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On the rate of convergence of interpolation polynomials of Hermite-Fejér type

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For the interpolation polynomial of Hermite-Fejér type $A_n[f]$ of degree less than or equal to $4n-1$ constructed on the nodes $x_k = \cos \frac{2k-1}{2n} \pi$, $k = 1, 2, \ldots, n$, it is shown that for $f \in C_M(\Omega)$ the inequality

$$|A_n[f](x)-f(x)| \le \frac{c_3 M}{n} \sum_{k=1}^{n} \Omega\left(\frac{\sqrt{1-x^2}}{k} + \frac{1}{k^2}\right), \quad -1 \le x \le 1,$$

holds where $C_M(\Omega)$ is the class of continuous functions on $[-1, 1]$ satisfying certain conditions, Ω is a certain modulus of continuity, and c_3 and M are positive constants.

The Hermite-Fejér interpolation polynomial $H_n[f]$ of degree less than or equal to $2n-1$ of a function f defined on $[-1, 1]$ is given by

$$H_n[f](x) = \sum_{k=1}^{n} f(x_{kn})(1-xx_{kn})\left[\frac{T_n(x)}{n(x-x_{kn})}\right]^2 \tag{1}$$

where

$$x_{kn} = \cos \frac{2k-1}{2n} \pi, \quad k = 1, 2, \ldots, n, \tag{2}$$

are the zeros of the Chebyshev polynomial $T_n(x) = \cos(n \arccos x)$. Fejér

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[2] proved that $H_n[f]$ converges uniformly to f on $[-1, 1]$ provided f is continuous on $[-1, 1]$. For the rate of convergence Moldovan [6] obtained the estimate

$$(3) \qquad \|H_n[f]-f\| \le 2\pi\omega_f\left(\frac{\log n}{n}\right), \quad n \ge 4,$$

where $\|f\| = \max_{-1\le x\le 1} |f(x)|$ and ω_f is the modulus of continuity of $f(x)$. This estimate was also established by Shisha and Mond [8].

Let Ω be an increasing, subadditive, and continuous function on $\{x : x \ge 0\}$ with $\Omega(0) = 0$, and let $C_M(\Omega)$ be the class of continuous functions on $[-1, 1]$ defined by

$$f \in C_M(\Omega) \Leftrightarrow \omega_f(h) \le M\Omega(h)$$

or equivalently, $f \in C_M(\Omega)$ iff $|f(x')-f(x'')| \le M\Omega(|x'-x''|)$ for all $x', x'' \in [-1, 1]$. We have then the following result due to Bojanic [1].

THEOREM 1 (Bojanic). *There exist constants* c *and* C $(0 < c < C < \infty)$ *such that for* $n \ge 2$,

$$(4) \qquad \frac{cM}{n}\sum_{k=2}^{n} \Omega\left(\frac{1}{k}\right) \le \sup_{f\in C_M(\Omega)} \|H_n[f]-f\| \le \frac{CM}{n}\sum_{k=1}^{n} \Omega\left(\frac{1}{k}\right).$$

Saxena [7] showed that a better local approximation can be obtained at the end points of the interval by proving the following:

THEOREM 2. *There exists a constant* C^* *such that for* $n \ge 2$ *and* $-1 \le x \le 1$,

$$(5) \qquad |H_n[f](x)-f(x)| \le \frac{C^*M}{n}\sum_{k=1}^{n} \Omega\left\{\frac{\left(1-x^2\right)^{\frac{1}{2}}}{k} + \frac{1}{k^2}\right\}.$$

Recently Stancu [9] has considered the polynomial $A_n[f]$ of degree less than or equal to $4n - 1$ uniquely determined by the conditions:

$$(6) \qquad A_n[f]\left(x_{in}\right) = f\left(x_{in}\right), \quad k = 1, 2, \ldots, n;$$

$$A_n^{(i)}[f]\left(x_{kn}\right) = 0, \quad i = 1, 2, 3, \quad k = 1, 2, \ldots, n.$$

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Stancu showed that $A_n[f]$ converges uniformly to f on $[-1, 1]$ if f is continuous on $[-1, 1]$. He also proved that

(7) $$\|A_n[f]-f\| = O(1)\omega_f\left(n^{-\frac{1}{2}}\right) ,$$

where the $O(1)$ is independent of f and n. Later Florica [3] improved the above result (7), showing that

(8) $$\|A_n[f]-f\| = O(1)\omega_f\left(\frac{\log n}{n}\right) , \text{ for } n > 1 .$$

This result of Florica was further improved by Mills [5] who proved the following:

THEOREM 3 (Mills). *There are positive constants* c_1 *and* c_2 *such that for* $n > 1$,

(9) $$\frac{c_1}{n} \sum_{r=2}^{n} \Omega\left(\frac{1}{r}\right) \le \sup_{f\in C(\Omega)} \|A_n[f]-f\| \le \frac{c_2}{n} \sum_{r=1}^{n} \Omega\left(\frac{1}{r}\right) .$$

Our aim here is to show that a better local approximation can be obtained in this case also. For this purpose we shall prove the following:

THEOREM 4. *There exists a constant* c_3 *such that for* $n \ge 2$ *and* $-1 \le x \le 1$,

(10) $$|A_n[f](x)-f(x)| \le \frac{c_3M}{n} \sum_{k=1}^{n} \Omega\left\{\frac{\left(1-x^2\right)^{\frac{1}{2}}}{k} + \frac{1}{k^2}\right\} .$$

Stancu [9] has proved that

(11) $$A_n[f](x) = \sum_{k=1}^{n} f\left(x_k\right)\rho_k(x) ,$$

where

(12) $$\rho_k(x) = u_k(x) + v_k(x) + w_k(x) ,$$

(13) $$u_k(x) = n^{-4}\left(1-x_k^2\right)\left(1-x^2\right)\left[\frac{T_n(x)}{x-x_k}\right]^4 ,$$

(14) $$v_k(x) = \frac{n^{-4}}{6} \left(4n^2-1\right)\left(x-x_k\right)^2\left(1-xx_k\right)\left[\frac{T_n(x)}{x-x_k}\right]^4 ,$$

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(15) $$w_k(x) = \frac{n^{-4}}{2}\left[\frac{T_n^2(x)}{x-x_k}\right]^2 ,$$

and $x_k = x_{kn}$ are as in (2). Now we shall prove the following:

LEMMA. *If*

$$\frac{j-1}{n}\pi \le \theta \le \frac{j}{n}\pi \quad (j = 1, 2, \ldots, n) , \quad x = \cos\theta ,$$

then

$$\rho_k(x) \le \begin{cases} 19 , & \text{if } k = j , \\ \dfrac{70}{(i-\frac{1}{2})^2} , & \text{if } j < k = j+i \le n \text{ or } 1 \le k = j-i < j . \end{cases}$$

Proof. Putting $x = \cos\theta$, $0 \le \theta \le \pi$, $x_k = \cos\theta_k$, $\theta_k = \frac{2k-1}{2n}\pi$, $k = 1, 2, \ldots, n$, and using the definition of j we see that

(16) $$\frac{1}{\sin\frac{1}{2}|\theta_k-\theta|} \le \frac{n}{(i-\frac{1}{2})} , \text{ for } k \ne j .$$

Further from (11), (12), (13), (14), and (15) it follows that

(17) $$\rho_k(x) = n^{-4}\left(1-\cos^2\theta_k\right)\left(1-\cos^2\theta\right)\left[\frac{\cos n\theta}{\cos\theta-\cos\theta_k}\right]^4$$
$$+ \frac{n^{-4}}{6}\left(4n^2-1\right)\left(\cos\theta-\cos\theta_k\right)^2\left(1-\cos\theta\cos\theta_k\right)\left[\frac{\cos n\theta}{\cos\theta-\cos\theta_k}\right]^4$$
$$+ \frac{n^{-4}}{2}\left[\frac{\cos^2 n\theta}{\cos\theta-\cos\theta_k}\right]^2$$
$$= I_1 + I_2 + I_3 .$$

Since

(18) $$\sin\theta \le \sin\theta + \sin\theta_k \le 2\sin\tfrac{1}{2}\left(\theta+\theta_k\right)$$

and

(19) $$\sin\theta_k \le \sin\theta_k + \sin\theta \le 2\sin\tfrac{1}{2}\left(\theta+\theta_k\right) ,$$

we have after simple computations,

(20) $$I_1 \le \frac{|\cos n\theta - \cos n\theta_k|^4}{n^4 [\sin\frac{1}{2}(\theta_k-\theta)]^4}$$

$$\le \frac{16}{n^4} \left[\frac{\sin\frac{1}{2}n(\theta_k-\theta)}{\sin\frac{1}{2}(\theta_k-\theta)}\right]^4 .$$

Next we have

(21) $$I_2 \le \frac{1}{n^2} \left(1 - \cos\theta\cos\theta_k\right) \left[\frac{\cos n\theta}{\cos\theta-\cos\theta_k}\right]$$

$$\le \frac{1}{n^2} \left(1 - \cos\theta\cos\theta_k + \sin\theta\sin\theta_k\right) \left[\frac{\cos n\theta-\cos n\theta_k}{\cos\theta-\cos\theta_k}\right]^2$$

$$\le \frac{2}{n^2} \left[\frac{\sin\frac{1}{2}n(\theta_k-\theta)}{\sin\frac{1}{2}(\theta_k-\theta)}\right]^2 .$$

Similarly,

(22) $$I_3 \le \frac{1}{2n^4} \left[\frac{\cos n\theta-\cos n\theta_k}{\cos\theta-\cos\theta_k}\right]^2$$

$$= \frac{1}{2n^4} \left[\frac{\sin\frac{1}{2}n(\theta+\theta_k)\sin\frac{1}{2}n(\theta_k-\theta)}{\sin\frac{1}{2}(\theta+\theta_k)\sin\frac{1}{2}(\theta_k-\theta)}\right]^2 .$$

Consequently from (17), (20), (21), and (22), it follows that

(23) $$\rho_k(x) \le \frac{16}{n^4} \left[\frac{\sin\frac{1}{2}n(\theta_k-\theta)}{\sin\frac{1}{2}(\theta_k-\theta)}\right]^4 + \frac{2}{n^2} \left[\frac{\sin\frac{1}{2}n(\theta_k-\theta)}{\sin\frac{1}{2}(\theta_k-\theta)}\right]^2 + \frac{1}{2n^4} \left[\frac{\sin\frac{1}{2}n(\theta+\theta_k)\sin\frac{1}{2}n(\theta_k-\theta)}{\sin\frac{1}{2}(\theta+\theta_k)\sin\frac{1}{2}(\theta_k-\theta)}\right]^2 .$$

Thus if $k \ne j$, on using the inequality

$$\frac{1}{\sin\frac{1}{2}(\theta+\theta_k)} \le \frac{1}{\sin\frac{1}{2}|\theta_k-\theta|} , \quad 0 \le \theta \le \pi , \quad 0 < \theta_k < \pi ,$$

and (16) we obtain, from (23),

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(24) $$\rho_k(x) \le \frac{17}{n^4[\sin\frac{1}{2}(\theta_k-\theta)]^4} + \frac{2}{n^2[\sin\frac{1}{2}(\theta_k-\theta)]^2}$$

$$\le \frac{70}{(i-\frac{1}{2})^2}\ , \text{ for } k \ne j\ .$$

Further if $k = j$ then from (23) we have on using the inequality $|\sin n\theta| \le n|\sin\theta|$,

(25) $$\rho_j(x) \le 19\ .$$

Hence, from (24) and (25), the lemma follows.

Proof of the theorem. For any $f \in C_M(\Omega)$ and $x \in [-1, 1]$ we have

$$|A_n[f](x)-f(x)| \le \sum_{k=1}^{n} |f(x_k)-f(x)|\rho_k(x)$$

$$\le M \sum_{k=1}^{n} \Omega(|x-x_k|)\rho_k(x)\ .$$

Putting $x = \cos\theta$, $0 \le \theta \le \pi$, $x_k = \cos\theta_k$, $\theta_k = \frac{2k-1}{2n}\pi$, $k = 1, 2, \ldots, n$, we then have

$$|A_n[f](x)-f(x)| \le M \sum_{k=1}^{n} \Omega(|\cos\theta - \cos\theta_k|)\rho_k(x)\ .$$

Since

$$\cos\theta - \cos\theta_k = (\theta_k-\theta)\sin\theta - \tfrac{1}{2}(\theta_k-\theta)^2\cos\eta\ , \quad \theta < \eta < \theta_k\ ,$$

it follows that

$$\Omega(|\cos\theta - \cos\theta_k|) \le \Omega(|\theta_k-\theta|\sin\theta) + \Omega\{|\theta_k-\theta|^2\}\ ,$$

and we obtain

(26) $$|A_n[f](x)-f(x)| \le M \sum_{k=1}^{n} \left[\Omega(|\theta_k-\theta|\sin\theta) + \Omega\{|\theta_k-\theta|^2\}\right]\rho_k(x)\ .$$

Now if

$$\frac{j-1}{n}\pi \le \theta \le \frac{j}{n}\pi\ , \quad j = 1, 2, \ldots, n\ ,$$

then from [4] it is known that

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$$(27)\qquad |\theta_k-\theta| \le \frac{\pi}{2n}\ ,\ \text{if}\ \ k = j\ ,$$

$$\le \frac{3i\pi}{2n}\ ,\ \text{if}\ \ j < k = j + i \le n\ \ \text{or}\ \ 1 \le k = j - i < j\ .$$

From (26), (27), and the lemma it follows that

$$(28)\qquad |A_n[f](x)-f(x)| \le 19M\left[\Omega\left(\frac{\pi\sin\theta}{2n}\right) + \Omega\left(\frac{\pi^2}{4n^2}\right)\right]$$
$$+ 70M \sum_{k\neq j} \left[\Omega\left(\frac{3i\pi\sin\theta}{2n}\right) + \Omega\left(\frac{9i^2\pi^2}{4n^2}\right)\right] \frac{1}{(i-\frac{1}{2})^2}$$
$$\le \lambda M \sum_{r=1}^{n} \frac{1}{r^2} \left[\Omega\left(\frac{r+1}{n+1}\pi \sin \theta\right) + \Omega\left(\frac{(r+1)^2\pi^2}{(n+1)^2}\right)\right] ,$$

where λ is a positive constant. On using the inequality (6) of [7] for $m = n + 1$, we find that

$$(29)\qquad \sum_{r=1}^{n} \frac{1}{r^2}\, \Omega\left(\frac{r+1}{n+1}\pi \sin \theta\right) \le \frac{8\pi}{n+1} \int_{\pi/n+1}^{\pi} \frac{\Omega(t\sin\theta)}{t^2}\, dt$$

and

$$(30)\qquad \sum_{r=1}^{n} \frac{1}{r^2}\, \Omega\left(\frac{(r+1)^2\pi^2}{(n+1)^2}\right) \le \frac{8\pi}{n+1} \int_{\pi/n+1}^{\pi} \frac{\Omega(t^2)}{t^2}\, dt\ .$$

Hence (28), (29), and (30) yield

$$(31)\qquad |A_n[f](x)-f(x)| \le \frac{8\pi M\lambda}{n+1}\left[\int_{\pi/n+1}^{\pi} \frac{\Omega(t\sin\theta)}{t^2}\, dt + \int_{\pi/n+1}^{\pi} \frac{\Omega(t^2)}{t^2}\, dt\right] .$$

But

$$(32)\qquad \int_{\pi/n+1}^{\pi} \frac{\Omega(t\sin\theta)}{t^2}\, dt = \frac{1}{\pi} \int_{1}^{n+1} \Omega\left(\frac{\pi\sin\theta}{t}\right) dt$$
$$\le 2 \int_{1}^{2n} \Omega\left(\frac{\sin\theta}{t}\right) dt$$
$$\le 8 \int_{1}^{n} \Omega\left(\frac{\sin\theta}{t}\right) dt$$
$$\le 8 \sum_{k=1}^{n} \Omega\left(\frac{\sin\theta}{k}\right) .$$

$$(33)\qquad \int_{\pi/n+1}^{\pi} \frac{\Omega(t^2)}{t^2}\,dt = \frac{1}{\pi}\int_1^{n+1} \Omega\left(\frac{\pi^2}{t^2}\right)dt \le \frac{1}{\pi}\int_1^{2n} \Omega\left(\frac{\pi^2}{t^2}\right)dt \le 16\int_1^{n} \Omega\left(\frac{1}{t^2}\right)dt \le 16\sum_{k=1}^{n} \Omega\left(\frac{1}{k^2}\right) .$$

Consequently from (31), (32), and (33) it follows that

$$|A_n[f](x)-f(x)| \le \frac{256\pi M\lambda}{n+1}\left[\sum_{k=1}^{n} \Omega\left(\frac{\sqrt{1-x^2}}{k}+\frac{1}{k^2}\right)\right] \le \frac{c_3 M}{n}\sum_{k=1}^{n} \Omega\left(\frac{\sqrt{1-x^2}}{k}+\frac{1}{k^2}\right) .$$

This completes the proof of the theorem.

Since the modulus of continuity ω_f of any continuous function f on $[-1, 1]$ has the same properties as Ω, we also conclude from the above theorem that for any continuous function f on $[-1, 1]$ the estimate

$$|A_n[f](x)-f(x)| \le \frac{C_1^*}{n}\sum_{k=1}^{n} \omega_f\left(\frac{\sqrt{1-x^2}}{k}+\frac{1}{k^2}\right)$$

is valid for $-1 \le x \le 1$.

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