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A note on spaces of almost periodic functions with values in Banach spaces

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Abstract. In this paper, we consider an equivalence relation on the space $AP(\mathbb{R}, X)$ of almost periodic functions with values in a prefixed Banach space X. In this context, it is known that the normality or Bochner-type property, which characterizes these functions, is based on the relative compactness of the family of translates. Now, we prove that every equivalence class is sequentially compact and the family of translates of a function belonging to this subspace is dense in its own class, i.e., the condition of almost periodicity of a function $f \in AP(\mathbb{R}, X)$ yields that every sequence of translates of f has a subsequence that converges to a function equivalent to f. This extends previous work by the same authors on the case of numerical almost periodic functions.

1 Introduction

The definition of an almost periodic function given by Bohr in his pioneering work [6] is based on two properly generalized concepts: the periodicity to the so-called almost periodicity, and the periodic distribution of periods to the so-called relative density of almost periods. Specifically, a continuous function $f : \mathbb{R} \to \mathbb{C}$ is almost periodic if for every $\varepsilon > 0$, there corresponds a number $l = l(\varepsilon) > 0$ such that each interval of length l contains a number τ satisfying $|f(t+\tau) - f(t)| < \varepsilon$ for all t. In particular, this notion yields that f is bounded and uniformly continuous. We will denote as $AP(\mathbb{R}, \mathbb{C})$ the space of almost periodic functions in the sense of this definition (Bohr's condition). It is noteworthy that almost periodic functions occur frequently and they are more often encountered in the study of various phenomena than the rather special periodic ones. For example, note that the sum of the two periodic functions of a real variable $t \mapsto e^{it}$ and $t \mapsto e^{i\sqrt{2}t}$ is not periodic, but it is almost periodic.

In the course of time, Bohr's work was studied by several mathematicians, such as Amerio, Besicovitch, Bochner, Corduneanu, Favard, Fink, Levitan, Lusternik, Pontryagin, Prouse, Stepanov, and Von-Neumann, Weyl, who also contributed to develop and expand this theory completely. Moreover, several variants and extensions of Bohr's concept were introduced, most notably by Besicovitch, Stepanov, and Weyl. In this context, we can cite, among others, the papers [1–3, 5, 7, 8, 10–15, 21, 22] and the references therein.

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Without a doubt, almost periodic functions have played an important role in various branches of mathematics since their introduction by Bohr during the 1920s. Indeed, as the literature about this topic shows, this theory has many important applications in problems of ordinary and partial differential equations, dynamical systems, stability theory, nonlinear oscillations theory, and so on, which have at the same time a wide scope of applications in various fields of science and technology. Moreover, the theory of almost periodic functions opened a way to study a wide class of trigonometric series of the general type and even exponential series.

In this paper, we focus our attention on the case when the values taken by such functions belong to a Banach space, which were defined and studied by Bochner in 1933 [4]. Let $(X, \|\cdot\|)$ be an arbitrary Banach space over $\mathbb{R}$ or $\mathbb{C}$, and we shall briefly present the basic properties of $AP(\mathbb{R}, X)$, which is defined, in terms of Bohr's property, as the set of continuous functions $f : \mathbb{R} \to X$ such that, fixed $\varepsilon > 0$, there corresponds a relatively dense set $\{\tau_j\}$ of real numbers satisfying $\|f(t+\tau_j) - f(t)\| < \varepsilon$ for $t \in \mathbb{R}$ and $\tau_j \in \{\tau_j\}$. This theory is, in its essential lines, similar to the theory of numerical almost periodic functions. It is plain that all functions f in $AP(\mathbb{R}, X)$ are bounded and uniformly continuous. Moreover, $AP(\mathbb{R}, X)$ is a Banach space equipped with the uniform convergence norm which, by abuse of language, is denoted as $\|f\| := \sup\{\|f(t)\| : t \in \mathbb{R}\}$.

On the one hand, as in the case of $AP(\mathbb{R}, \mathbb{C})$, Bohr's definition of almost periodicity of a function $f \in AP(\mathbb{R}, X)$ is also equivalent in this case to the property of relative compactness, called normality or Bochner-type property, for the family $\{f(t+h) : h \in \mathbb{R}\}$ of translates of f [10, Section 3.5].

On the other hand, another very important result of this theory is the approximation theorem according to which the class of almost periodic functions $AP(\mathbb{R}, X)$ coincides with the class of limit functions of uniformly convergent sequences of trigonometric polynomials of the type

$$a_1 e^{i\lambda_1 t} + \cdots + a_n e^{i\lambda_n t}, \tag{1.1}$$

with arbitrary real exponents λ_j and arbitrary coefficients $a_j \in X$ [10, Sections 3.5 and 4.5]. These approximating exponential polynomials can be found by Bochner–Fejér's summation (see, in this regard, [3, Chapter 1, Section 9], [10, Section 4.5], or [1, Chapter 2, Section 3]). In fact, if a function $f(t)$ belongs to $AP(\mathbb{R}, X)$, then there exists a (Bochner–Fejér's) sequence $P_k^f(t) = \sum_{m=1}^n r_{m,k} a_m e^{i\lambda_m t}$ of trigonometric polynomials of type (1.1) which satisfies the condition $\|f(t) - P_k^f(t)\| \to 0$ as $k \to \infty$, where the rational numbers $r_{m,k}$ depend on m and k, but not on a_m, and $r_{m,k} \to 1$ as $k \to \infty$.

Moreover, for any function $f \in AP(\mathbb{R}, X)$, the mean value

$$M(f(t)) = \lim_{l \to \infty} \frac{1}{l} \int_a^{a+l} f(t)\, dt$$

exists uniformly with respect to $a \in \mathbb{R}$, and, at most, a countable set of values of $\lambda_k \in \mathbb{R}$ such that $a_k = a(f, \lambda_k) = M(f(t)e^{-\lambda_k t}) \in X$ is different from the null element in X [10, Sections 3.5 and 4.5]. In this way, the series $\sum_{k \geq 1} a_k e^{i\lambda_k t}$ is called the Fourier series of f [10, Section 4.5]. The elements a_k and λ_k are also called the Fourier coefficients

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and exponents of f, respectively. It is worth noting that if two functions $f_1(t)$ and $f_2(t)$ in $AP(\mathbb{R}, X)$ have identical Fourier series, then they are equal (see [1, p. 25] or [10, Section 4.5]).

In this paper, we consider an equivalence relation on the functions with values in a Banach space X which can be represented with a Fourier-like series (see the comments before Definition 3.2), and which satisfies the important property consisting of that an equivalence class is completely contained in the space $AP(\mathbb{R}, X)$ when at least one of its functions is almost periodic (see Corollary 3.3). In this way, by analogy with the recent developments which we made for the space $AP(\mathbb{R}, \mathbb{C})$, the Besicovitch almost periodic functions, and other spaces of generalized almost periodic functions (see [16, 17, 19], respectively), this equivalence relation leads to refine the Bochner-type property or normality in the sense that the condition of almost periodicity in $AP(\mathbb{R}, X)$ implies that every sequence of translates has a subsequence that converges, with respect to the topology of $AP(\mathbb{R}, X)$, to an equivalent function (see Theorem 3.7 and Corollary 3.8). This extends our previous work on the case of numerical almost periodic functions (see [16, 17], but also [18, 20]). Moreover, we point out that the proof given here of the main result, and specifically that of Lemma 3.2, is different from those of previous papers.

2 Preliminaries

Let $(X, \|\cdot\|)$ be an arbitrary Banach space over $\mathbb{C}$. We shall refer to the expressions of the type

$$a_1 e^{i\lambda_1 p} + \cdots + a_j e^{i\lambda_j p} + \cdots$$

as exponential sums, where the λ_j's are real numbers, the a_j's are in X, and p is the variable. For our purposes, we next consider the following classes.

Definition 2.1 Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots\}$ be an arbitrary countable set of distinct real numbers, which we will call a set of exponents or frequencies. We will say that an exponential sum is in the class $\mathcal{S}_\Lambda^X$ if it is a formal series of type

$$\sum_{j\geq 1} a_j e^{i\lambda_j p},\ a_j \in X,\ \lambda_j \in \Lambda.$$

In the next section of this paper, we are going to consider some functions which are associated with a concrete subclass of these exponential sums, where the parameter p will be changed by $t \in \mathbb{R}$.

By analogy with [17, Definition 2], we next consider the following equivalence relation on the classes $\mathcal{S}_\Lambda^X$.

Definition 2.2 Given an arbitrary countable set $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots\}$ of distinct real numbers, consider $A_1(p)$ and $A_2(p)$ two exponential sums in the class $\mathcal{S}_\Lambda^X$, say $A_1(p) = \sum_{j\geq 1} a_j e^{i\lambda_j p}$ and $A_2(p) = \sum_{j\geq 1} b_j e^{i\lambda_j p}$. We will say that A_1 is *-equivalent to A_2 (in that case, we will write $A_1 \overset{*}{\sim} A_2$) if for each integer value $n \geq 1$, with $n \leq \sharp\Lambda$, there exists a $\mathbb{Q}$-linear map $\psi_n : V_n \to \mathbb{R}$, where V_n is the $\mathbb{Q}$-vector space generated by

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$\{\lambda_1, \lambda_2, \dots, \lambda_n\}$, such that

$$b_j = a_j e^{i\psi_n(\lambda_j)},\ j = 1, \dots, n.$$

As we next show, this equivalence relation can be characterized in terms of a basis for the set of exponents Λ. Let $G_\Lambda = \{g_1, g_2, \dots, g_k, \dots\}$ be a basis of the vector space over the rationals generated by a set Λ of exponents, which yields that G_Λ is linearly independent over the rational numbers and each λ_j is expressible as a finite linear combination of terms of G_Λ, say

$$\lambda_j = \sum_{k=1}^{q_j} r_{j,k} g_k, \text{ for some } r_{j,k} \in \mathbb{Q}. \tag{2.1}$$

By abuse of notation, we will say that G_Λ is a basis for Λ. Moreover, we will say that G_Λ is an integral basis for Λ when $r_{j,k} \in \mathbb{Z}$ for any j, k. For completeness, we will provide the proof of the following result that allows us to characterize the equivalence relation introduced in Definition 2.2.

Proposition 2.1 *Given $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_j, \dots\}$ a set of exponents, consider $A_1(p)$ and $A_2(p)$ two exponential sums in the class $\mathcal{S}_\Lambda^X$, say $A_1(p) = \sum_{j\geq 1} a_j e^{i\lambda_j p}$ and $A_2(p) = \sum_{j\geq 1} b_j e^{i\lambda_j p}$. Fixed a basis G_Λ for Λ, for each $j \geq 1$, let $\mathbf{r}_j$ be the vector of rational components satisfying (2.1). Then $A_1 \overset{*}{\sim} A_2$ if and only if for each integer value $n \geq 1$, with $n \leq \sharp\Lambda$, there exists a vector $\mathbf{x}_n = (x_{n,1}, x_{n,2}, \dots, x_{n,k}, \dots) \in \mathbb{R}^{\sharp G_\Lambda}$ such that $b_j = a_j e^{<\mathbf{r}_j, \mathbf{x}_n>i}$ for $j = 1, 2, \dots, n$. Furthermore, if G_Λ is an integral basis for Λ, then $A_1 \overset{*}{\sim} A_2$ if and only if there exists $\mathbf{x}_0 = (x_{0,1}, x_{0,2}, \dots, x_{0,k}, \dots) \in \mathbb{R}^{\sharp G_\Lambda}$ such that $b_j = a_j e^{<\mathbf{r}_j, \mathbf{x}_0>i}$ for every $j \geq 1$.*

Proof For each integer value $n \geq 1$, let V_n be the $\mathbb{Q}$-vector space generated by $\{\lambda_1, \dots, \lambda_n\}$, V the $\mathbb{Q}$-vector space generated by Λ, and $G_\Lambda = \{g_1, g_2, \dots, g_k, \dots\}$ a basis of V. If $A_1 \overset{*}{\sim} A_2$ and n is a positive integer number with $n \leq \sharp\Lambda$, by Definition 2.2, there exists a $\mathbb{Q}$-linear map $\psi_n : V_n \to \mathbb{R}$ such that $b_j = a_j e^{i\psi_n(\lambda_j)}$, $j = 1, 2 \dots, n$. Hence, $b_j = a_j e^{i\sum_{k=1}^{i_j} r_{j,k}\psi_n(g_k)}$, $j = 1, 2, \dots, n$, or, equivalently, $b_j = a_j e^{i<\mathbf{r}_j, \mathbf{x}_n>}$, $j = 1, 2, \dots, n$, with $\mathbf{x}_n := (\psi_n(g_1), \psi_n(g_2), \dots, \psi_n(g_p), 0, \dots)$, where $p = \max\{i_1, i_2, \dots, i_n\}$. Conversely, suppose the existence, for each integer value $n \geq 1$, of a vector $\mathbf{x}_n = (x_{n,1}, x_{n,2}, \dots, x_{n,k}, \dots) \in \mathbb{R}^{\sharp G_\Lambda}$ such that $b_j = a_j e^{<\mathbf{r}_j, \mathbf{x}_n>i}$, $j = 1, 2, \dots, n$. Thus, a $\mathbb{Q}$-linear map $\psi_n : V_n \to \mathbb{R}$ can be defined from $\psi_n(g_k) := x_{n,k}$, $k \geq 1$. Therefore, $\psi_n(\lambda_j) = \sum_{k=1}^{i_j} r_{j,k}\psi(g_k) =< \mathbf{r}_j, \mathbf{x}_n >$, $j = 1, 2, \dots, n$, and the result follows.

Now, suppose that G_Λ is an integral basis for Λ and $A_1 \overset{*}{\sim} A_2$. By above, for each given integer value $n \geq 1$, let $\mathbf{x}_n = (x_{n,1}, x_{n,2}, \dots) \in \mathbb{R}^{\sharp G_\Lambda}$ be a vector such that $b_j = a_j e^{i<\mathbf{r}_j, \mathbf{x}_n>}$, $j = 1, 2, \dots, n$. Since each component of $\mathbf{r}_j$ is an integer number, without loss of generality, we can take $\mathbf{x}_n \in [0, 2\pi)^{\sharp G_\Lambda}$ as the unique vector in $[0, 2\pi)^{\sharp G_\Lambda}$ satisfying the above equalities, where we assume $x_{n,k} = 0$ for any k such that $r_{j,k} = 0$, for $j = 1, \dots, n$. Therefore, under this assumption, if $m > n$, then $x_{m,k} = x_{n,k}$ for any k, so that $x_{n,k} \neq 0$. In this way, we can construct a vector $\mathbf{x}_0 = (x_{0,1}, x_{0,2}, \dots, x_{0,k}, \dots) \in [0, 2\pi)^{\sharp G_\Lambda}$ such that $b_j = a_j e^{<\mathbf{r}_j, \mathbf{x}_0>i}$, for every $j \geq 1$. Indeed, if $r_{1,k} \neq 0$, then the

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component $x_{0,k}$ is chosen as $x_{1,k}$, and if $r_{1,k} = 0$, then each component $x_{0,k}$ is defined as $x_{n+1,k}$, where $r_{j,k} = 0$, for $j = 1, \ldots, n$, and $r_{n+1,k} \neq 0$. Conversely, if there exists $\mathbf{x}_0 = (x_{0,1}, x_{0,2}, \ldots, x_{0,k}, \ldots) \in \mathbb{R}^{\sharp G_\Lambda}$ such that $b_j = a_j e^{<\mathbf{r}_j, \mathbf{x}_0> i}$, for every $j \geq 1$, then it is clear that $A_1 \overset{*}{\sim} A_2$ under Definition 2.2. ∎

3 Main results

Depending on the set of Fourier exponents, we next consider the following classes of almost periodic functions in $AP(\mathbb{R}, X)$.

Definition 3.1 Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots\}$ be an arbitrary countable set of distinct real numbers. We will say that a function $f : \mathbb{R} \to X$ is in the class $\mathcal{F}_\Lambda^X$ if it is an almost periodic function in $AP(\mathbb{R}, X)$ whose associated Fourier series is of the form

$$\sum_{j \geq 1} a_j e^{i\lambda_j t}, \; a_j \in X, \; \lambda_j \in \Lambda. \tag{3.1}$$

In the particular case that $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ is finite, the functions in $\mathcal{F}_\Lambda^X$ are finite exponential sums of the form

$$a_1 e^{i\lambda_1 t} + \cdots + a_n e^{i\lambda_n t}, \; a_j \in X, \; \lambda_j \in \Lambda, \; j = 1, \ldots, n.$$

In terms of Definition 2.2, we next define an equivalence relation on a certain space of functions with values in a Banach space X which can be represented by exponential sums of the form (3.1), and in particular on the classes $\mathcal{F}_\Lambda^X$. More specifically, the next equivalence relation is defined on the set, say $\mathcal{A}(\mathbb{R}, X)$, of functions f from $\mathbb{R}$ to X for which there exists the mean value

$$M\{f(t)\} := \lim_{l \to \infty} \frac{1}{2l} \int_{-l}^{l} f(t) dt \in X,$$

$f(t)e^{-i\lambda t} \in \mathcal{A}(\mathbb{R}, X)$ for any $\lambda \in \mathbb{R}$, and there always exists at most a countable infinite set of real values λ_k for which $a(f, \lambda_k) := M\{f(t)e^{-it\lambda_k}\} \neq 0$. In this way, we can associate to $f \in \mathcal{A}(\mathbb{R}, X)$ a unique exponential sum $\sum_{\lambda_k \in \Lambda} a(f, \lambda_k) e^{i\lambda_k t}$, where $\Lambda = \{\lambda_k \in \mathbb{R} : a(f, \lambda_k) \neq 0\}$, which is of the form (3.1) and which we will call its Fourier series. It is clear that $AP(\mathbb{R}, X) \subset \mathcal{A}(\mathbb{R}, X)$, where X stands for a Banach space, satisfies this property (diverse types of almost periodic functions with values in Banach spaces which can be represented by its Fourier-like series can be seen in [9]). Furthermore, if $X = \mathbb{C}$, every function in the Besicovitch space $B(\mathbb{R}, \mathbb{C}) \supset AP(\mathbb{R}, \mathbb{C})$ satisfies this property (see [10, Section 3.4]).

Definition 3.2 Given $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots\}$ a set of exponents, let f_1 and f_2 denote two functions in $\mathcal{A}(\mathbb{R}, X)$ which are, respectively, associated with exponential sums of the form

$$\sum_{j \geq 1} a_j e^{i\lambda_j t} \text{ and } \sum_{j \geq 1} b_j e^{i\lambda_j t}, \; a_j, b_j \in X, \; \lambda_j \in \Lambda.$$

We will say that f_1 is *-equivalent to f_2 if for each integer value $n \geq 1$, with $n \leq \sharp\Lambda$, there exists a $\mathbb{Q}$-linear map $\psi_n : V_n \to \mathbb{R}$, where V_n is the $\mathbb{Q}$-vector space generated by

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$\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, such that

$$b_j = a_j e^{i\psi_n(\lambda_j)}, \ j = 1, \ldots, n.$$

In that case, we will write $f_1 \overset{*}{\sim} f_2$.

Recall that if two functions $f_1(t)$ and $f_2(t)$ in $AP(\mathbb{R}, X)$ have identical Fourier series, then they are equal. As we can see, restricted to actual series of almost periodic functions, the *-equivalence of formal series (Definition 2.2) reduces to that on the functions in the classes $\mathcal{F}_\Lambda^X$. That is why we use the same notation for them.

In the context of finite exponential sums (when Λ is finite), the following proposition is useful to get the subsequent results in this paper. By analogy with the case of $AP(\mathbb{R}, \mathbb{C})$ with the topology of the uniform convergence, its proof is based on the fact that the functions in $\mathcal{F}_\Lambda^X$ are also equipped with the uniform convergence norm. Hence, the proof of the next result is analogous to that of [16, Theorem 1].

Proposition 3.1 *Given $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ a finite set of exponents, let $f_1(t) = \sum_{j=1}^n a_j e^{i\lambda_j t}$ and $f_2(t) = \sum_{j=1}^n b_j e^{i\lambda_j t}$ be two *-equivalent functions in the class $\mathcal{F}_\Lambda^X$. Fixed $\varepsilon > 0$, there exists a relatively dense set of real numbers $\{\tau\}$ such that*

$$\|f_1(t+\tau) - f_2(t)\| < \varepsilon \ \forall \tau \in \{\tau\}.$$

We next prove the following result which is essential in our subsequent development. Given an arbitrary set Λ of exponents and two exponential sums $A_1(t), A_2(t) \in \mathcal{S}_\Lambda^X$, suppose that $A_1(t)$ is associated with an almost periodic function in $AP(\mathbb{R}, X)$ and $A_1(t) \overset{*}{\sim} A_2(t)$ (under Definition 2.2), then is the exponential sum $A_2(t)$ associated with an almost periodic function in $AP(\mathbb{R}, X)$? The next lemma answers affirmatively this question.

Lemma 3.2 *Let $f_1(t) \in AP(\mathbb{R}, X)$ be an almost periodic function whose Fourier series is given by $\sum_{j\geq 1} a_j e^{i\lambda_j t}$, $a_j \in X$, where $\{\lambda_1, \ldots, \lambda_j, \ldots\}$ is a set of distinct exponents. Consider $b_j \in X$ such that $\sum_{j\geq 1} b_j e^{i\lambda_j t}$ and $\sum_{j\geq 1} a_j e^{i\lambda_j t}$ are *-equivalent. Then $\sum_{j\geq 1} b_j e^{i\lambda_j t}$ is the Fourier series associated with an almost periodic function $f_2(t) \in AP(\mathbb{R}, X)$ satisfying $f_1 \overset{*}{\sim} f_2$.*

Proof Take $\Lambda = \{\lambda_1, \ldots, \lambda_j, \ldots\}$. By hypothesis, $f_1 \in \mathcal{F}_\Lambda^X \subset AP(\mathbb{R}, X)$ is determined by a series of the form $\sum_{j\geq 1} a_j e^{i\lambda_j t}$, $a_j \in X$, $\lambda_j \in \Lambda$. In virtue from [1, p. 29] or [10, Section 4.5], let $P_k^{f_1}(t) = \sum_{j\geq 1} p_{j,k} a_j e^{i\lambda_j t}$, $k = 1, 2, \ldots$, be the Bochner–Fejér trigonometric polynomials which converge to f_1 with respect to the topology of $AP(\mathbb{R}, X)$ (and converge formally to its Fourier series on $\mathbb{R}$). Now, take $\tau \in \mathbb{R}$, then $P_k^{f_{1,\tau}}(t) = \sum_{j\geq 1} p_{j,k} a_j e^{i\lambda_j (t+\tau)}$, $k = 1, 2, \ldots$, are the Bochner–Fejér trigonometric polynomials which converge to $f_{1,\tau}(t) := f_1(t+\tau)$, $t \in \mathbb{R}$. On the other hand, consider the finite exponential sums $\sum_{j\geq 1} p_{j,k} b_j e^{i\lambda_j t}$, with $t \in \mathbb{R}$ and $k = 1, 2, \ldots$. Take a positive sequence $\{\varepsilon_l\}_{l\geq 1}$ tending to 0. By Proposition 3.1, given $l \in \{1, 2, \ldots\}$, there exists a

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relatively dense set of real numbers τ_l such that

$$\left\| \sum_{j\geq 1} p_{j,k} a_j e^{i\lambda_j (t+\tau_l)} - \sum_{j\geq 1} p_{j,k} b_j e^{i\lambda_j t} \right\| < \frac{\varepsilon_l}{3} \quad \forall t \in \mathbb{R}\ \forall \tau_l.$$

Furthermore, by the Bochner–Féjer property, for each $l = 1, 2, \ldots$, there exists $k_l \in \mathbb{N}$ such that

$$\left\| P_k^{f_1,\tau_l}(t) - f_1(t+\tau_l) \right\| = \left\| \sum_{j\geq 1} p_{j,k} a_j e^{i\lambda_j (t+\tau_l)} - f_1(t+\tau_l) \right\| \leq \frac{\varepsilon_l}{3}, \quad \text{for each } k > k_l.$$

Finally, by the property of normality applied to the sequence $\{f_1(t+\tau_l)\}_{\tau_l}$, we can extract a subsequence $\{f_1(t+\tau_{l_m})\}_{\tau_{l_m}}$ which converges to a certain function f_2 with respect to the topology of $AP(\mathbb{R}, X)$. In this way, we assure the existence of $m_l \in \mathbb{N}$ such that

$$\| f_2(t) - f_1(t+\tau_{l_m}) \| \leq \frac{\varepsilon}{3}, \quad \text{for each } m > m_l.$$

Now, given $l \geq 1$, let $k > k_l$ and $m > m_l$. We deduce from above that

$$\begin{aligned} \left\| \sum_{j\geq 1} p_{j,k} b_j e^{i\lambda_j t} - f_2(t) \right\| &\leq \left\| \sum_{j\geq 1} p_{j,k} b_j e^{i\lambda_j t} - \sum_{j\geq 1} p_{j,k} a_j e^{i\lambda_j (t+\tau_{l_m})} \right\| \\ &\quad + \left\| \sum_{j\geq 1} p_{j,k} a_j e^{i\lambda_j (t+\tau_{l_m})} - f_1(t+\tau_{l_m}) \right\| + \| f_1(t+\tau_{l_m}) - f_2(t) \| \leq \varepsilon_l. \end{aligned}$$

Making l tend to infinity, we deduce that $\sum_{j\geq 1} p_{j,k} b_j e^{i\lambda_j t}$, $k = 1, 2, \ldots$, converges to f_2 with respect to the topology of $AP(\mathbb{R}, X)$. Likewise, since $AP(\mathbb{R}, X)$ is the closure of the trigonometric polynomials with respect to the uniform convergence norm, we get that $f_2 \in AP(\mathbb{R}, X)$. Finally, by Definition 3.2, it is clear that $f_1 \overset{*}{\sim} f_2$. ∎

The reader can note that the proof of the previous result given here is different from those of [16, Lemma 2] and [17, Lemma 1].

As an immediate consequence of Lemma 3.2, the following result justifies the fact that an equivalence class (under Definition 3.2) is completely contained in $\mathcal{F}_\Lambda^X / \overset{*}{\sim}$ when one of its functions is in $\mathcal{F}_\Lambda^X$.

Corollary 3.3 *Let Λ be a set of exponents and $\mathcal{G}$ an equivalence class of functions in $\mathcal{A}(\mathbb{R}, X)$ (under Definition 3.2). If $f \in \mathcal{G} \cap \mathcal{F}_\Lambda^X$, then $\mathcal{G} \subset \mathcal{F}_\Lambda^X / \overset{*}{\sim}$.*

As it can be seen below, the statements of many of the subsequent results include the condition that $\mathcal{G}$ is an equivalence class in $\mathcal{F}_\Lambda^X / \overset{*}{\sim}$. It is worth noting that Corollary 3.3 assures that this condition is satisfied when one of the functions in $\mathcal{G}$ belongs to $AP(\mathbb{R}, X)$.

We next prove that if $f_1(t)$ and $f_2(t)$ are two *-equivalent almost periodic functions in some $\mathcal{F}_\Lambda^X$, then $f_2(t)$ can be approximated by translates $\{f_1(t+\tau) : \tau \in \mathbb{R}\}$ of $f_1(t)$. This generalizes Proposition 3.1 for the case of a (nonnecessarily finite) arbitrary set of exponents. The main ingredient in order to prove this result for the

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case of functions with an infinite Fourier expansion is the Bochner–Fejér's summation method, which was mentioned in the introduction of this article (and already used in Lemma 3.2).

Theorem 3.4 *Let Λ be a set of exponents, $\mathcal{G}$ an equivalence class in $\mathcal{F}_\Lambda^X/\overset{*}{\sim}$, and $f_1, f_2 \in \mathcal{G}$. Fixed $\varepsilon > 0$, there exists a relatively dense set of real numbers $\{\tau\}$ such that*

$$\|f_1(t+\tau) - f_2(t)\| \leq \varepsilon \ \forall \tau \in \{\tau\}.$$

Proof Let $f_1(t), f_2(t) \in AP(\mathbb{R}, X)$ be two *-equivalent almost periodic functions whose Fourier series are given by $\sum_{j\geq 1} a_j e^{i\lambda_j t}$, $a_j \in X$, and $\sum_{j\geq 1} b_j e^{i\lambda_j t}$, $b_j \in X$, respectively. Likewise, consider $P_N^{f_1}(t) = \sum_{m=1}^N r_{m,N} a_m e^{i\lambda_m t}$ and $P_N^{f_2}(t) = \sum_{m=1}^N r_{m,N} b_m e^{i\lambda_m t}$ the (Bochner–Fejér's) sequences of trigonometric polynomials associated with $f_1(t)$ and $f_2(t)$, respectively, where the rational numbers $r_{m,N}$ depend on m and N, but not on a_m and b_m, and $r_{m,k} \to 1$ as $k \to \infty$. In this way, fixed $\varepsilon > 0$, there exists N_ε which satisfies the condition

$$\|f_1(t) - P_{N_\varepsilon}^{f_1}(t)\| < \varepsilon/3 \quad \text{and} \quad \|f_2(t) - P_{N_\varepsilon}^{f_2}(t)\| < \varepsilon/3. \tag{3.2}$$

Furthermore, since $P_{N_\varepsilon}^{f_1}(t)$ and $P_{N_\varepsilon}^{f_2}(t)$ are *-equivalent (because $f_1, f_2 \in \mathcal{G}$) by Proposition 3.1, there exists a relatively dense set of real numbers $\{\tau\}$ such that

$$\|P_{N_\varepsilon}^{f_1}(t+\tau) - P_{N_\varepsilon}^{f_2}(t)\| < \varepsilon/3 \ \forall \tau \in \{\tau\}. \tag{3.3}$$

Consequently, from (3.2) and (3.3), we conclude for any $\tau \in \{\tau\}$ that

$$\|f_1(t+\tau) - f_2(t)\| \leq \|f_1 - P_{N_\varepsilon}^{f_1}\| + \|P_{N_\varepsilon}^{f_1}(t+\tau) - P_{N_\varepsilon}^{f_2}(t)\| + \|P_{N_\varepsilon}^{f_2} - f_2\| < \varepsilon.$$

That is, $f_2(t)$ can be approximated by translates of $f_1(t)$. ∎

What is more, by taking Lemma 3.2 into account, Theorem 3.4 can be more generally stated as follows.

Corollary 3.5 *Let $\Lambda = \{\lambda_1, \ldots, \lambda_j, \ldots\}$ be a set of exponents and $f_1 \in \mathcal{F}_\Lambda^X$ whose Fourier series is given by $\sum_{j\geq 1} a_j e^{i\lambda_j t}$, $a_j \in X$. Consider $b_j \in X$ such that $\sum_{j\geq 1} b_j e^{i\lambda_j t}$ and $\sum_{j\geq 1} a_j e^{i\lambda_j t}$ are *-equivalent. Fixed $\varepsilon > 0$, there exists a relatively dense set of real numbers $\{\tau\}$ such that*

$$\|f_1(t+\tau) - f_2(t)\| \leq \varepsilon \ \forall \tau \in \{\tau\},$$

where $f_2 \in \mathcal{F}_\Lambda^X$ is an almost periodic function whose Fourier series is given by $\sum_{j\geq 1} b_j e^{i\lambda_j t}$.

The following result, which extends [16, Proposition 3], concerns the concept of convergence in $AP(\mathbb{R}, X)$. With respect to the topology of $AP(\mathbb{R}, X)$, it is satisfied that the equivalence classes of $\mathcal{F}_\Lambda^X/\overset{*}{\sim}$ are closed. In fact, more specifically, they are sequentially compact. The proof is based on Lemma 3.2, and it is analogous to that of [16, Proposition 3] (where the last statement is deduced from Helly's selection principle).

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Proposition 3.6 *Let Λ be a set of exponents and $\mathcal{G}$ an equivalence class in $\mathcal{F}_\Lambda^X/\overset{*}{\sim}$. Then $\mathcal{G}$ is sequentially compact.*

Regarded as members of a metric space, sequential compactness is the same as compactness (in the topology induced by the metric), and it implies being closed. So, as a consequence of Proposition 3.6 and with respect to the topology of $AP(\mathbb{R}, X)$, we state that the family of translates of a function $f \in \mathcal{F}_\Lambda^X$ is closed on its equivalence class of $\mathcal{F}_\Lambda^X/\overset{*}{\sim}$. In fact, this result can be improved in the sense that, fixed a function $f \in \mathcal{F}_\Lambda^X$, the limit points of the set of the translates $\mathcal{T}_f = \{f(t+\tau) : \tau \in \mathbb{R}\}$ of f are precisely the almost periodic functions which are *-equivalent to *f*. The proof is rather similar to that of [16, Theorem 2].

Theorem 3.7 *Let Λ be a set of exponents, $\mathcal{G}$ an equivalence class in $\mathcal{F}_\Lambda^X/\overset{*}{\sim}$, and $f \in \mathcal{G}$. Then the set of functions $\mathcal{T}_f = \{f_\tau(t) := f(t+\tau) : \tau \in \mathbb{R}\}$ is dense in $\mathcal{G}$.*

We have seen in the introduction that the functions in $AP(\mathbb{R}, X)$ satisfy the Bochner-type property consisting of the relative compactness of the set $\{f(t+\tau)\}$, $\tau \in \mathbb{R}$, associated with an arbitrary function $f \in AP(\mathbb{R}, X)$. As an important consequence of Theorem 3.7, we have refined this property in the sense that we show that the condition of almost periodicity of a function $f(t) \in AP(\mathbb{R}, X)$ yields that every sequence $\{f(t+\tau_n)\}$, $\tau_n \in \mathbb{R}$, of translates of *f* has a subsequence that converges with the topology of $AP(\mathbb{R}, X)$ to a function which is *-equivalent to *f*.

Corollary 3.8 *Let $f \in \mathcal{A}(\mathbb{R}, X)$. Then f is in $AP(\mathbb{R}, X)$ if and only if the closure of its set of translates is compact. Furthermore, in this case, this closure coincides with its equivalence class.*

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