lambda V_d(\mathbf{x}_1, \mathbf{x}_2, 2)} \, d\mathbf{x}_1 \, d\mathbf{x}_2. \tag{3.10}$$

Using (3.2), (3.5), (3.9), and (3.10), (3.7) now becomes

$$\begin{aligned}
\mathrm{cov}(\nu_d(\mathcal{V}_{\mathrm{o}}), \nu_d(\mathcal{S}_{\mathrm{o}})) = {} & \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-\lambda U_d(\mathbf{x}_1, \mathbf{x}_2)} \, d\mathbf{x}_1 \, d\mathbf{x}_2 - \frac{1}{2\lambda^2} + \frac{1}{2(2^d \lambda)^2} \\
& - \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-\lambda U_d(\mathbf{x}_1, \mathbf{x}_2)} \, d\mathbf{x}_1 \, d\mathbf{x}_2 \\
& + \iint_{\|\mathbf{x}_1\| \leq \|\mathbf{x}_2\|} e^{-\lambda V_d(\mathbf{x}_1, \mathbf{x}_2, 2)} \, d\mathbf{x}_1 \, d\mathbf{x}_2 + \frac{(2^d - 1)^2}{2(2^d \lambda)^2} \\
= {} & \frac{1 - 2^d}{(2^d \lambda)^2} + \iint_{\|\mathbf{x}_1\| \leq \|\mathbf{x}_2\|} e^{-\lambda V_d(\mathbf{x}_1, \mathbf{x}_2, 2)} \, d\mathbf{x}_1 \, d\mathbf{x}_2. \tag{3.11}
\end{aligned}$$

Finally, (3.2), (3.5), and (3.7) imply that

$$\mathrm{corr}(\nu_d(\mathcal{V}_{\mathrm{o}}), \nu_d(\mathcal{S}_{\mathrm{o}})) = \frac{(1 - 2^d)/(2^d \lambda)^2 + \iint_{\|\mathbf{x}_1\| \leq \|\mathbf{x}_2\|} e^{-\lambda V_d(\mathbf{x}_1, \mathbf{x}_2, 2)} \, d\mathbf{x}_1 \, d\mathbf{x}_2}{(1/2^d \lambda) \sqrt{\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-\lambda U_d(\mathbf{x}_1, \mathbf{x}_2)} \, d\mathbf{x}_1 \, d\mathbf{x}_2 - 1/\lambda^2}}.$$

The change of variables $(x_{i1}, \ldots, x_{id}) \mapsto (y_{i1}\lambda^{-1/d}, \ldots, y_{id}\lambda^{-1/d})$, $i = 1, 2$, has the Jacobian λ^{-1}, giving a factor of λ^{-2} outside each double integral. We also have

$$\lambda U_d(\boldsymbol{x}_1, \boldsymbol{x}_2) = U_d(\boldsymbol{y}_1, \boldsymbol{y}_2) \quad \text{and} \quad \lambda V_d(\boldsymbol{x}_1, \boldsymbol{x}_2, 2) = V_d(\boldsymbol{y}_1, \boldsymbol{y}_2, 2).$$

Furthermore, the change of variables

$$(x_{11}, \ldots, x_{1d}) \mapsto (z_{11}2^{-1}, \ldots, z_{1d}2^{-1})$$

has the Jacobian 2^{-d} and it further holds that $V_d(\boldsymbol{x}_1, \boldsymbol{x}_2, 2) = V_d(\boldsymbol{z}_1, \boldsymbol{x}_2, 1) = V_d(\boldsymbol{z}_1, \boldsymbol{x}_2)$. The integration limits become $\|\boldsymbol{z}_1\| \le 2\|\boldsymbol{x}_2\|$. This completes the proof.

For $d = 1$, the integrals in (3.6) can be calculated explicitly, but instead we will use the fact that $\nu_1(\mathcal{S}_o)$ and $\nu_1(\tilde{\mathcal{V}}_o)$ are independent, which follows from the basic properties of the Poisson process. The above directly yields

$$\text{corr}(\nu_1(\mathcal{V}_o), \nu_1(\mathcal{S}_o)) = \sqrt{\frac{\text{var}(\nu_1(\mathcal{S}_o))}{\text{var}(\nu_1(\mathcal{V}_o))}} = \frac{1}{\sqrt{2}}.$$

The last equality follows from (3.4) and (3.8). For $d \ge 2$, consider the following. Let $\alpha = \angle(o, \boldsymbol{x}_1, \boldsymbol{x}_2)$, $r = \|\boldsymbol{x}_1 - \boldsymbol{x}_2\|$, and $R = \|\boldsymbol{x}_1\|/\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|$. Then, by using the results in [3], it follows that

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-U_d(\boldsymbol{x}_1, \boldsymbol{x}_2)} \, d\boldsymbol{x}_1 \, d\boldsymbol{x}_2$$
$$= d(d-1)b_d b_{d-1} \int_0^\infty \int_0^\pi \int_0^\infty \sin^{d-2}(\alpha) e^{-b_{d-1}S_d(R,\alpha)r^d} r^{2d-1} R^{d-1} \, dR \, d\alpha \, dr,$$

where $S_d(R, \alpha)$ is as defined in (2.3). By integrating over r, we obtain the double integral of the denominator in (2.1). Furthermore, (2.2) holds since $B_d(o, \|\boldsymbol{x}_1\|) \cup B_d(\boldsymbol{x}_2, \|\boldsymbol{x}_2\|) = \mathcal{S}(\|\boldsymbol{x}_1\|, t_1) \cup \mathcal{S}(\|\boldsymbol{x}_2\|, t_2)$, where $\mathcal{S}(\|\boldsymbol{x}_1\|, t_1)$ and $\mathcal{S}(\|\boldsymbol{x}_2\|, t_2)$ are two disjoint, truncated d-dimensional spheres of radii $\|\boldsymbol{x}_1\|$ and $\|\boldsymbol{x}_2\|$, respectively, truncated at distances

$$t_1 = \frac{\|\boldsymbol{x}_1\|^2}{2\|\boldsymbol{x}_2\|} \quad \text{and} \quad t_2 = \frac{2\|\boldsymbol{x}_2\|^2 - \|\boldsymbol{x}_1\|^2}{2\|\boldsymbol{x}_2\|}$$

from their respective centres. Geometric considerations and straightforward calculations show that the volume of each of these spheres is equal to b_{d-1} times the corresponding term in (2.2). Changing to polar coordinates yields the expression in the denominator of (2.1).

3.4. Proofs of Propositions 2.2 and 2.3

We will use (3.6) to obtain bounds on the covariance. All we require are bounds on $\iint_{\|\boldsymbol{x}_1\| \le 2\|\boldsymbol{x}_2\|} e^{-V_d(\boldsymbol{x}_1, \boldsymbol{x}_2)} \, d\boldsymbol{x}_1 \, d\boldsymbol{x}_2$. As $V_d(\boldsymbol{x}_1, \boldsymbol{x}_2) \le b_d(\|\boldsymbol{x}_1\|^d + \|\boldsymbol{x}_2\|^d)$, we have

$$\iint_{\|\boldsymbol{x}_1\| \le 2\|\boldsymbol{x}_2\|} e^{-V_d(\boldsymbol{x}_1, \boldsymbol{x}_2)} \, d\boldsymbol{x}_1 \, d\boldsymbol{x}_2 \ge \iint_{\|\boldsymbol{x}_1\| \le 2\|\boldsymbol{x}_2\|} e^{-b_d(\|\boldsymbol{x}_1\|^d + \|\boldsymbol{x}_2\|^d)} \, d\boldsymbol{x}_1 \, d\boldsymbol{x}_2$$
$$= \frac{1}{1 + 2^d}. \tag{3.12}$$

To obtain an upper bound we first use the fact that

$$\iint_{\|\boldsymbol{x}_1\|\leq 2\|\boldsymbol{x}_2\|} e^{-V_d(\boldsymbol{x}_1,\boldsymbol{x}_2)}\,d\boldsymbol{x}_1\,d\boldsymbol{x}_2 = \iint_{\|\boldsymbol{x}_1\|\leq\|\boldsymbol{x}_2\|} e^{-V_d(\boldsymbol{x}_1,\boldsymbol{x}_2)}\,d\boldsymbol{x}_1\,d\boldsymbol{x}_2$$
$$+ \iint_{\|\boldsymbol{x}_2\|\leq\|\boldsymbol{x}_1\|\leq 2\|\boldsymbol{x}_2\|} e^{-V_d(\boldsymbol{x}_1,\boldsymbol{x}_2)}\,d\boldsymbol{x}_1\,d\boldsymbol{x}_2$$

and, furthermore, since $V_d(\boldsymbol{x}_1,\boldsymbol{x}_2) \geq b_d\|\boldsymbol{x}_2\|^d$, the facts that

$$\iint_{\|\boldsymbol{x}_1\|\leq\|\boldsymbol{x}_2\|} e^{-V_d(\boldsymbol{x}_1,\boldsymbol{x}_2)}\,d\boldsymbol{x}_1\,d\boldsymbol{x}_2 \leq \frac{2\pi^{d/2}}{\Gamma(d/2)}\int_{\mathbb{R}^d} e^{-b_d\|\boldsymbol{x}_2\|^d}\int_0^{\|\boldsymbol{x}_2\|} r^{d-1}\,dr\,d\boldsymbol{x}_2 = 1$$

and

$$\iint_{\|\boldsymbol{x}_2\|\leq\|\boldsymbol{x}_1\|\leq 2\|\boldsymbol{x}_2\|} e^{-V_d(\boldsymbol{x}_1,\boldsymbol{x}_2)}\,d\boldsymbol{x}_1\,d\boldsymbol{x}_2 \leq \frac{2\pi^{d/2}}{\Gamma(d/2)}\int_{\mathbb{R}^d}\int_{\|\boldsymbol{x}_2\|}^{2\|\boldsymbol{x}_2\|} r^{d-1}e^{-b_d r^d}\,dr\,d\boldsymbol{x}_2$$
$$= 1 - \frac{1}{2^d}.$$

Using the above, together with (3.11), proves Proposition 2.2(i). Furthermore, by symmetry, we have

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-U_d(\boldsymbol{x}_1,\boldsymbol{x}_2)}\,d\boldsymbol{x}_1\,d\boldsymbol{x}_2 = 2\iint_{\|\boldsymbol{x}_1\|\leq\|\boldsymbol{x}_2\|} e^{-U_d(\boldsymbol{x}_1,\boldsymbol{x}_2)}\,d\boldsymbol{x}_1\,d\boldsymbol{x}_2 \leq 2, \qquad (3.13)$$

where we have used the fact that $U_d(\boldsymbol{x}_1,\boldsymbol{x}_2) \geq b_d\|\boldsymbol{x}_2\|^d$. The lower bound in (3.12) and the upper bound in (3.13) yield Proposition 2.2(ii). Finally, as

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-\lambda U_d(\boldsymbol{x}_1,\boldsymbol{x}_2)}\,d\boldsymbol{x}_1\,d\boldsymbol{x}_2 = E[\nu_d(\mathcal{V}_o)^2],$$

(3.13) together with Remark 2.1 yield Proposition 2.3.

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