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ARTICLE

Large monochromatic components in expansive hypergraphs

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Abstract
A result of Gyárfás [12] exactly determines the size of a largest monochromatic component in an arbitrary r-colouring of the complete k-uniform hypergraph K_n^k when $k \geq 2$ and $k \in \{r-1, r\}$. We prove a result which says that if one replaces K_n^k in Gyárfás' theorem by any 'expansive' k-uniform hypergraph on n vertices (that is, a k-uniform hypergraph G on n vertices in which $e(V_1, \ldots, V_k) > 0$ for all disjoint sets $V_1, \ldots, V_k \subseteq V(G)$ with $|V_i| > \alpha$ for all $i \in [k]$), then one gets a largest monochromatic component of essentially the same size (within a small error term depending on r and α). As corollaries we recover a number of known results about large monochromatic components in random hypergraphs and random Steiner triple systems, often with drastically improved bounds on the error terms.
Gyárfás' result is equivalent to the dual problem of determining the smallest possible maximum degree of an arbitrary r-partite r-uniform hypergraph H with n edges in which every set of k edges has a common intersection. In this language, our result says that if one replaces the condition that every set of k edges has a common intersection with the condition that for every collection of k disjoint sets $E_1, \ldots, E_k \subseteq E(H)$ with $|E_i| > \alpha$, there exists $(e_1, \ldots, e_k) \in E_1 \times \cdots \times E_k$ such that $e_1 \cap \cdots \cap e_k \neq \emptyset$, then the smallest possible maximum degree of H is essentially the same (within a small error term depending on r and α). We prove our results in this dual setting.

Keywords: Monochromatic components; hypergraphs; colourings

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1. Introduction

We say that a hypergraph G is connected if the 2-shadow of G is connected (the 2-shadow of G is the graph on vertex set $V(G)$ and edge set $\{e \in \binom{V(G)}{2} : \exists f \in E(G), e \subseteq f\}$). A component in a hypergraph is a maximal connected subgraph. Given a hypergraph G and a positive integer r, let $\mathrm{mc}_r(G)$ be the largest integer t such that every r-colouring of the edges of G contains a monochromatic component of order at least t. Let K_n^k denote the complete k-uniform hypergraph on n vertices (and $K_n = K_n^2$ as usual). A well-studied problem has been determining the value of $\mathrm{mc}_r(K_n^k)$; however, this problem is still open for most values of r and k. On the other hand, Gyárfás proved the following well-known results.

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Theorem 1.1 (Gyárfás [12]).

(i) For all $n \geq r \geq 2$, $\mathrm{mc}_r(K_n) \geq \frac{n}{r-1}$. This is best possible when $(r-1)^2$ divides n and there exists an affine plane of order $r-1$.

(ii) For all $n \geq r \geq 2$, $\mathrm{mc}_r(K_n^r) = n$.

(iii) For all $n \geq r \geq 4$, $\mathrm{mc}_r(K_n^{r-1}) \geq \frac{(r-1)n}{r}$. This is best possible for all such r and n.

A natural question which has received attention lately has been to determine conditions under which a k-uniform hypergraph G on n vertices satisfies $\mathrm{mc}_r(G) = \mathrm{mc}_r(K_n^k)$ or $\mathrm{mc}_r(G) \geq (1-o(1))\mathrm{mc}_r(K_n^k)$; or, if this is too restrictive, determining the value of $\mathrm{mc}_r(G)$ in terms of some natural parameters of G.

Perhaps the first such result is due to Füredi [9] who proved that for all graphs G on n vertices, $\mathrm{mc}_r(G) \geq \frac{n}{(r-1)\alpha(G)}$, which is best possible when an affine plane of order $r-1$ exists (see Section 2.1.2 for more details). But note that the value of $\mathrm{mc}_r(G)$ is far from $\mathrm{mc}_r(K_n)$ in this case. In a sense that will be made precise in the coming pages, our paper is essentially a variant of Füredi's result using a different (but related) parameter in place of independence number for which we can guarantee that $\mathrm{mc}_r(G)$ is close to $\mathrm{mc}_r(K_n)$.

Note that for $1 \leq r \leq k$, $\mathrm{mc}_r(G) = n = \mathrm{mc}_r(K_n^k)$ if and only if the r-shadow of G is complete.[1] On the other hand, as first noted by Gyárfás and Sárközy [11], when $r > k = 2$ it is surprisingly possible for $\mathrm{mc}_r(G) = \mathrm{mc}_r(K_n)$ provided G has large enough minimum degree. See [11], [17], and [10] for the best known results on this minimum degree threshold in the case $k = 2$, and [3] for a precise result on the minimum codegree threshold in the case $r = k+1 \geq 4$.

For hypergraphs, Bennett, DeBiasio, Dudek, and English [5] proved that if G is an $(r-1)$-uniform hypergraph on n vertices with $e(G) \geq (1-o(1))\binom{n}{r-1}$, then $\mathrm{mc}_{r-1}(G) \geq (1-o(1))n$ and $\mathrm{mc}_r(G) \geq (\frac{r-1}{r} - o(1))n$.

As for random graphs, it was independently determined in [2], [8] that with high probability,[2] $\mathrm{mc}_r(G(n,p)) \geq (1-o(1))\frac{n}{r-1}$ provided $p = \frac{\omega(1)}{n}$, and it was determined (using the result mentioned in the previous paragraph) in [5] that $\mathrm{mc}_r(H^r(n,p)) \geq (1-o(1))n$ provided $p = \frac{\omega(1)}{n^{r-1}}$, and $\mathrm{mc}_r(H^{r-1}(n,p)) \geq (1-o(1))\frac{(r-1)n}{r}$ provided $p = \frac{\omega(1)}{n^{r-2}}$. All of these results for random graphs use the sparse regularity lemma and thus only provide weak bounds on the error terms. Additionally, it was determined in [6] that for almost all Steiner triple systems S on n vertices, $\mathrm{mc}_3(S) = (1-o(1))n$. In this case, there is an explicit bound on the error term, but their result is specific to 3 colours and 3-uniform hypergraphs in which every pair of vertices is contained in at least one edge.

In this paper, we study a common generalisation which implies all of the results from the previous two paragraphs with more precise error terms.

1.1. Relationship between monochromatic components and partite holes

Given a hypergraph G, a *k-partite hole of size α* is a collection of pairwise disjoint sets $X_1, \ldots, X_k \subseteq V(G)$ such that $|X_1| = \cdots = |X_k| = \alpha$ and no edge $e \in E(G)$ satisfies $e \cap X_i \neq \emptyset$ for all $i \in [k]$. Define the *k-partite hole number* $\alpha_k(G)$ to be the largest integer α such that G contains an k-partite hole of size α. Note that if G is a k-uniform hypergraph with $\alpha_k(G) \leq \alpha$, then for all disjoint sets $V_1, \ldots, V_k \subseteq V(G)$ with $|V_i| > \alpha$ for all $i \in [k]$ there exists $e \in E(G)$ such that $e \cap V_i \neq \emptyset$ for all $i \in [k]$. Note that this implies that G is 'expansive' in a certain sense which will be made explicit in

[1] Indeed, if every r-set of G is contained an edge, then since $\mathrm{mc}_r(K_n^r) = n$, we have $\mathrm{mc}_r(G) = n$. Furthermore, if some r-set $\{x_1, \ldots, x_r\}$ is not contained in an edge, then we can colour the edges of G with r-colours such that colour i is never used on x_i and thus $\mathrm{mc}_r(G) < n$ (c.f. Observation 3.1).

[2] An event is said to happen *with high probability* or *w.h.p.* if the probability that the event occurs tends to 1 as $n \to \infty$.

Section 2.2. As a point of comparison, note that small (ordinary) independence number does not imply expansiveness; that is, if G is a k-uniform hypergraph, then $\alpha(G) \le \alpha$ does not even imply that G has a connected component of order larger than $\frac{n}{\alpha}$ (again, see Section 2.2 for a discussion about the relationship between $\alpha(G)$ and $\alpha_k(G)$).

All of the results mentioned above regarding random hypergraphs and random Steiner triple systems either implicitly or explicitly make use of the fact that the k-partite hole number is bounded (for further discussion, see Section 5). In this paper we consider the following general problem which attempts to pin down a more precise relationship between a k-uniform hypergraph having bounding k-partite hole number (i.e. being 'expansive') and having large monochromatic components in arbitrary r-colourings.

Problem 1.2. *Prove that for all integers $r, k \ge 2$, there exists $c_{r,k}, d_{r,k} > 0$ such that for all k-uniform hypergraphs G on n vertices, if $\alpha_k(G) < c_{r,k}n$, then*

$$\mathrm{mc}_r(G) \ge \mathrm{mc}_r(K_n^k) - d_{r,k}\alpha_k(G).$$

Furthermore, determine the optimal values of $c_{r,k}, d_{r,k}$.

We solve Problem 1.2 for all $1 \le r \le k+1$, give the optimal values of $c_{r,k}, d_{r,k}$ in the case of $k = 2 = r$, give the optimal value of $c_{r,k}$ in the case $k = 3 = r$, and give reasonable estimates on $c_{r,k}, d_{r,k}$ in the other cases. The formal statements will be given below.

Our first result covers the case $k = r = 2$ and thus generalises Theorem 1.1(i) in the case $r = 2$ (i.e. a graph or its complement is connected). This is the result for which we have the tightest bounds.

Theorem 1.3.

(i) (a) For all graphs G on n vertices, if $\alpha_2(G) < n/6$, then $\mathrm{mc}_2(G) \ge n - 2\alpha_2(G)$.
(b) Furthermore, the bound on $\alpha_2(G)$ is best possible in the sense that there exists a graph on n vertices with $\alpha_2(G) = n/6$ such that $\mathrm{mc}_2(G) \le n/3$.

(ii) For all graphs G on n vertices, $\mathrm{mc}_2(G) \le n - \alpha_2(G)$.

(iii) For all integers n and a with $0 \le a \le n/4$, there exists a graph on n vertices with $\alpha_2(G) = a$ such that $\mathrm{mc}_2(G) \le n - 2a$.

Our second result covers the case when $k = 2$ and $r = 3$ and thus generalises Theorem 1.1(i) in the case $r = 3$. (Extending this result to the case $r \ge 4$ is the main open problem raised by this paper. See Conjecture 4.12 and the discussion which precedes it for more details about this open case.)

Theorem 1.4.

(i) For all graphs G on n vertices, if $\alpha_2(G) \le \frac{n}{3^9}$, then $\mathrm{mc}_3(G) \ge \frac{n}{2} - 2\alpha_2(G)$.

(ii) For all $0 \le a \le n/2$, there exists a graph on n vertices with $\alpha_2(G) = a$ such that $\mathrm{mc}_3(G) \le \frac{n-a}{2}$.

Our third result covers the case when $k = r \ge 3$ and thus generalises Theorem 1.1(ii).

Theorem 1.5. *Let r be an integer with $r \ge 3$.*

(i) There exists $c_r > 0$ such that for all r-uniform hypergraphs G on n vertices, if $\alpha_r(G) < c_r n$, then $n - \alpha_r(G) \ge \mathrm{mc}_r(G) \ge n - (r-1)\alpha_r(G)$.

(ii) For all $0 \le a \le n/(r+2)$, there exists a r-uniform hypergraph G on n vertices with $\alpha_r(G) = a$ such that $\mathrm{mc}_r(G) \le n - 2\alpha_r(G)$.

Our fourth result covers the case when $r = k + 1 \ge 4$ and thus generalises Theorem 1.1(iii).

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Theorem 1.6. *Let r be an integer with $r \geq 4$.*

(i) There exists $c_r > 0$ such that for all $(r-1)$-uniform hypergraphs G on n vertices, if $\alpha_{r-1}(G) < c_r n$, then $\mathrm{mc}_r(G) \geq \frac{r-1}{r}n - \binom{r}{2}\alpha_{r-1}(G)$.

(ii) For all $0 \leq a \leq n/(r-1)$, there exists an $(r-1)$-uniform hypergraph on n vertices with $\alpha_{r-1}(G) = a$ such that $\mathrm{mc}_r(G) \leq \frac{r-1}{r}(n-a)$.

While this is not the main focus of the current paper, it would also be interesting to improve the error terms in the above theorems. In particular, for Theorem 1.5 we currently have (in the context of Problem 1.2) that $2 \leq d_{r,r} \leq (r-1)$.

Regarding upper bounds on $\mathrm{mc}_r(G)$, note that because of the results mentioned above regarding (hyper)graphs with large minimum (co)degree, when $r = k+1 \geq 3$ we can't necessarily get a $d'_{r,k} > 0$ such that $\mathrm{mc}_r(G) \leq \mathrm{mc}_r(K_n) - d'_{r,k}\alpha_k(G)$ (because it is possible to have large minimum (co)degree and large $\alpha_k(G)$). However, when $r = k$, it is the case that $\mathrm{mc}_r(G) \leq \mathrm{mc}_r(K_n^r) - \alpha_r(G)$ (see Observation 3.1).

1.2. Corollaries

As mentioned earlier, there have been a number of results showing that $\mathrm{mc}_r(H^k(n,p)) = (1-o(1))\mathrm{mc}_r(K_n^k)$ where $H^k(n,p)$ is the binomial random k-uniform hypergraph. However, those results have all used the sparse regularity lemma and thus there are no reasonable estimates on the error terms. Since the value of $\alpha_k(H^k(n,p))$ is easy to estimate, we automatically recover $\mathrm{mc}_r(H^k(n,p)) = (1-o(1))\mathrm{mc}_r(K_n^k)$ (for all values of k and r for which Theorems 1.3–1.6 hold) with very good estimates on the error terms.

Corollary 1.7. *For all $r \geq 2$ and $p = \frac{d}{n^{r-1}}$ with $d \to \infty$, we have that with high probability,*

$$\mathrm{mc}_r(H^r(n,p)) = n - \Theta_r\left(\left(\frac{\log d}{d}\right)^{\frac{1}{r-1}} n\right).$$

Additionally,

$$\mathrm{mc}_3(H^2(n,p)) \geq \frac{n}{2} - O_r\left(\frac{\log d}{d} n\right)$$

and for all $r \geq 3$,

$$\mathrm{mc}_{r+1}(H^r(n,p)) \geq \frac{r}{r+1}n - O_r\left(\left(\frac{\log d}{d}\right)^{\frac{1}{r-1}} n\right)$$

Proof. The statements above follow from Theorems 1.3–1.6 and the fact that for p as in the statement, w.h.p., $\alpha_r(H^r(n,p)) = \Theta_r\left(\left(\frac{\log d}{d}\right)^{\frac{1}{r-1}} n\right)$. The upper bound can be shown using a standard first moment calculation. The lower bound follows by taking an independent set of size $\Theta_r\left(\left(\frac{\log d}{d}\right)^{\frac{1}{r-1}} n\right)$ and partitioning it into r equal sized sets (see Observation 2.2). Independent sets of this size are known to exist (see e.g. [15]). □

See Observation 3.6 and Problem 3.7 for a discussion about upper bounds on the terms in the second and third statements.

Let $\mathcal{S}_n$ be the family of all Steiner triple systems on n vertices. DeBiasio and Tait [6] proved that for all 3-uniform hypergraphs G on n vertices in which every pair of vertices is contained in at least one edge, $\mathrm{mc}_3(G) \geq n - 2\alpha_3(G)$ (note that Theorem 1.5(i) is stronger in the sense that

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there is no requirement that every pair of vertices is contained in at least one edge). They used this to prove that for all $S \in \mathcal{S}_n$, $\mathrm{mc}_3(S) \geq 2n/3 + 1$, and furthermore there exists $\delta > 0$ such that for almost all $S \in \mathcal{S}_n$, $\mathrm{mc}_3(S) \geq n - n^{1-\delta}$. This latter result was proved by showing that for almost all $S \in \mathcal{S}_n$, $\alpha_3(S) \leq n^{1-\delta}$. Gyárfás [13] proved in particular that for all $S \in \mathcal{S}_n$, $\mathrm{mc}_4(S) \geq \frac{n}{3}$ (and this is best possible for infinitely many n). On the other hand, we show that for almost all $S \in \mathcal{S}_n$, the value of $\mathrm{mc}_4(S)$ is much larger. More precisely, using the fact (from [6]) that for almost all $S \in \mathcal{S}_n$, $\alpha_3(S) \leq n^{1-\delta}$, we obtain the following corollary of Theorem 1.6(i) (with $r = 4$).

Corollary 1.8. *There exists $\delta > 0$ such that for almost all $S \in \mathcal{S}_n$, $\mathrm{mc}_4(S) \geq \frac{3n}{4} - O(n^{1-\delta})$.*

1.3. Outline of paper

In Section 2.1, we discuss a reformulation of our problem in the dual language of r-partite r-uniform hypergraphs which we will work with for the remainder of the paper. In Section 2.2, we discuss a reformulation of the notion of having bounded k-partite holes in terms of expansion in hypergraphs. In Section 3, we provide examples which show the tightness of our results. In particular, this section contains proofs of Theorem 1.3(i)(b), (ii), (iii), Theorem 1.4 (ii), Theorem 1.5 (ii) and Theorem 1.6 (ii). In Section 4, we prove Theorem 1.3(i)(a), Theorem 1.4(i), Theorem 1.5(i), and Theorem 1.6(i).

2. Duality and expansion

2.1. Duality

Throughout the rest of the paper we will be talking about multi-hypergraphs and we will always assume that all of the edges are distinguishable (and more generally, we assume that all of the elements in a multi-set are distinguishable). This means, for example, that if an edge has multiplicity 5, we can partition those five edges into two disjoint sets of say 3 and 2 edges respectively.

Let $r, k \geq 2$ be integers. Given an r-partite r-uniform multi-hypergraph H and multisets of edges $E_1, \ldots, E_k$, we say that $E_1, \ldots, E_k$ is *cross-intersecting* if there exists $(e_1, e_2, \ldots, e_k) \in E_1 \times E_2 \times \cdots \times E_k$ such that $e_1 \cap \cdots \cap e_k \neq \emptyset$. Furthermore, if $S \subseteq V(H)$, we say that $E_1, \ldots, E_k$ is *cross-intersecting in* S if there exists $(e_1, e_2, \ldots, e_k) \in E_1 \times E_2 \times \cdots \times E_k$ such that $S \cap e_1 \cap \cdots \cap e_k \neq \emptyset$.

Let $\nu_k(H)$ be the largest integer m such that there exists multisets of edges $E_1, \ldots, E_k$ with $|E_i| = m$ for all $i \in [k]$ and $E_i \cap E_j = \emptyset$ for all distinct $i, j \in [k]$ such that $E_1, \ldots, E_k$ is not cross-intersecting; that is, $e_1 \cap e_2 \cap \cdots \cap e_k = \emptyset$ for all $e_1 \in E_1, e_2 \in E_2, \ldots, e_k \in E_k$.

2.1.1. Monochromatic components and k-partite holes

The following observation precisely describes what we mean by 'duality'.

Observation 2.1 (Duality). *Let $n \geq 1$, $r, k \geq 2$, and $s, t \geq 0$. The following are equivalent:*

(i) Let H be an r-partite r-uniform multi-hypergraph with n edges. If $\nu_k(H) \leq s$, then $\Delta(H) \geq t$.

(ii) Let G be a k-uniform hypergraph on n vertices. If $\alpha_k(G) \leq s$, then $\mathrm{mc}_r(G) \geq t$

Proof. ((i)⇒(ii)) Suppose that (i) holds and let G be a k-uniform hypergraph with $\alpha_k(G) \leq s$. Suppose we are given an r-colouring of G. We will use the r-coloured hypergraph G to define an r-partite r-uniform multi-hypergraph having the property that $\nu_k(H) \leq \alpha_k(G) \leq s$ and every vertex in H with degree d corresponds to a monochromatic component in G with order d.

For all $i \in [r]$, let $C^i_1, \ldots, C^i_{k_i}$ be the components of G of colour i (note that a vertex which is incident with no edges of colour i is itself a component of colour i). Let H be an r-partite r-uniform multi-hypergraph with parts $C^i = \{C^i_1, \ldots, C^i_{k_i}\}$ for all $i \in [r]$ where $\{C^1_{j_1}, \ldots, C^r_{j_r}\} \in E(H)$ is

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an edge of multiplicity m if and only if $|\bigcap_{i\in[r]} V(C^i_{j_i})| = m$ (in G); note that an edge of multiplicity 0 just means a non-edge. Note that $V(G) = E(H)$ since every vertex in G is in exactly one component of each colour. If there exists $E_1, E_2, \ldots, E_k \subseteq E(H)$ such that $E_i \cap E_j = \emptyset$ for all distinct $i, j \in [k]$ and $e_1 \cap e_2 \cap \cdots \cap e_k = \emptyset$ for all $(e_1, \ldots, e_k) \in E_1 \times \cdots \times E_k$, then $e_G(E_1, E_2, \ldots, E_k) = 0$ because any such edge intersecting all of $E_1, \ldots, E_k$ (in G) would violate $e_1 \cap e_2 \cap \cdots \cap e_k = \emptyset$ (in H). So we have $\nu_k(H) \le \alpha_k(G) \le s$ which by the assumption implies $\Delta(H) \ge t$. Without loss of generality, suppose $d_H(C^1_1) = \Delta(H) \ge t$ which means C^1_1 is a component of colour 1 in G with at least t vertices.

((ii)⇒(i)) Suppose that (ii) holds and let H be an r-partite r-uniform multi-hypergraph with $\nu_k(H) \le s$ and let the parts of H be labelled as $C^i = \{C^i_1, \ldots, C^i_{k_i}\}$ for all $i \in [r]$ (so we have $V(H) = \cup_{i\in[r]} C^i$). We will use the r-partite r-uniform multi-hypergraph H to define an r-coloured k-uniform hypergraph G having the property that $\alpha_k(G) \le \nu_k(H) \le s$ and where the components of colour i in G correspond to the vertices $C^i = \{C^i_1, \ldots, C^i_{k_i}\}$ in such a way that the order of the component in G corresponding to $C^i_{i_j}$ is equal to the degree (in H) of $C^i_{i_j}$.

Let G be an r-edge coloured k-uniform hypergraph with $V(G) = E(H)$ where $\{e_1, \ldots, e_k\} \in E(G)$ and of colour i if and only if $(e_1 \cap \cdots \cap e_k) \cap C^i \ne \emptyset$ (in H); note that an edge of G may receive multiple colours. Consider a component of colour i in G (the vertex set of which corresponds to a collection of edges in H). By the definition of connectivity in hypergraphs, these edges of H must all pairwise intersect in C^i and since H is r-partite, they must pairwise intersect in a single vertex $C^i_{i_j}$ of C^i (and in this way, we can say that there is a bijection between the monochromatic components of G and the vertices of H). Now, if $E_1, E_2, \ldots, E_k \subseteq V(G)$ such that $e_G(E_1, E_2, \ldots, E_k) = 0$, then $\bigcap_{i\in[k]} (\bigcup_{e\in E_i} e) = \emptyset$ (in H). So we have $\alpha_k(G) \le \nu_k(H) \le s$, which by the assumption implies that G has a monochromatic component of order at least t. Without loss of generality suppose this monochromatic component corresponds to C^1_1 and thus by the comments above, we have that C^1_1 has degree at least t. □

2.1.2. Monochromatic components and independence number

For expository reasons and as a comparison to the result in the last subsection, we describe Füredi's classic example of the use of duality.

For a hypergraph H, let $\tau(H)$ denote the vertex cover number, let $\nu(H)$ denote the matching number and let $\tau^*(H)$ and $\nu^*(H)$ denote the respective fractional versions. Ryser conjectured that for every r-partite (multi)hypergraph H, $\tau(H) \le (r-1)\nu(H)$. Füredi [9] proved a fractional version; that is, for every r-partite (multi)hypergraph H, $\tau^*(H) \le (r-1)\nu(H)$. Now since

$$\frac{n}{\Delta(H)} \le \nu^*(H) = \tau^*(H) \le (r-1)\nu(H),$$

it follows that for every r-partite (multi)hypergraph H with n edges, $\Delta(H) \ge \frac{n}{(r-1)\nu(H)}$. In the dual language, this says for every graph G on n vertices, $\mathrm{mc}_r(G) \ge \frac{n}{(r-1)\alpha(G)}$.

2.2. Expansion

The purpose of this section is formalise what we mean when we say that k-uniform hypergraphs with small k-partite hole number are 'expansive'.

Let G be a k-uniform hypergraph G on n vertices and let $S_1, \ldots, S_{k-1} \subseteq V(G)$. Define $N(S_1, \ldots, S_{k-1}) = \{v : \{v_1, \ldots, v_{k-1}, v\} \in E(G), v_i \in S_i \text{ for all } i \in [k-1]\}$ and $N^+(S_1, \ldots, S_{k-1}) = \{v \in V(G) \setminus (S_1 \cup \cdots \cup S_{k-1}) : \{v_1, \ldots, v_{k-1}, v\} \in E(G), v_i \in S_i \text{ for all } i \in [k-1]\}$.

We say that a k-uniform hypergraph G on n vertices is a *(p, q)-expander* if for all sets $S_1, \ldots, S_{k-1} \subseteq V(G)$ with $|S_i| > p$ for all $i \in [k-1]$, we have $|N(S_1, \ldots, S_{k-1})| \ge q$.

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We say that a k-uniform hypergraph G on n vertices is a (p,q)-*outer-expander* if for all *disjoint* sets $S_1,\ldots,S_{k-1}\subseteq V(G)$ with $|S_i|>p$ for all $i\in[k-1]$, we have $|N^+(S_1,\ldots,S_{k-1})|+|S_1\cup\cdots\cup S_{k-1}|\geq q$.

Given a hypergraph G and an integer $r\geq 2$, let $\hat{\alpha}_r(G)$ be the largest integer a such that there exists (not-necessarily disjoint) sets $V_1,\ldots,V_r$ with $|V_i|=a$ for all $i\in[r]$ such that there are no edges e such that $e\cap V_i\neq\emptyset$ for all $i\in[r]$.

We first make an observation regarding the relationship between $\alpha_k(G)$, $\hat{\alpha}_k(G)$, $\alpha(G)$. One takeaway from this observation is that it would make very little difference in our results if we considered bounding $\hat{\alpha}_k(G)$ instead of $\alpha_k(G)$. However, it is possible for $\alpha(G)$ to be small and $\alpha_k(G)$ to be large (the disjoint union of cliques of order n/k for instance), so it makes a big difference if we were to consider bounding $\alpha(G)$ instead of $\alpha_k(G)$.

Observation 2.2. *For all k-uniform hypergraphs G,*

$$\left\lfloor\frac{\alpha(G)}{k}\right\rfloor\leq\left\lfloor\frac{\hat{\alpha}_k(G)}{k}\right\rfloor\leq\alpha_k(G)\leq\hat{\alpha}_k(G).$$

Proof. First note that if S is an independent set, then by letting $V_1=\cdots=V_k=S$, we have $\hat{\alpha}_k(G)\geq|S|$. So $\alpha(G)\leq\hat{\alpha}_k(G)$. Also we clearly have $\alpha_k(G)\leq\hat{\alpha}_k(G)$ since $\hat{\alpha}_k$ is computed over a strictly larger domain than α_k (all collections of sets vs. all collections of disjoint sets).

Now let $V_1,\ldots,V_k\subseteq V(G)$ (not-necessarily-disjoint) be sets such that $|V_1|=\cdots=|V_k|$, and there are no edges e such that $e\cap X_i\neq\emptyset$. For all $i\in[k]$, there exists $V_i'\subseteq V_i$ with $|V_i'|\geq\lfloor\frac{|V_i|}{k}\rfloor$ such that $V_i'\cap V_j'=\emptyset$ for all distinct $i,j\in[k]$. Since there are no edges which intersect all of $V_1,\ldots,V_k$, there are no edges which intersect all of $V_1',\ldots,V_k'$ and thus we have $\alpha_k(G)\geq\lfloor\frac{\hat{\alpha}_k(G)}{k}\rfloor$. □

We now make an observation which provides the relationship between small k-partite holes and expansion.

Observation 2.3. *Let $G=(V,E)$ be a k-uniform hypergraph on n vertices.*

(i) G is a $(p,n-p)$-expander if and only if $\hat{\alpha}_k(G)\leq p$.

(ii) G is a $(p,n-p)$-outer-expander if and only if $\alpha_k(G)\leq p$.

Proof. (i) Let $S_1,\ldots,S_{k-1}\subseteq V$ with $|S_i|>p$ for all $i\in[k-1]$. If $|N(S_1,\ldots,S_{k-1})|<n-p$, then $|V\setminus N(S_1,\ldots,S_{k-1})|>p$ and there are no edges touching all of $S_1,\ldots,S_{k-1}$, $V\setminus N(S_1,\ldots,S_{k-1})$ which implies $\hat{\alpha}_k(G)>p$.

Now suppose G is a $(p,n-p)$-expander and let $S_1,\ldots,S_k\subseteq V$ with $|S_i|>p$ for all $i\in[k]$. Since $|N(S_1,\ldots,S_{k-1})|\geq n-p$, we have $S_k\cap N(S_1,\ldots,S_{k-1})\neq\emptyset$; that is, there is an edge which touches all of $S_1,\ldots,S_k$ and thus $\hat{\alpha}_k(G)\leq p$.

(ii) Let $S_1,\ldots,S_{k-1}\subseteq V$ be disjoint sets with $|S_i|>p$ for all $i\in[k-1]$. If $|N^+(S_1,\ldots,S_{k-1})|+|S_1\cup\cdots\cup S_{k-1}|<n-p$, then $|V\setminus(N^+(S_1,\ldots,S_{k-1})\cup(S_1\cup\cdots\cup S_{k-1}))|>p$ and there are no edges touching all of $S_1,\ldots,S_{k-1}$, $V\setminus(N^+(S_1,\ldots,S_{k-1})\cup(S_1\cup\cdots\cup S_{k-1}))$ which implies $\alpha_k(G)>p$.

Now suppose G is a $(p,n-p)$-outer-expander and let $S_1,\ldots,S_k\subseteq V$ be disjoint sets with $|S_i|>p$ for all $i\in[k]$. Since $|N^+(S_1,\ldots,S_{k-1})|+|S_1\cup\cdots\cup S_{k-1}|\geq n-p$, we have $S_k\cap N^+(S_1,\ldots,S_{k-1})\neq\emptyset$; that is, there is an edge which touches all of $S_1,\ldots,S_k$ and thus $\alpha_k(G)\leq p$. □

3. Examples

The first example provides the upper bound in Theorem 1.3.(ii).

Observation 3.1. *Let $2\leq r\leq k$. For all k-uniform hypergraphs G on n vertices, $\mathrm{mc}_r(G)\leq n-\alpha_r(G)$.*

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Proof. Let $X_1, \ldots, X_r$ be disjoint sets which witness the value of $\alpha_r(G)$; that is, disjoint sets with $|X_i| = \alpha_r(G)$ for all $i \in [r]$ such that $e(X_1, \ldots, X_r) = \emptyset$. For all $i \in [r]$, colour all edges not incident with X_i with colour i (so edges may receive many colours). Since every edge misses some X_i, every edge receives at least one colour. So every component of colour i avoids X_i and thus has order at most $n - \alpha_r(G)$. □

The next example provides the proof of Theorem 1.3.(iii) and Theorem 1.5.(ii).

Example 3.2. *For all integers $n \geq r \geq 2$ and $0 \leq a \leq n/(r+2)$, there exists a r-uniform hypergraph G on n vertices with $\alpha_r(G) = a$ such that $\mathrm{mc}_r(G) \leq n - 2\alpha_r(G)$.*

Proof. Let V be a set of order n and let $\{V_0, V_1, \ldots, V_r, V_{r+1}\}$ be a partition of V with $|V_0| = n - (r+1)a$, $|V_1| = \cdots = |V_r| = |V_{r+1}| = a$. For $j \in [r]$, define $\mathcal{X}_j = V_j \cup V_{r+1}$ and $\mathcal{Y}_j = V \setminus \mathcal{X}_j = V_0 \cup \bigcup_{i \in [r] \setminus \{j\}} V_i$. Let G be an r-uniform hypergraph on V with edge set $\bigcup_{j \in [r]} \binom{\mathcal{X}_j}{r} \cup \binom{\mathcal{Y}_j}{r}$. If $e \in \binom{\mathcal{X}_j}{r} \cup \binom{\mathcal{Y}_j}{r}$ colour e with j (so edges can receive more than one colour). Note that for all $j \in [r]$, $\mathcal{X}_j$, and $\mathcal{Y}_j$ form disjoint monochromatic components of colour j and thus the largest monochromatic component has order $\max\{n - 2a, 2a\} = n - 2a$ as desired.

We now check that $\alpha_r(G) = a$. Note that $V_1, \ldots, V_r$ is an r-partite hole of size a. Suppose $U_1, \ldots, U_r$ is an r-partite hole of size $a + 1$. First note that for all $j \in [r]$, there exists $j \in [r]$ such that $U_i \subseteq \mathcal{X}_j$. If not, then there exists $j \in [r]$ such that every U_i intersects $\mathcal{Y}_j$, but since every r-set in $\mathcal{Y}_j$ is an edge, this is a contradiction. Also note that we cannot have $U_i, U_j \subseteq \mathcal{X}_k$ for $i \neq j$ since $|\mathcal{X}_k| = 2a < |U_i \cup U_j|$. So without loss of generality, we may assume that for all $j \in [r]$, $U_j \subseteq \mathcal{X}_j$. So for all $j \in [r]$, U_j intersects V_{r+1} since $|U_j| > |V_j|$. But since every r-set in V_{r+1} is an edge, this is a contradiction. □

For expository reasons, we give the same example as above in the dual language.

Example 3.3. *For all integers $n \geq r \geq 2$ and $a \geq 0$ with $a \leq n/(r+2)$, there exists an r-uniform hypergraph H on n vertices with $\nu_r(H) = a$ such that $\Delta(H) = n - 2\nu_r(H)$.*

Proof. Let H be an r-partite hypergraph with two vertices u_i, v_i in each part. Let $\{v_1, \ldots, v_r\}$ be an edge of multiplicity a. For all $i \in [r]$, let $\{u_1, \ldots, u_{i-1}, v_i, u_{i+1}, \ldots, u_r\}$ be an edge of multiplicity a. Finally, let $\{u_1, \ldots, u_r\}$ be an edge of multiplicity $n - (r+1)a$. Note that every vertex in $\{u_1, \ldots, u_r\}$ has degree $n - 2a$ and every vertex in $\{v_1, \ldots, v_n\}$ has degree $2a$, so $\Delta(H) = \max\{n - 2a, 2a\} = n - 2a$ as desired.

For $i \in [r]$, let F_i be the multiset of a edges $\{u_1, \ldots, u_{i-1}, v_i, u_{i+1}, \ldots, u_r\}$. Then the collection $F_1, \ldots, F_r$ shows that $\nu_r(H) \geq a$. Now suppose there is a non-cross intersecting collection $E_1 \ldots, E_r$ with $|E_i| = a + 1$ for all $i \in [r]$. For all $j \in [r]$ there is some $i \in [r]$ such that no element of E_i contains u_j. So without loss of generality, we may assume that for all $j \in [r]$, no element of E_j contains u_j. But now each E_j must contain an edge of the form $\{v_1, \ldots, v_r\}$ since there are only a edges of the form $\{u_1, \ldots, u_{j-1}, v_j, u_{j+1}, \ldots, u_r\}$. Thus the sets $E_1, \ldots, E_r$ are in fact cross-intersecting (in all of $v_1, \ldots, v_r$), a contradiction. □

The next example provides the proof of Theorem 1.6(ii).

Example 3.4. *For all $r \geq 2$ and $1 \leq a \leq n/k$, there exists a k-uniform hypergraph G on n vertices with $\alpha_k(G) = a$ such that $\mathrm{mc}_r(G) = \mathrm{mc}_r(K^k_{n-a}) < \mathrm{mc}_r(K^k_n)$.*

Proof. Let G be a complete k-uniform hypergraph on $n - a$ vertices together with a isolated vertices. We have $\mathrm{mc}_r(G) = \mathrm{mc}_r(K^k_{n-a}) < \mathrm{mc}_r(K^k_n)$. □

The next example provides the proof of Theorem 1.3.(i)(b). For instance when $s = 3$, $t = 4$, this gives an example of a graph G with $\alpha_2(G) = \frac{n}{6}$ and a 2-colouring in which the largest monochromatic component has order $\frac{n}{3}$ and thus $\mathrm{mc}_2(G) \leq \frac{n}{3}$.

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Example 3.5. *Let $n \geq t \geq s$ be positive integers such that st divides n. The (s, t)-grid on n vertices, denoted $G_n(s, t)$, is the graph obtained by partitioning $[n]$ into st sets $A_{11}, \ldots, A_{1t}, A_{21}, \ldots, A_{2t}, \ldots, A_{s1}, \ldots, A_{st}$, each of order $\frac{n}{st}$. For all $i \in [s]$ let $A_{i1} \cup \cdots \cup A_{it}$ be a clique, and for all $j \in [t]$ let $A_{1j} \cup \cdots \cup A_{sj}$ be a clique.*

The natural 2-colouring of $G_n(s, t)$ is defined by colouring all of the edges inside the 'rows' $A_{i1} \cup \cdots \cup A_{it}$ red and all of the edges inside the 'columns' $A_{1j} \cup \cdots \cup A_{sj}$ blue (the edges inside the sets A_{ij} can be coloured with either colour).

We have $\alpha_2(G_n(s, t)) = \min\{\frac{\lceil s/2\rceil\lfloor t/2\rfloor}{st} n, \frac{\lfloor s/2\rfloor\lceil t/2\rceil}{st} n\}$ and the largest monochromatic component in the natural colouring of $G_n(s, t)$ has order $\frac{n}{s}$.

Proof. Set $G := G_n(s, t)$ and take the natural 2-colouring of G. The fact that the largest monochromatic component has order $\frac{n}{s}$ is evident by the way the graph and its colouring is defined since $s \leq t$. To see that $\alpha_2(G) = \frac{\lceil s/2\rceil\lfloor t/2\rfloor}{st} n$, let $X, Y \subseteq V(G)$ be maximal disjoint sets witnessing the value of $\alpha_2(G)$; that is, $\min\{|X|, |Y|\} = \alpha_2(G)$ and $e(X, Y) = 0$. By the maximality of X, Y and the structure of G, we have that if $X \cap A_{ij} \neq \emptyset$ then $X \cap A_{ij} = A_{ij}$ and likewise $Y \cap A_{ij} \neq \emptyset$ implies $Y \cap A_{ij} = A_{ij}$. Let $I = \{i \in [s] : X \cap A_{ij} \neq \emptyset$ for some $j \in [t]\}$ and $J = \{j \in [t] : X \cap A_{ij} \neq \emptyset$ for some $i \in [s]\}$. This implies that if $Y \cap A_{ij} \neq \emptyset$, then $i \in [s] \setminus I$ and $j \in [t] \setminus J$. So we have $|X| = \frac{|I||J|n}{st}$ and $|Y| = \frac{(s-|I|)(t-|J|)n}{st}$ and thus $\alpha_2(G) = \min\{|X|, |Y|\}$ is maximised when $|I| = \lceil s/2\rceil$ and $|J| = \lfloor t/2\rfloor$ (equivalently, $|I| = \lfloor s/2\rfloor$ and $|J| = \lceil t/2\rceil$). □

For random graphs $G(n, p)$, it was shown in [2] and [8] that for $p = \frac{\omega(1)}{n}$, we have w.h.p., $\mathrm{mc}_r(G(n, p)) \geq \left(\frac{1}{r-1} - o(1)\right) n$ and thus (whenever an affine plane of order $r-1$ exists) we have $\mathrm{mc}_r(G(n, p)) = (1 - o(1))\mathrm{mc}_r(K_n)$. Analogously, for random hypergraphs it was shown in [5] that for $r \geq 4$ and $p = \frac{\omega(1)}{n^{r-2}}$, we have w.h.p., $\mathrm{mc}_r(H^{r-1}(n, p)) \geq (\frac{r-1}{r} - o(1))n$ and thus $\mathrm{mc}_r(H^{r-1}(n, p)) = (1 - o(1))\mathrm{mc}_r(K_n^{r-1})$.

The following observation shows that for sufficiently small p (but above the thresholds mentioned above), we have $\mathrm{mc}_r(G(n, p))$ is bounded away from $\mathrm{mc}_r(K_n)$ by a constant and $\mathrm{mc}_r(H^{r-1}(n, p))$ is bounded away from $\mathrm{mc}_r(K_n^{r-1})$ by a constant.

Observation 3.6. *Let r and C be integers with $r \geq 2$ and $C \geq 1$.*

(i) If an affine plane of order $r-1$ exists, then for $\frac{\omega(1)}{n} = p < \frac{1}{2Cr(r-1)^2}$ we have w.h.p.,

$$\left(\frac{1}{r-1} - o(1)\right) n = \mathrm{mc}_r(G(n, p)) \leq \frac{n}{r-1} - C.$$

(ii) If $r \geq 4$, then for $\frac{\omega(1)}{n^{r-2}} = p = \frac{o(\sqrt{n})}{n^{r-2}}$ we have w.h.p.,

$$\left(\frac{r-1}{r} - o(1)\right) n = \mathrm{mc}_r(H^{r-1}(n, p)) \leq \frac{r-1}{r} n - C(r-1).$$

Proof. (i) Note that p is small enough so that with high probability, $G(n, p)$ has an independent set X with Cr vertices such that $|\bigcup_{v \in X} N(v)| \leq \frac{n-Cr}{(r-1)^2}$. Partition X into r sets $\{X_1, \ldots, X_r\}$ each of order C and partition the vertices of $V(G) - X$ into sets of size $\frac{n-Cr}{(r-1)^2}$, with one of those sets containing $\bigcup_{v \in X} N(v)$, and colour the edges of $G - X$ according to the affine plane colouring. Now colour all edges incident with X_i with colour i for all $i \in [r]$. So every component of colour i has order at most $\frac{n-Cr}{r-1} + C \leq \frac{n}{r-1} - C$.

(ii) Note that p is small enough so that with high probability there exists an independent set X of order Cr. Partition X into r sets $\{X_1, \ldots, X_r\}$ each of order C and let $E_i = \{e : e \cap X_i \neq \emptyset\}$ for all $i \in [r]$. Additionally, p is small enough so that with high probability for all $i \in [r]$, $|\bigcup_{e \in E_i} e| \leq \frac{n}{r}$, and for all distinct $i, j \in [r]$, all $e_i \in E_i$, and all $e_j \in E_j$, we have $e_i \cap e_j = \emptyset$.

Let $\{A_1, \ldots, A_r\}$ be a partition of $V(G)$ into sets which are as equally sized as possible having the property that for all $i \in [r]$, $\bigcup_{e \in E_i} e \subseteq A_i$. Now for all $i \in [r]$ colour the edges in E_i with colour i. For all $e \in E(G) \setminus \cup_{i \in [r]} E_i$, note that since $|e| = r - 1$, there exists $i \in [r]$ such that $e \cap A_i = \emptyset$ and thus we assign any such colour i to the edge e. So we have that for all $i \in [r]$, there is a component of colour i containing X_i and having order at most n/r and there is a component colour i which avoids X and A_i and thus has order at most $\frac{r-1}{r}n - (r-1)C$. □

On the other hand, when p is sufficiently close to 1, the minimum degree of $G(n,p)$ is close to n and the results of [10] apply to give $\mathrm{mc}_r(G(n,p)) = \mathrm{mc}_r(K_n)$. Likewise when p is sufficiently close to 1, the minimum co-degree of $H^{r-1}(n,p)$ is close to n and the results of [3] apply to give $\mathrm{mc}_r(H^{r-1}(n,p)) = \mathrm{mc}_r(K_n^{r-1})$. This observation together with Observation 3.6 leads us to the following problem.

Problem 3.7. *Determine the smallest p such that $\mathrm{mc}_r(G(n,p)) = \mathrm{mc}_r(K_n)$, and for all $r \geq 4$, determine the smallest p such that $\mathrm{mc}_r(H^{r-1}(n,p)) = \mathrm{mc}_r(K_n^{r-1})$.*

4. Main results in the dual language

All of the results of this section are of the type 'For all k, r there exists $c_{k,r}, d_{k,r}$ such that if G is a k-uniform hypergraph on n vertices with $\alpha_k(G) < c_{k,r}n$, then $\mathrm{mc}_r(G) \geq \mathrm{mc}_r(K_n^k) - d_{k,r}\alpha_k(G)$'; however we prove these statements in the equivalent dual form 'For all k, r there exists $c_{k,r}, d_{k,r} > 0$ such that if H is an r-partite r-uniform multihypergraph with n edges and $\nu_k(H) < c_{k,r}$, then $\Delta(H) \geq \mathrm{mc}_r(K_n^k) - d_{k,r}\nu_k(G)$'.

Theorem 4.1 (Dual of Theorem 1.3(i)(a)). *Let H be a bipartite multigraph with n edges. If $\nu_2(H) < n/6$, then $\Delta(H) \geq n - 2\nu_2(H)$.*

Theorem 4.2 (Dual of Theorem 1.4(i)). *Let H be an 3-partite 3-uniform multi-hypergraph with n edges. If $\nu_2(H) \leq \frac{n}{3^9}$, then $\Delta(H) \geq \frac{n}{2} - 2\nu_2(H)$.*

Theorem 4.3 (Dual of Theorem 1.5(i)). *Let $r \geq 3$ and let H be an r-partite r-uniform hypergraph with n edges. If $\nu_r(H) \leq \frac{n}{3^{\binom{r+1}{2}+r}}$, then $\Delta(H) \geq n - (r-1)\nu_r(H)$.*

Theorem 4.4 (Dual of Theorem 1.6(i)). *Let $r \geq 3$ and let H be an r-partite r-uniform hypergraph with n edges. If $\nu_{r-1}(H) \leq \frac{n}{3^{\binom{r+1}{2}+r}}$, then $\Delta(H) \geq \frac{(r-1)n}{r} - \binom{r}{2}\nu_{r-1}(H)$.*

4.1. General lemmas

In this section we collect a number of general lemmas. We begin with an elementary lemma that will be used throughout the proofs in this section. This lemma basically says that if we have a collection of edge sets in an r-partite r-uniform hypergraph, then for any part V_i of the partition, either there is a vertex in V_i which is incident with a large number of edges from each set, or else there is a large subset of each edge set which does not cross-intersect in V_i.

Lemma 4.5. *Let $\nu, \ell, a_1, \ldots, a_\ell$ be positive integers. Let H be an r-partite r-uniform multi-hypergraph with parts $V_1, \ldots, V_r$ and let $F_1, \ldots, F_\ell \subseteq E(H)$ such that $|F_1| \geq 3a_1\nu + 1$ and $|F_j| \geq 2a_j\nu + 1$ for all $j \in [2, \ell]$. For all $i \in [r]$, either*

(B1) there exists $u \in V_i$ such that for all $j \in [\ell]$, u is incident with at least $|F_j| - a_j\nu$ edges of F_j, or

(B2) there exists a subset $F_1' \subseteq F_1$ with $|F_1'| \geq a_1\nu + 1$ and a subset $F_j' \subseteq F_j$ for some $j \in [2, \ell]$ with $|F_j'| \geq a_j\nu + 1$ such that $F_1', F_2, \ldots, F_{j-1}, F_j', F_{j+1}, \ldots, F_\ell$ is not cross-intersecting in V_i.

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Proof. Let $i \in [r]$ and suppose (B1) doesn't hold. If there exists $u \in V_i$ such that u is incident with at least $|F_1| - a_1\nu$ edges from F_1, then by the assumption, there exists $j \in [2, \ell]$ such that u is incident with at most $|F_j| - a_j\nu - 1$ edges of F_j and thus at least $a_j\nu + 1$ edges of F_j intersect $V_i - u$ and thus (B2) is satisfied.

So suppose that every $u \in V_i$ is incident with at most $|F_1| - a_1\nu - 1$ edges of F_1. Let $V_i^1 \subseteq V_i$ be a minimal set of vertices incident with at least $a_1\nu + 1$ edges of F_1. By minimality, and the fact that every $u \in V_i$ is incident with at most $|F_1| - a_1\nu - 1$ edges of F_1, we have that both V_i^1 and $V_i^2 := V_i \setminus V_i^1$ are incident with at least $a_1\nu + 1$ edges of F_1. Now either V_i^1 or V_i^2 is incident with at least $\lfloor |F_2|/2 \rfloor \geq a_2\nu + 1$ edges of F_2, and either way (B2) is satisfied. $\square$

A simpler version of the above lemma which suffices whenever we don't care about the exact bounds is as follows.

Lemma 4.6. *Let $\nu, \ell, a_1, \ldots, a_\ell$ be positive integers. Let H be an r-partite r-uniform multi-hypergraph with parts $V_1, \ldots, V_r$, and let $F_1, \ldots, F_\ell \subseteq E(H)$ such that $|F_j| \geq 3a_j\nu + 1$ for all $j \in [\ell]$. For all $i \in [r]$, either*

- *(B1′) there exists $u \in V_i$ such that for all $j \in [\ell]$, u is incident with at least $|F_j| - a_j\nu$ edges of F_j, or*
- *(B2′) for all $j \in [\ell]$ there exists a subset $F_j' \subseteq F_j$ with $|F_j'| \geq a_j\nu + 1$ such that $F_1', \ldots, F_\ell'$ is not cross-intersecting in V_i.*

The following observation explicitly gives a relationship between ν_s and ν_t for $s \leq t$.

Observation 4.7. *Let $2 \leq s < t \leq r$ and let H be an r-partite r-uniform multi-hypergraph on n edges. If $\nu_t(H) \leq \frac{n}{t} - 1$, then $\nu_s(H) \leq \nu_t(H)$.*

Proof. Suppose $\nu_t(H) \leq \frac{n}{t} - 1$ and suppose for contradiction that $\nu_s(H) > \nu_t(H)$. So there exists disjoint sets $E_1, \ldots, E_s$ with $|E_i| = \nu_t(H) + 1$ for all $i \in [s]$ such that $E_1, \ldots, E_s$ is not cross-intersecting. Since $\nu_t(H) \leq \frac{n}{t} - 1$, we have $|E_i| \leq \frac{n}{t}$ for all $i \in [s]$ and thus $|E(H) \setminus (E_1 \cup \cdots \cup E_s)| \geq n - s\frac{n}{t} = (t - s)\frac{n}{t}$ and thus there is a partition of $E(H) \setminus (E_1 \cup \cdots \cup E_s)$ into $t - s$ sets $E_{s+1}, \ldots, E_t$ each of order greater than $\nu_t(H)$ such that $E_1, \ldots, E_s, E_{s+1}, \ldots, E_t$ is not cross-intersecting. $\square$

We now show that if H is an r-partite r-uniform multi-hypergraph on n edges with $\nu_s(H)$ small enough in terms of n and r, then there must be a vertex of fairly large degree.

Lemma 4.8. *Let Δ, r, s be positive integers with $2 \leq s \leq r$. Let H be an r-partite r-uniform multi-hypergraph with n edges and set $\nu := \nu_s(H)$. If $\nu \leq \frac{n}{3^r\Delta}$, then $\Delta(H) \geq \Delta\nu + 1$.*

Proof. Suppose $\nu \leq \frac{n}{3^r\Delta}$ and suppose for contradiction that $\Delta(H) \leq \Delta\nu$. Note that by Observation 4.7 we have $\nu_2(H) \leq \nu \leq \frac{n}{3^r\Delta}$.

Let $V_1, \ldots, V_r$ be the parts of H. Let $V_1' \subseteq V_1$ be a minimum set of vertices incident with at least $3^{r-1}\Delta\nu + 1$ edges. By minimality, we have

$$3^{r-1}\Delta\nu + 1 \leq e(V_1') \leq 3^{r-1}\Delta\nu + \Delta\nu$$

and consequently, since $\nu \leq \frac{n}{3^r\Delta}$, we have

$$e(V_1 \setminus V_1') = n - e(V_1') \geq n - 3^{r-1}\Delta\nu - \Delta\nu > n - 3^{r-1}\Delta\nu - 3^{r-1}\Delta\nu \geq 3^{r-1}\Delta\nu.$$

Let F_1^1 and F_2^1 be the sets of edges incident with V_1' and $V_1 \setminus V_1'$ respectively. Now we apply Lemma 4.6 (with $a_1 = a_2 = 3^{r-2}\Delta$ and $i = 2$), and since we are assuming $\Delta(H) \leq \Delta\nu$, (B2′) must happen. Now we have sets $F_1^2 \subseteq F_1^1$ and $F_2^2 \subseteq F_2^1$ such that $|F_1^2|, |F_2^2| \geq 3^{r-2}\Delta\nu + 1$ and F_1^2 and F_2^2 are not cross-intersecting in $V_1 \cup V_2$. Now we repeatedly apply Lemma 4.6 until we have sets F_1^{r-1} and F_2^{r-1} with $|F_1^{r-1}|, |F_2^{r-1}| \geq 3\Delta\nu + 1$ and F_1^{r-1} and F_2^{r-1} are not cross-intersecting in $V_1 \cup \cdots \cup V_{r-1}$. In the final step (where we apply Lemma 4.6 with $a_1 = a_2 = \Delta$ and $i = r$),

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either (B2′) happens and we have a contradiction to $\nu_2(H) \leq \nu$, or (B1′) happens and we have $\Delta(H) \geq \Delta\nu + 1$, contradicting the assumption. □

For the last result in this subsection we show that if there is a vertex of fairly large degree, then either we have an edge of multiplicity at least $\nu + 1$ or there is a vertex of even larger degree.

Lemma 4.9. *Let r be an integer with $r \geq 3$ and let $s \in \{2, r-1, r\}$. Let H be an r-partite r-uniform multi-hypergraph with n edges and set $\nu := \nu_s(H)$. If $\nu \leq \frac{n}{2r}$ and $\Delta(H) \geq 3^{\binom{r+1}{2}}\nu + 1$, then either H has an edge of multiplicity at least $\nu + 1$ or*

(i) if $s = 2$, then $\Delta(H) \geq \frac{n-2\nu}{r-1}$.
(ii) if $s = r$, then $\Delta(H) \geq n - 2\nu$.
(iii) if $s = r - 1 \geq 3$, then $\Delta(H) \geq \frac{(r-1)n}{r} - 2(r-1)\nu$.

Proof. Let $V_1, \ldots, V_r$ be the parts of H. For a set $U \subseteq V(H)$, let $d(U)$ denote the number of edges, counting multiplicity, which contain U (i.e. $d(U)$ is the degree of U). Note that since H is r-partite, $d(U) > 0$ implies that U contains at most one vertex from each part V_i. Let $U \subseteq V(H)$ be maximum such that $d(U) \geq 3^{\binom{r+2-|U|}{2}}\nu + 1$ and note that $U \neq \emptyset$ by the degree condition. Without loss of generality, suppose $U = \{u_1, \ldots, u_\ell\}$ with $u_i \in V_i$ for all $i \in [\ell]$ and let E be the set of edges containing U. If $\ell = r$, we have an edge of multiplicity at least $3\nu + 1 \geq \nu + 1$ and we are done; so suppose $1 \leq \ell \leq r - 1$.

Case (i) ($s = 2$). Let $F = \{f \in E(H) : f \cap U = \emptyset\}$. If $|F| \leq \frac{(r-1-\ell)n+2\ell\nu}{r-1}$, then for some $i \in [\ell]$,

$$d(u_i) \geq \frac{n - \frac{(r-1-\ell)n+2\ell\nu}{r-1}}{\ell} = \frac{n-2\nu}{r-1}$$

and we are done; so suppose $|F| > \frac{(r-1-\ell)n+2\ell\nu}{r-1} \geq 2^{r-\ell}\nu$ (where the last inequality holds since $\nu \leq \frac{n}{2r}$) and thus we have $|F| \geq 2^{r-\ell}\nu + 1$.

Applying Lemma 4.5 at most $r - \ell$ times with E, F and using the fact that $\nu_2(H) \leq \nu$, it must be the case that (B1) holds within $r - \ell$ steps and we obtain a vertex which is incident with at least $\frac{3^{\binom{r+2-\ell}{2}-1}}{3^{r-\ell}}\nu + 1 = 3^{\binom{r+2-(\ell+1)}{2}}\nu + 1$ edges of E contradicting the maximality of U.

Case (ii) and (iii) ($r - 1 \leq s \leq r$).

Since $|E| = d(U) \geq 3^{\binom{r+2-\ell}{2}} \geq 2(3^{\binom{r+2-\ell}{2}-1}\nu + 1)$, we can choose disjoint subsets E_1 and E_2 of E, each with at least $3^{\binom{r+2-\ell}{2}-1}\nu + 1$ edges. Applying Lemma 4.6 at most $r - \ell$ times with E_1, E_2, we will either find a vertex which is contained in at least $\frac{3^{\binom{r+2-\ell}{2}-1}}{3^{r-\ell}}\nu + 1 = 3^{\binom{r+2-(\ell+1)}{2}}\nu + 1$ edges from both E_1 and E_2, which would violate the maximality of U, or else we will get sets $E_1' \subseteq E_1$ and $E_2' \subseteq E_2$ with

$$|E_1'|, |E_2'| \geq 3^{\binom{r+1-\ell}{2}}\nu + 1 \geq 3\nu + 1 \text{ such that for all } e_1 \in E_1', e_2 \in E_2',\ e_1 \cap e_2 = U \tag{1}$$

(where the last inequality holds since $\ell \leq r - 1$).

Case (ii) ($s = r$). We have the desired degree condition unless for all $i \in [\ell]$, the set F_i of edges which avoids u_i has order at least $2\nu + 1$. If $\ell \leq r - 2$, then $E_1', E_2', F_1, \ldots, F_\ell$ is a collection of $\ell + 2 \leq r$ sets each of order at least $\nu + 1$ which are not cross intersecting, violating the bound on $\nu_r(H)$.

So suppose $\ell = r - 1$. Applying Lemma 4.5 with $E, F_1, \ldots, F_\ell$ and using the fact that $\ell + 1 \leq r$ and $\nu_r(H) \leq \nu$, it must be the case that (B1) holds and we obtain a vertex which is incident with at least $3\nu + 1$; that is, an edge of multiplicity at least $\nu + 1$.

Case (iii) ($s = r - 1 \geq 3$). We have the desired degree condition unless for all $i \in [\ell]$, the set F_i of edges which avoids u_i has order at least $\frac{n}{r} + 2(r-1)\nu + 1$. If $\ell \leq r - 3$, then $E_1', E_2', F_1, \ldots, F_\ell$ is a family of $\ell+2 \le r-1$ sets of at least $\nu+1$ edges each which are not cross intersecting and thus $\nu_{r-1}(H) \ge \nu_{\ell+2}(H) \ge \nu+1$, contradicting the assumption. If $\ell = r-2$, then applying Lemma 4.5 at most twice with $E, F_1, \dots, F_\ell$ and using the fact that $\ell+1 \le r-1$ and $\nu_{r-1}(H) \le \nu$, it must be the case that (B1) holds within two steps and we obtain a vertex which is incident with at least $\frac{3^{\binom{4}{2}-1}}{3^2}\nu + 1 = 3^3\nu + 1$ edges of E contradicting the maximality of U.

So finally suppose $\ell = r-1$. If there exists distinct $i, j \in [r-1]$ such that $|F_i \cap F_j| \ge 2\nu+1$, without loss of generality say $|F_{r-2} \cap F_{r-1}| \ge 2\nu+1$, then we apply Lemma 4.5 with $E, F_1, \dots, F_{r-3}, F_{r-2} \cap F_{r-1}$ and since (B2) can't happen, we have (B1) which gives us an edge of multiplicity at least $\nu+1$. So suppose $|F_i \cap F_j| \le 2\nu$ for all distinct $i, j \in [r-1]$. For all $i \in [r-1]$, let $F_i^* = F_i \setminus (\bigcup_{j \in [r-1]\setminus\{i\}} F_j)$ and note that by the previous sentence and the bound on $|F_i|$, we have $|F_i^*| \ge |F_i| - 2(r-2)\nu \ge \frac{n}{r} + 2\nu + 1$. Note that $F_1^*, \dots, F_{r-1}^*$ must be cross-intersecting and by the way the sets are defined, the cross-intersection must happen in V_r. Now applying Lemma 4.5 with $F_1^*, \dots, F_{r-1}^*$, we get a vertex in V_r which is adjacent with at least $|F_i^*| - \nu \ge \frac{n}{r}$ edges from each of $F_1^*, \dots, F_{r-1}^*$ giving us a vertex of degree at least $\frac{r-1}{r}n \ge \frac{r-1}{r}n - 2(r-1)\nu$ as desired. □

4.2. *Theorem 4.1 and Theorem 4.2*

Lemma 4.10. *Let $r \ge 2$ and let H be an r-partite multi-hypergraph with n edges and set $\nu := \nu_2(H)$. If H has an edge $e = \{u_1, \dots, u_r\}$ of multiplicity at least $\nu+1$, then*

(i) there are at least $n-\nu$ edges incident with e,

(ii) for all $e' \subseteq e$ with $1 \le |e'| \le r-1$, either the number of edges incident with every vertex in e' and no vertex in $e \setminus e'$ is at most ν, or the number of edges incident with every vertex in $e \setminus e'$ and no vertex in e' is at most ν, and

(iii) either $\Delta(H) \ge \frac{n}{r-1} - 2\nu$, or for all $e' \subseteq e$ with $1 \le |e'| \le r-1$, there are at least $\frac{(r-1-|e'|)n}{r-1} + (2|e'|-1)\nu + 1$ edges incident with $e \setminus e'$ but not e'.

Proof. Note that (i) and (ii) just follow from the condition on $\nu_2(H)$. To see (iii), let $e' \subseteq e$ with $1 \le |e'| =: t \le r-1$. If the number of edges incident with e' is at least $\frac{tn}{r-1} - 2t\nu$, then some $u \in e'$ satisfies $d(u) \ge \frac{n}{r-1} - 2\nu$ and we are done. So suppose that e' is incident with fewer than $\frac{tn}{r-1} - 2t\nu$ edges, which means there are at least

$$n - \nu - \left(\frac{tn}{r-1} - 2t\nu\right) + 1 = \frac{(r-1-t)n}{r-1} + (2t-1)\nu + 1$$

edges which are incident with $e \setminus e'$ but not e'. □

Now we prove that if H is a bipartite multigraph with n edges and $\nu_2(H) < n/6$, then $\Delta(H) \ge n - 2\nu_2(H)$.

Proof of Theorem 4.1. Let V_1, V_2 be the parts of H and set $\nu := \nu_2(H) < n/6$.
Case 1 (There exists an edge u_1u_2 of multiplicity at least $\nu+1$). By Lemma 4.10(i) and (ii), there are at least $n-\nu$ edges incident with $\{u_1, u_2\}$ and without loss of generality, there are at most ν edges which are incident with u_2 but not u_1. Thus there are at least $n-2\nu$ edges incident with u_1; that is, $\Delta(H) \ge n - 2\nu$.
Case 2 (Every edge has multiplicity at most ν). Suppose first that there exists $u_1 \in V_1, u_2 \in V_2$ so that $d(u_1), d(u_2) \ge 2\nu+1$. Since u_1u_2 has multiplicity at most ν, there are at least $\nu+1$ edges incident with u_1 but not u_2 and at least $\nu+1$ edges incident with u_2 but not u_1, a violation of the fact that $\nu_2(H) \le \nu$. So suppose without loss of generality that

$$d(u) \le 2\nu \text{ for all } u \in V_2. \tag{2}$$

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Now let $V_2' \subseteq V_2$ be minimal such that $e(V_2', V_1) \geq 2\nu + 1$. By (2) and minimality, we have $2\nu + 1 \leq e(V_2', V_1) \leq 4\nu$. Since $6\nu < n$, we also have $e(V_2 \setminus V_2', V_1) = n - e(V_2', V_1) \geq 2\nu + 1$. Furthermore, by pigeonhole and the fact that $6\nu < n$, we have either $e(V_2', V_1) \geq 3\nu + 1$ or $e(V_2 \setminus V_2', V_1) \geq 3\nu + 1$. So by applying Lemma 4.5 (with $a_1 = a_2 = 1$, $i = 1$, $F_1 = E(V_2', V_1)$, and $F_2 = E(V_2 \setminus V_2', V_1)$), we either have (B2) (that is, there exists $F_1' \subseteq F_1$ with $|F_1'| \geq \nu + 1$ and $F_2' \subseteq F_2$ with $|F_2'| \geq \nu + 1$ such that F_1' and F_2' are not cross intersecting in V_1) which violates the fact that $\nu_2(H) \leq \nu$, or (B1) which implies that there exists a vertex in V_1 which is incident with at least $|F_1| + |F_2| - 2\nu = n - 2\nu$ edges; that is, $\Delta(H) \geq n - 2\nu$. □

Proposition 4.11. *Let H be an 3-partite 3-uniform multi-hypergraph with n edges and set $\nu := \nu_2(H)$. If H has an edge of multiplicity at least $\nu + 1$, then $\Delta(H) \geq \frac{n}{2} - 2\nu$.*

Proof. Let $e = \{u_1, u_2, u_3\}$ be an edge of multiplicity at least $\nu + 1$. For all distinct $i, j, k \in [3]$, let E_i be the set of edges incident with u_i and let $E_i' = E_i \setminus (E_j \cup E_k)$. By Lemma 4.10.(iii) we have $|(E_1 \cup E_2) \setminus E_3| \geq \frac{n}{2} + \nu + 1$ and for all $i \in [3]$, $|E_i'| \geq 3\nu + 1$. So by Lemma 4.10.(ii), $|(E_1 \cap E_2) \setminus E_3| \leq \nu$. Thus $|E_1'| + |E_2'| \geq \frac{n}{2} + 1$. Now applying Lemma 4.6 with E_1' and E_2', we can't have (B2′), thus (B1′) holds and we have a vertex in V_3 which is incident with more than $\frac{n}{2} - 2\nu$ edges. □

Now we prove that if H is a 3-partite 3-uniform multi-hypergraph with n edges and $\nu_2(H) \leq \frac{n}{3^9} = \frac{n}{19683}$, then $\Delta(H) \geq \frac{n}{2} - 2\nu_2(H)$.

Proof of Theorem 4.2. Set $\nu := \nu_2(H)$. By Lemma 4.8 (with $\Delta = 729 = 3^6$), we have $\Delta(H) \geq 729\nu + 1 = 3^{\binom{3+1}{2}}\nu + 1$. Now by Lemma 4.9, we are done or we have an edge of multiplicity at least $\nu + 1$ in which case we are done by Proposition 4.11. □

In this subsection we solved Problem 1.2 in the case $k = 2$ and $2 \leq r \leq 3$. Because of Lemma 4.8 and Lemma 4.9, in order to solve Problem 1.2 in the case $k = 2$ and $r \geq 4$ it suffices to prove the following generalisation of Proposition 4.11.

Conjecture 4.12. *Let $r \geq 4$ and let H be an r-partite r-uniform multi-hypergraph with n edges and set $\nu := \nu_2(H)$. There exists $d_r > 0$ such that if H has an edge of multiplicity at least $\nu + 1$, then $\Delta(H) \geq \frac{n}{r-1} - d_r\nu$.*

The following is essentially a much weaker version of the previous conjecture.

Proposition 4.13. *Let $r \geq 4$ and let H be an r-partite r-uniform multi-hypergraph with n edges. If $\nu_2(H) \leq \frac{n}{3^{\binom{r+1}{2}+r}}$, then $\Delta(H) \geq \frac{n - \nu_2(H)}{r}$.*

Proof. Set $\nu := \nu_2(H)$. By Lemma 4.8 we have $\Delta(H) \geq 3^{\binom{r+1}{2}}\nu + 1$. Now by Lemma 4.9, we are done or we have an edge e of multiplicity at least $\nu + 1$. Thus by Lemma 4.10.(i), we have $n - \nu$ edges incident with e so, by averaging, one of these vertices has degree at least $\frac{n-\nu}{r}$. □

4.3. Theorem 4.3

Proposition 4.14. *Let $r \geq 3$ and let H be an r-partite r-uniform hypergraph with n edges and set $\nu := \nu_r(H)$. If H has an edge of multiplicity at least $\nu + 1$, then $\Delta(H) \geq n - (r-1)\nu$.*

Proof. Assume there exists an edge $e = \{u_1, \ldots, u_r\}$ of multiplicity at least $\nu + 1$. For all $i \in [r]$, let F_i be the set of edges which avoid u_i. If $|F_i| \leq (r-1)\nu$ for some $i \in [r]$, then $d(u_i) \geq n - (r-1)\nu$ and we are done; so suppose $|F_i| \geq (r-1)\nu + 1$ for all $i \in [r]$.

Claim 4.15. *For all distinct $i, j \in [r]$, $|F_i \cap F_j| \leq \nu$.*

Proof. Suppose for contradiction that $|F_i \cap F_j| \geq \nu + 1$ for some distinct $i, j \in [r]$ and without loss of generality suppose $\{i, j\} = [2]$. Now $e, F_1 \cap F_2, F_3, \ldots, F_r$ is a collection of r sets violating $\nu_r(H) \leq \nu$. □

Now for all $i \in [r-1]$, let $F_i' = F_i \setminus \bigcup_{j\in[r]\setminus\{i,i+1\}} F_j$ and let $F_r' = F_r \setminus \bigcup_{j\in[r]\setminus\{r,1\}} F_j$. Note that by Claim 4.15 we have that for all $i \in [r]$, $|F_i'| \geq (r-1)\nu + 1 - (r-2)\nu = \nu + 1$. Furthermore, by construction, we have $F_i' \cap F_j' = \emptyset$ for all distinct $i, j \in [r]$. So we have r disjoint sets $F_1', \ldots, F_r{}'$ each of order at least $\nu + 1$ which are not cross intersecting, violating the assumption. Indeed, let $e_i \in F_i'$ for all $i \in [r]$ and suppose for contradiction that $\bigcap_{i\in[r]} e_i \neq \emptyset$. Let $u \in \bigcap_{i\in[r]} e_i$ and suppose without loss of generality that $u \in V_1$. We cannot have $u = u_1$ since $e_1 \in F_1' \subseteq F_1$ misses the vertex u_1, but also we cannot have $u \neq u_1$ since $e_2 \in F_2'$ and $F_2' \cap F_1 = \emptyset$ and thus e_2 touches u_1. □

Now we prove that if $r \geq 3$ and H is an r-partite r-uniform hypergraph with n edges and $\nu_r(H) \leq \frac{n}{3^{\binom{r+1}{2}+r}}$, then $\Delta(H) \geq n - (r-1)\nu_r(H)$.

Proof of Theorem 4.3. Set $\nu := \nu_r(H)$. By applying Lemma 4.8 with $\Delta = 3^{\binom{r+1}{2}}$, we have $\Delta(H) \geq 3^{\binom{r+1}{2}}\nu + 1$. Now by Lemma 4.9, we are done or we have an edge of multiplicity at least $\nu + 1$ in which case we are done by Proposition 4.14. □

4.4. Theorem 4.4

Proposition 4.16. *Let $r \geq 4$ and let H be an r-partite multi-hypergraph with n edges and set $\nu := \nu_{r-1}(H)$. If H has an edge of multiplicity at least $\nu + 1$, then $\Delta(H) \geq \frac{r-1}{r}n - \binom{r}{2}\nu$.*

Proof. Assume there exists an edge $e = \{u_1, \ldots, u_r\}$ of multiplicity at least $\nu + 1$. For all $i \in [r]$, let F_i be the set of edges which avoid u_i. If $|F_i| \leq \frac{n}{r} + \binom{r}{2}\nu$ for some $i \in [r]$, then $\Delta(H) \geq \frac{r-1}{r}n - \binom{r}{2}\nu$ and we are done; so suppose $|F_i| > \frac{n}{r} + \binom{r}{2}\nu \geq (r-1)\nu + 1$ for all $i \in [r]$. Let $F = F_1 \cup \cdots \cup F_r$.

Claim 4.17. *For all distinct $h, i, j \in [r]$, $|F_h \cap F_i \cap F_j| \leq \nu$.*

Proof. Suppose for contradiction that $|F_h \cap F_i \cap F_j| \geq \nu + 1$ for some distinct $h, i, j \in [r]$ and without loss of generality suppose $\{h, i, j\} = [3]$. Now $e, F_1 \cap F_2 \cap F_3, F_4, \ldots, F_r$ is a collection of $r-1$ sets violating $\nu_{r-1}(H) \leq \nu$. □

Claim 4.18. *For all distinct $h, i, j, k \in [r]$, $|F_h \cap F_i| \leq \nu$ or $|F_j \cap F_k| \leq \nu$.*

Proof. Suppose for contradiction that $|F_h \cap F_i| \geq \nu + 1$ and $|F_j \cap F_k| \geq \nu + 1$ for some distinct $h, i, j, k \in [r]$ and without loss of generality suppose $\{h, i, j, k\} = [4]$. Now $e, F_1 \cap F_2, F_3 \cap F_4, F_5, \ldots, F_r$ is a collection of $r-1$ sets violating $\nu_{r-1}(H) \leq \nu$. □

For all $S \subseteq [r]$, let $(\cap_{i\in S} F_i)^* = (\cap_{i\in S} F_i) \setminus \left(\cup_{j\in[r]\setminus S} F_j\right)$. In other words $(\cap_{i\in S} F_i)^*$ is the collection of elements which are in all of the sets F_i, $i \in S$, but none of the other sets F_j, $j \in [r] \setminus S$.

Claim 4.19. *For all distinct $h, i, j \in [r]$, $|(F_h \cap F_i)^*| \leq \nu$ or $|F_j^*| \leq \nu$.*

Proof. Suppose for contradiction that $|(F_h \cap F_i)^*| \geq \nu + 1$ and $|F_j^*| \geq \nu + 1$ for some distinct $h, i, j \in [r]$ and without loss of generality suppose $\{h, i, j\} = [3]$. Now the sets $(F_1 \cap F_2)^*, F_3^*, F_4, \ldots, F_r$ is a collection of $r-1$ sets violating $\nu_{r-1}(H) \leq \nu$. □

Since $|F_i| > \frac{n}{r} + \binom{r}{2}\nu$ for all $i \in [r]$, inclusion-exclusion implies that $|F_i \cap F_j| \geq (r-1)\nu + 1$ for some distinct $i, j \in [r]$; without loss of generality, say $i = r-1$ and $j = r$. Furthermore, by Claim 4.17 we must have that $|(F_{r-1} \cap F_r)^*| \geq \nu + 1$. Thus by Claim 4.18, we have that for all distinct $i, j \in [r-2]$, $|F_i \cap F_j| \leq \nu$, and by Claim 4.19 we have that for all $i \in [r-2]$, $|F_i^*| \leq \nu$.

So for all $i \in [r-2]$, we have $|F_i \setminus (F \setminus F_i)| \leq \nu$, $|F_i \cap F_{r-1} \cap F_r| \leq \nu$, and for all $j \in [r-2] \setminus \{i\}$, $|F_i \cap F_j| \leq \nu$, thus

$$|F_i \cap F_{r-1}| + |F_i \cap F_r| \geq |F_i| - (r-1)\nu \geq \frac{n}{r} + \binom{r}{2}\nu - (r-1)\nu \geq \frac{n}{r} + 2\nu.$$

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Let $i \in [r-2]$. Without loss of generality, suppose $|F_i \cap F_r| \geq \frac{1}{2}(|F_i \cap F_{r-1}| + |F_i \cap F_r|) \geq \frac{n}{2r} + \nu > \nu$. Thus by Claim 4.18 we have that for all $j \in [r-2] \setminus \{i\}$, $|F_j \cap F_{r-1}| \leq \nu$ which in turn implies that for all $i \in [r-2]$, $|(F_i \cap F_r)^*| \geq |F_i| - r\nu > \frac{n}{r} + \nu$. By Claim 4.19 this implies that $|F^*_{r-1}| \leq \nu$. Thus $|(F_{r-1} \cap F_r) \setminus (F_1 \cup \cdots \cup F_{r-2})| \geq |F_{r-1}| - \nu - (r-2)\nu \geq \frac{n}{r} + \nu$. Now we have a collection of $r-1$ sets, $(F_1 \cap F_r)^*, \ldots, (F_{r-2} \cap F_r)^*, (F_{r-1} \cap F_r)^*$ all with more than $\frac{n}{r} + \nu$ elements. Now applying Lemma 4.6 (with $a_1 = \cdots = a_{r-1} = 1$) to the collection of $r-1$ sets, we cannot have (B2′) by the bound on $\nu_{r-1}(H)$, so we must have (B1′) which gives us a vertex in $V_r \setminus \{u_r\}$ with degree at least $(r-1)((\frac{n}{r} + \nu) - \nu) = \frac{r-1}{r}n$. □

Now we prove that if $r \geq 3$ and H is an r-partite r-uniform hypergraph with n edges and $\nu_{r-1}(H) \leq \frac{n}{3^{\binom{r+1}{2}+r}}$, then $\Delta(H) \geq \frac{(r-1)n}{r} - \binom{r}{2}\nu_{r-1}(H)$.

Proof of Theorem 4.4. Set $\nu := \nu_{r-1}(H)$. By Lemma 4.8 (with $\Delta = 3^{\binom{r+1}{2}}$), we have $\Delta(H) \geq 3^{\binom{r+1}{2}}\nu + 1$. Now by Lemma 4.9 we are done, or we have an edge of multiplicity at least $\nu + 1$ in which case we are done by Proposition 4.16. □

5. Conclusion

We were able to solve Problem 1.2 in all cases corresponding to Theorem 1.1 except when $k = 2$ and $r \geq 4$. However because of Lemma 4.8 and Lemma 4.9, in order to solve the case $k = 2$ and $r \geq 4$ it suffices to prove Conjecture 4.12. It would be very interesting to prove Conjecture 4.12 even in the case $k = 4$.

Another possible direction for further study involves replacing large monochromatic components with long monochromatic paths. Letzter [16] showed that in every 2-colouring of $G(n, p)$ with $p = \frac{\omega(1)}{n}$, there is w.h.p., a monochromatic cycle (path) of order at least $(2/3 - o(1))n$. Bennett, DeBiasio, Dudek, and English [5] generalised this result showing that if $p = \frac{\omega(1)}{n^{k-1}}$, then a.a.s. there is a monochromatic loose-cycle (loose-path) of order at least $(\frac{2k-2}{2k-1} - o(1))n$ in every 2-colouring of $H^k(n, p)$. Both of those results use sparse regularity and implicitly only use the fact $\alpha_k(H) = o(n)$, so we can retroactively rephrase their result as follows.

Theorem 5.1 (Bennett, DeBiasio, Dudek, and English [5]). *If H is a k-uniform hypergraph on n vertices with $\alpha_k(H) = o(n)$, then in every 2-colouring of the edges of H, there exists a monochromatic loose-cycle (loose-path) of order at least $(\frac{2k-2}{2k-1} - o(1))n$.*

The idea is that it would be nice to extend the above theorem to hold when $\alpha_k(G)$ can be considerably larger (especially in the case $k = 2$).

There are two results in the literature which implicitly broach this subject. Balogh, Barát, Gerbner, Gyárfás, Sárközy [4] proved that in every 2-colouring of the edges of a graph G on n vertices there exist two vertex disjoint monochromatic paths covering at least $n - 1000(50\alpha_2(G))^{\alpha_2(G)}$ vertices. Letzter [16] implicitly proved that in every 2-colouring of every graph G on n vertices there is a monochromatic path of order at least $\frac{n}{2} - 2\alpha_2(G)$.

So a particular case of the general problem we are interested in is the following.

Problem 5.2. *Given n sufficiently large, determine the largest value of α such that if G is a graph on n vertices with $\alpha_2(G) \leq \alpha$, then in every 2-colouring of G there is a monochromatic path of order greater than $n/2$.*

Finally, we mention that the best upper bounds on the size-Ramsey number of a path come from random d-regular graphs $G(n, d)$ (see [7]). An upper bound on $\mathrm{mc}_2(G(n, d))$ would give an upper bound on the longest monochromatic path. However, determining an upper bound on the largest monochromatic component in an arbitrary 2-colouring of $G(n, d)$ for small d falls outside

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the purview of this paper (partly since α_2 can be large in this case). So we raise the following problem.

Problem 5.3. *Determine bounds on* $\mathrm{mc}_2(G(n,d))$ *for* $d \geq 5$. *More generally, determine bounds on* $\mathrm{mc}_r(G(n,d))$ *for* $d \geq 2r+1$.

Note that a result of Anastos and Bal [1] implies that $\mathrm{mc}_r(G(n,d)) = o(n)$ when $d \leq 2r$.

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