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On Denjoy's theorem for endomorphisms

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Abstract. In this paper we give an extension to C^2 endomorphisms of the circle of the well known Denjoy's theorem on C^2 diffeomorphisms. We also give a simple proof of an extension of Block and Franke's theorem on the existence of periodic points for maps of the circle.

Introduction

One of the fundamental results in one-dimensional dynamics is Denjoy's theorem:

THEOREM (Denjoy [**3**]). *If f is a C^2 diffeomorphism of the circle $S^1 = \mathbb{R}/\mathbb{Z}$ then either f is topologically conjugate to an irrational rotation or it has periodic points.*

The aim of this paper is to prove a similar result for endomorphisms of S^1. Let $\mathrm{End}^r(S^1)$ be the set of endomorphisms of class C^r, $r \geq 0$, of S^1 and let $\mathrm{End}^r_k(S^1)$ be the set of C^r-endomorphisms of degree k. Recall that a point $z \in S^1$ is a critical point of an endomorphism $f \in \mathrm{End}^1(S^1)$ if $f'(z) = 0$. We shall say that z is a *non-flat critical point* if there exists an integer $k > 1$ such that f is C^k in a neighbourhood of z and $f^{(k)}(z) \neq 0$.

The first extension of Denjoy's theorem is due to Yoccoz [**7**]:

THEOREM (Yoccoz). *If f is a C^2 homeomorphism of S^1 all of whose critical points are non-flat, then either f is topologically conjugate to an irrational rotation or it has periodic points.*

The non-flatness hypothesis is impossible to avoid because of the example constructed by G. R. Hall [**4**] of a C^∞ homeomorphism of S^1 that has no periodic points and is not topologically equivalent to a rotation. Anyway, it is a reasonably weak assumption, satisfied generically. It is also satisfied by any real analytic endomorphism of the circle.

In order to state our main result, let us denote by $P(f)$, $R(f)$ and $\Omega(f)$ respectively, the sets of periodic, recurrent and non-wandering points of f.

THEOREM A. *If f is a C^2 endomorphism of the circle and all its critical points (if any) are non-flat then either f is topologically conjugate to an irrational rotation or* $\overline{P(f)} = \overline{R(f)}$.

The results used to prove theorem A enable us to give a simple proof of an extension of Block and Franke's theorem [**2**] on the existence of periodic points for maps of the circle. If $z \in S^1$ and $f \in \mathrm{End}^0(S^1)$, let I be the maximal interval containing z such that $f|_I$ is constant. We say that z is a turning point if $f(z)$ is an extreme value of f in a neighbourhood of I.

THEOREM B. *Let $f \in \mathrm{End}_1^2(S^1)$ and suppose that all non-turning critical points of f are non-flat. If f has a turning critical point, then f has periodic points.*

We also provide a simple example which shows that theorem A is not true if we replace the recurrent set $R(f)$ by the non-wandering set $\Omega(f)$ of f.

I would like to thank J. Gheiner, M. J. Pacífico and J. C. Yoccoz for useful talks on this subject.

Statements and proofs

Let f be an endomorphism of S^1. A *lift* for f is a function $F: \mathbb{R} \to \mathbb{R}$ such that $f\Pi = \Pi F$, where $\Pi: \mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z}$ is the canonical projection. If deg (f), the degree of f, is k then $F(x+1) = F(x) + k$ for every x.

To prove theorem A when $P(f) = \emptyset$, we use Yoccoz's theorem and the following result due to Auslander and Katznelson [**1**], which is an extension to continuous endomorphisms of a well known theorem of Poincaré [**6**].

PROPOSITION 1. *If f is a continuous endomorphism of the circle with no periodic points, then f is semi-conjugate to an irrational rotation, i.e. there exists a continuous degree one map $h: S^1 \circlearrowleft$ and an irrational rotation R of the circle such that $Rh = hf$. Moreover, the semi-conjugacy h is monotone.*

Let $f \in \mathrm{End}^0(S^1)$ with $P(f) = \emptyset$ and $h: S^1 \circlearrowleft$ be the semi-conjugacy given by proposition 1. We say that a non-degenerate interval $I \subset S^1$ is a *plateau* if h is constant in I and I is a maximal interval with this property. That is, if $J \supset I$ and h is constant in J then $J = I$. It follows immediately from proposition 1 that:

(i) The plateaux depend only on f.

(ii) If $I = [a, b]$ is a plateau, then $f(I) = [f(a), f(b)]$ is also a plateau.

(iii) If A is the union of the interior of the plateaux, one has $\Omega(f) = R(f) = S^1 \setminus A$, which is S^1 or a Cantor set.

With these results we prove the following lemma:

LEMMA 1. *Let $f \in \mathrm{End}^1(S^1)$ be such that $P(f) = \emptyset$ and $R(f)$ does not contain flat critical points. Then there exists a homeomorphism $g \in \mathrm{End}_1^1(S^1)$ such that $R(g) = R(f)$ and all its critical points are non-flat.*

Proof. The non-flatness hypothesis implies that the critical points in $R(f)$, if any, are isolated amongst critical points so there are at most finitely many plateaux having critical points of f. From (ii) we can modify f in these plateaux in order to obtain g with the required properties.

We now prove theorem B.

Proof of theorem B. Suppose $P(f)=\emptyset$. Then proposition 1 applies and since f has a turning point we conclude that $R(f)$ is a proper subset of S^1. On the other hand, from (ii), (iii) and the monotonicity of the semi-conjugacy given by proposition 1, we conclude that any critical point in $R(f)$ cannot be a turning point. So $R(f)$ does not contain flat critical points. Now, lemma 1 gives a contradiction with Yoccoz's theorem. □

To prove theorem A, we need one more result, which is a version for continuous endomorphisms of the circle of Young's theorem [**8**] that states that $\overline{P(f)}=\overline{R(f)}$ holds for any piecewise monotone map of the interval. The version we present here evolved from discussions during the expositions of J. Gheiner on Young's theorem in a seminar on maps of the circle and the interval at *IMPA*.

Let $f\in \mathrm{End}^0(S^1)$. A set $\{c_0,\dots,c_k=c_0\}$ will be called a partition by turning points of f if c_j is a turning point of f and the restriction of f to the interval $I_j=[c_{j-1},c_j]$ is a monotone map, for $1\le j\le k$.

The following lemma can be proved directly from lemmas 1-5 in § 2 of [**8**].

LEMMA 2. *If* $f\in\mathrm{End}^0(S^1)$ *and there exists a partition* $\{c_0,\dots,c_k\}$ *by turning points of* f *such that* $c_j\in\Omega(f)$ *for* $0\le j\le k$, *then* $\overline{P(f)}=\overline{R(f)}$.

PROPOSITION 2. *If* $f\in\mathrm{End}^0(S^1)$ *is a piecewise monotone endomorphism, then either* $P(f)=\emptyset$ *or* $\overline{P(f)}=\overline{R(f)}$.

Proof. We will modify f in such a way to obtain $g\in\mathrm{End}^0(S^1)$ with the following properties:

(i) $g=f$ on $R(f)$;

(ii) $g\ne f$ at most only on a finite union of open intervals $\bigcup_{i=1}^k J_i$ and $g|_{J_i}$ is constant for $1\le i\le k$;

(iii) $P(g)\setminus P(f)$ has at most finitely many points.

In fact if $z_0\in P(g)\setminus P(f)$ then the orbit of z_0, $O(z_0,g)$ needs to intersect J_i for some i. We may assume that $z_0\in J_i$ and $g^n(z_0)=z_0$. Then if $z\in J_i$ we have that $g^n(z)=z_0$, that is, z is eventually periodic but not periodic. Thus to each $z\in P(g)\setminus P(f)$ we can attach one J_i and there are only finitely many J_i's.

(iv) Either there is a partition by turning points $\{c_0,\dots,c_k\}$ with $c_i\in\Omega(g)$ for $0\le i\le k$ or g is monotone.

Let us construct g. If f is monotone, take $g=f$. Otherwise, let $z_0\in R(f)$ and $a_0\in[0,1]$ be such that $\Pi(a_0)=z_0$. Let F be a lift of f and let $a_0\le x_1$ be, amongst turning points, the smallest turning point of F. If x_1 is not isolated, let $I_1=[x_1,y_1]$ the maximal interval such that $F|_{I_1}$ is constant.

If $\Pi(I_1)\cap R(f)\ne\emptyset$, do nothing. Note that this is the case if $x_1=a_0$. If not, suppose that it is a maximum.

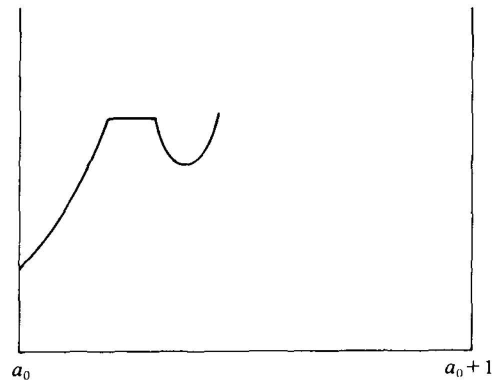

FIGURE 1

Push down F on I_1 until

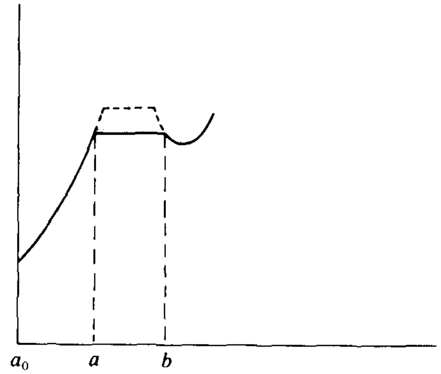

FIGURE 2

$\Pi(a)$ or $\Pi(b)$ belongs to $\overline{R(f)}$, or

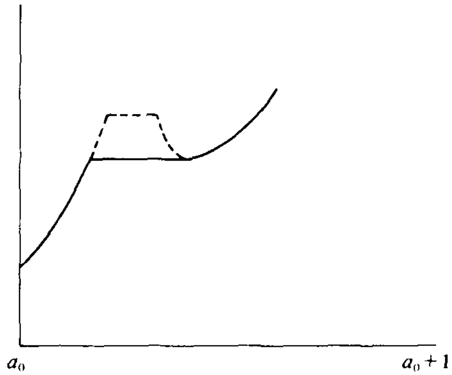

FIGURE 3

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the second turning point (or turning interval) disappears. Now work in the same way on the next turning point: x_2, if figure 2 prevails and x_3 if figure 3 prevails. Since F is piecewise monotone, after a finite number of such steps we obtain a function G such that $\deg(G) = \deg(F)$. (Note that we never touch a_0 and a_0+1, so

$$G(a_0+1) = F(a_0+1) = F(a_0) + \deg(F) = G(a_0) + \deg(F).)$$

Let g be the projection of G to S^1. Clearly g satisfies properties (i)–(iv) and so it suffices to prove that $\overline{P(g)} = \overline{R(g)}$ or $P(g) = \emptyset$.

If g is not monotone, (iv) and lemma 2 give the result. Suppose then that g is monotone. If $\deg g = 1$ there are two possibilities: the rotation number of g is a rational or an irrational number. In the first case $P(g) = R(g)$; in the second $P(g) = \emptyset$ and so $P(f) = \emptyset$. If $\deg g = -1$ then $\deg g^2 = 1$ and g^2 has a fixed point, and so $P(g) = R(g)$. There remains the case where $|\deg g| \geq 2$. In this case we know, from [**5**], that if $\phi_k: S^1 \circlearrowleft$ is defined by $\phi_k(z) = z^k$, where $k = \deg g$, then g is semi-conjugate to ϕ_k by a monotone continuous map $h: S^1 \circlearrowleft$ of degree one. To prove that $\overline{P(g)} = \overline{R(g)}$ let $z_0 \in R(g)$ and suppose that $z_0 \notin P(g)$. Then, given any neighbourhood V of z_0, there exists an interval $J \subset V$ with non-empty interior, such that z_0 is an end point of J and if z_1 is the other end point, then $h(z_0) \neq h(z) \neq h(z_1)$ for every $z \in \operatorname{int} J$. This implies that $h(J)$ is an interval with non-empty interior and if $w \in h(J)$, then $h^{-1}(w) \subset J$. Now, let $w_0 \in h(J)$ such that w_0 is a periodic point of ϕ_k. Then $h^{-1}(w_0) \cap P(g) \neq \emptyset$ and so $P(g) \cap V \neq \emptyset$.

Proof of theorem A. The proof goes by contradiction. From proposition 2 it suffices to prove that if $P(f) = \emptyset$ then f is conjugate to an irrational rotation. If f has no periodic points proposition 1 says that f is semi-conjugate to an irrational rotation R. Suppose, by contradiction, that f is not conjugate to R. Then $S^1 \setminus R(f) \neq \emptyset$ and from lemma 1 we conclude that there exists a homeomorphism g of class C^2 having no flat critical points such that $R(g) = R(f)$ which contradicts Yoccoz's theorem. □

The following example, which is a simple adaptation of the one given by L. S. Young in [**8**], shows that theorem A is not true if we replace the recurrent set $R(f)$ by the non-wandering set $\Omega(f)$.

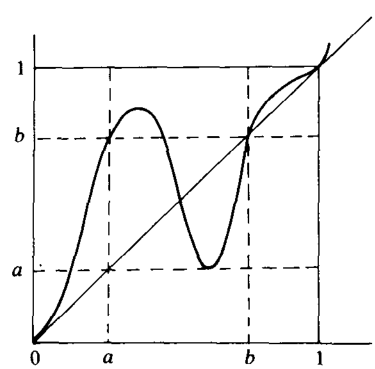

FIGURE 4

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