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WEAKLY COMPACT, OPERATOR-VALUED DERIVATIONS OF TYPE I VON NEUMANN ALGEBRAS

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1. Introduction. In [**18**], the author initiated an investigation of compact, Banach-module-valued derivations of C^*-algebras. In collaboration with C. A. Akemann [**3**] and S.-K. Tsui [**16**], he determined the structure of all compact and weakly compact, A-valued derivations of a C^*-algebra A, and of all compact, $B(H)$-valued derivations of a C^*-subalgebra of $B(H)$, the algebra of bounded linear operators on a Hilbert space H. In this paper, we begin the study of weakly compact, $B(H)$-valued derivations of C^*-subalgebras of $B(H)$.

Let R be a C^*-subalgebra of $B(H)$, $\delta:R \rightarrow B(H)$ a weakly compact derivation, i.e., a weakly compact linear map which has

$$\delta(ab) = a\delta(b) + \delta(a)b \text{ for each } a, b \in R.$$

Since δ has a unique weakly compact extension to a derivation of the closure of R in the weak operator topology (WOT) on $B(H)$ (consult the proof of Theorem 3.1 of [**16**]), we may assume with no loss of generality that R is a von Neumann subalgebra of $B(H)$. In this paper, we determine in Lemma 4.1 and Theorems 4.3 and 4.10 the structure of δ when R is type I, using I. E. Segal's multiplicity theory [**14**] for type I von Neumann algebras and results of E. Christensen [**6**], [**7**] on $B(H)$-valued derivations of von Neumann algebras.

In [**10**], Johnson and Parrott investigated derivations of a von Neumann subalgebra of $B(H)$ with range contained in the closed, two-sided ideal $C(H)$ of compact operators in $B(H)$. Such derivations are weakly compact, and the results of [**10**] show that they have a particularly simple structure in most cases, one of which is the type I case. For this reason, the structure of von Neumann subalgebras of $B(H)$ all of whose weakly compact, $B(H)$-valued derivations have range in $C(H)$ are of interest to us, and we determine their structure in Theorem 3.8.

We now record some notation that will be useful later. If A is a C^*-algebra, S a subset of A, we denote by $\mathrm{Re}(S)$ and S_+ the set of all self-adjoint and positive elements of S, respectively. Let H be a Hilbert space, κ a cardinal number. We denote by $\kappa \cdot H$ the Hilbert-space direct

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sum of κ copies of H. If $T \in B(H)$ and $K = \kappa \cdot H$, $\kappa \cdot T$ denotes the κ-fold ampliation of T, i.e., the element of $B(K)$ defined by

$$\kappa \cdot T:(x_\alpha) \to (Tx_\alpha), \quad (x_\alpha) \in K.$$

If A is a subalgebra of $B(H)$, $\kappa \cdot A$ will denote the subalgebra of $B(K)$ consisting of all elements of the form $\kappa \cdot a$, $a \in A$. If X and Y are Banach spaces, then $B(X, Y)$ denotes the linear space of all bounded linear transformations of X into Y, and we set $B(X) = B(X, X)$. If $T \in B(X, Y)$, we define the mappings $l_T:B(X) \to B(X, Y)$ (resp., $r_T:B(Y) \to B(X, Y)$) by $l_T:A \to TA$, $A \in B(X)$ (resp., $r_T:A \to AT$, $A \in B(Y)$). If $T \in B(X)$, then $\mathrm{ad}_T:B(X) \to B(X)$ is the mapping

$$A \to TA - AT, \quad A \in B(X).$$

If $S \subseteq X$, we set

$$\mathrm{Ball}(S) = \{x \in S:\|x\| \leqq 1\},$$

and we denote the distance of $x \in X$ to S by $\mathrm{dist}(x, S)$. All subspaces of $B(H)$ are assumed to contain the identity operator I on H.

All of the basic results on weak compactness contained in Sections V.4, V.6 and VI.4 of **[9]** that are used in what follows will be invoked where needed without specific reference.

2. Generalities on weakly compact maps of von Neumann algebras. In this section, we present the main abstract technical lemmas on which the subsequent sections will be based.

2.1. PROPOSITION. *Let R be a von Neumann algebra, X a Banach space, $\varphi:R \to X^*$ a $\sigma(R, R_*) - \sigma(X^*, X)$ continuous linear map. Then the following are equivalent*:

(i) *For any sequence $\{p_n\}$ of pairwise orthogonal projections in R,*

$$\lim_n \|\varphi(p_n)\| = 0.$$

(ii) *φ is weakly compact.*

(iii) *For any decreasing sequence $\{p_n\}$ of projections in R with infimum zero,*

$$\lim_n \|\varphi(p_n)\| = 0.$$

Proof. (i) $\Rightarrow$ (ii). By the assumed continuity of φ, there is a bounded linear map $T:X \to R_*$ such that $\varphi = T^*$. It thus suffices to show that T is weakly compact. Let $C = T(\mathrm{Ball}(X))$, and let $\{p_n\}$ be a sequence of orthogonal projections in R. For each $x \in \mathrm{Ball}(X)$,

$$p_n(Tx) = (T^*p_n)(x) = \varphi(p_n)(x),$$

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and so

$$|p_n(Tx)| \leqq \|\varphi(p_n)\|.$$

Since

$$\lim_n \|\varphi(p_n)\| = 0,$$

we hence conclude that

$$\lim_n p_n(f) = 0 \text{ uniformly for } f \in C,$$

and so by Theorem II.2 of [**1**], C is weakly precompact in R_*.

(ii) $\Rightarrow$ (iii). Let $\{p_n\}$ be a decreasing sequence of projections in R with infimum zero. Since $T(\text{Ball}(X))$ is weakly precompact,

$$\lim_n \varphi(p_n)(x) = \lim_n p_n(Tx) = 0 \text{ uniformly for } x \in \text{Ball}(X)$$

by Corollary II.5 of [**1**], i.e.,

$$\lim_n \|\varphi(p_n)\| = 0.$$

(iii) $\Rightarrow$ (i). Let $\{p_n\}$ be a pairwise orthogonal sequence of projections in R. Set $p = \oplus_n p_n$. Then the series $\sum_n \varphi(p_n)$ converges in norm to $\varphi(p)$, and so

$$\lim_n \|\varphi(p_n)\| = 0.$$

2.2. COROLLARY. *Let M and N be von Neumann algebras, $\varphi:M \to N$ a σ-continuous linear map. Then* (i), (ii), *and* (iii) *of Proposition* 2.1 *are equivalent.*

Remark. A problem worthy of further investigation would be to generalize Proposition 2.1 by replacing X^* by X. An outstanding unsolved problem in the duality theory of von Neumann algebras is the characterization of the weakly compact subsets of the dual of a von Neumann algebra (see [**1**], [**2**], [**13**] and [**17**] for information and partial progress on this problem). The main conjecture in this area is the following: if R is a von Neumann algebra, then a bounded subset S of R^* is weakly precompact if and only if for each sequence $\{p_n\}$ of orthogonal projections in R,

$$\lim_n f(p_n) = 0 \text{ uniformly for } f \in S.$$

If this conjecture is true, the simple arguments in the proof of Proposition 2.1 would show that that proposition holds for any bounded linear map of R into a Banach space. This stronger form of Proposition 2.1 can be easily deduced for abelian von Neumann algebras from Theorems I. 1.13 and VI.

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1.1 of [**8**], and also for a direct sum of type I factors (with the help of Theorem IV. 2 of [**1**]), at least when φ satisfies certain natural continuity assumptions.

The next lemma is well known and easy to prove: we state it merely for convenience of reference.

2.3. LEMMA. *Let R be a von Neumann algebra, X a normed linear space, $\varphi:R \to X$ a bounded linear map. Set*

$$\|\varphi\|_+ = \sup\{ \|\varphi(a)\|:a \in \mathrm{Ball}(R)_+\},$$

$$\|\varphi\|_{++} = \sup\{ \|\varphi(p)\|:p \text{ a projection in } R\}.$$

Then $\|\varphi\|_{++} = \|\varphi\|_+ \leqq \|\varphi\| \leqq 4\|\varphi\|_+$.

2.4. LEMMA. *Let $\{H_\alpha:\alpha \in \mathscr{A}\}$ be a family of Hilbert spaces, let R_α be a von Neumann subalgebra of $B(H_\alpha)$ for $\alpha \in \mathscr{A}$, and set $H = \oplus_\alpha H_\alpha$, $R = \oplus_\alpha R_\alpha$. For each finite subset σ of $\mathscr{A}$, let*

$$R_\sigma = \oplus \{R_\alpha:\alpha \notin \sigma\}.$$

Suppose $\varphi:R \to B(H)$ is a $\sigma(R, R_*)$ − ultraweakly continuous linear map. Then φ is weakly compact if and only if

$$\lim_\sigma \|\varphi|_{R_\sigma}\| = 0 \quad \text{and}$$

$$\varphi|_{R_\alpha}:R_\alpha \to B(H) \text{ is weakly compact for each } \alpha \in \mathscr{A}.$$

Proof. ($\Leftarrow$). This is clear.

($\Rightarrow$). It is clear that $\varphi|_{R_\alpha}:R_\alpha \to B(H)$ is weakly compact for each $\alpha \in \mathscr{A}$. Suppose that

$$\lim_\sigma \|\varphi|_{R_\sigma}\| \neq 0.$$

Let P_α = projection of H onto H_α, $P_\sigma = \oplus \{P_\alpha:\alpha \in \sigma\}$ for each finite subset σ of $\mathscr{A}$. Then since

$$\varphi(a) = \text{ultraweak-}\lim_\sigma \varphi(aP_\sigma) \quad \text{for } a \in R$$

and each P_σ is central in R, we conclude from Lemma 2.3 that there is a sequence $\{p_n\}$ of orthogonal projections in R for which

$$\lim_n \|\varphi(p_n)\| \neq 0,$$

contradicting Corollary 2.2.

3. Algebras with the Johnson-Parrott property. Let R be a von Neumann subalgebra of $B(H)$. If $\delta:R \to B(H)$ is a derivation, then it is automatically $\sigma(R, R_*)$-ultraweakly continuous, and if $\delta(R) \subseteq C(H)$, then

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it is an easy consequence of the WOT-compactness of Ball(R) and the normal-singular decomposition of $B(H)^*$ that δ is weakly compact. In [**10**], Johnson and Parrott showed that in most cases, all derivations of R into $C(H)$ are generated by compact operators. It follows that the weakly compact, $B(H)$-valued derivations of R that have the simplest and most natural structure are the derivations of R with range in $C(H)$. A natural question which therefore arises in our investigations is the one which asks: which von Neumann subalgebras of $B(H)$ have all of their weakly compact, $B(H)$-valued derivations of this particularly simple form? The purpose of this section is to answer this question for all von Neumann subalgebras of $B(H)$ (for a precise statement of our answer, see Theorem 3.8).

We isolate the property of R that will concern us here with the following definition: we say that R has the *Johnson-Parrott property* (property JP) if the following condition holds: if $T \in B(H)$ and ad_T is weakly compact on R, then $\mathrm{ad}_T(R) \subseteq C(H)$. Thus R has property JP if all weakly compact, $B(H)$-valued derivations of R which are generated by elements of $B(H)$ have their range contained in $C(H)$.

A concept which plays a fundamental role in our analysis is that of a weakly compact multiplier. An operator $T \in B(H)$ is said to be a *left* (resp., *right*) *weakly compact multiplier of* R if l_T (resp., r_T) is weakly compact on R. By Proposition 2.2 of [**4**], $T \in B(H)$ is a weakly compact multiplier of $B(H)$ if and only if $T \in C(H)$, and we will call elements of $C(H)$ *trivial* weakly compact multipliers. In what follows, we will suppress the adjectives right and left, and assume that all weakly compact multipliers are left ones. Indeed, the two types can be interchanged by simply taking adjoints.

3.1. LEMMA. *Let H and K be Hilbert spaces, with S a subspace of $B(H)$ and $T \in B(H, K)$. Then l_T is weakly compact on S if and only if l_{T^*T} is weakly compact on S.*

Proof. The "only if" implication is clear, and the "if" implication is a straightforward consequence of the spectral theorem and polar decomposition of T.

In the lemmas which follow, R will denote a fixed von Neumann subalgebra of $B(H)$.

3.2. LEMMA. *Let $T \in B(H)$, and let $\lambda \to E(\lambda)$ denote the spectral measure of T^*T. Then T is a weakly compact multiplier of R if and only if $E([\delta, +\infty))$ is a weakly compact multiplier of R for each $\delta > 0$.*

Proof. ($\Leftarrow$). Since $E_\delta = E([\delta, +\infty))$ is a weakly compact multiplier of R, so is T^*TE_δ, for $\delta > 0$. But

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$$\|T^*T - T^*TE_\delta\| \leqq \delta,$$

and so T^*T, and hence T by Lemma 3.1, is a weakly compact multiplier of R.

($\Rightarrow$). Fix $\delta > 0$, and let $\{p_n\}$ be an orthogonal sequence of projections in R. Then

$$\lim_n \|T^*Tp_n\| = 0,$$

and straightforward estimates using the spectral theorem show that

$$\lim_n \|E_\delta p_n\| = 0.$$

Thus by Corollary 2.2, E_δ is a weakly compact multiplier of R.

The next lemma, the fundamental one of this section, provides the connection between property JP and weakly compact multipliers.

3.3. LEMMA. *R has no nontrivial weakly compact multipliers if and only if R has property* JP *and contains no minimal projections which are central of infinite rank.*

Proof. ($\Rightarrow$). It is clear that R contains no minimal projections which are central of infinite rank.

Suppose $T \in B(H)$ and ad_T is weakly compact on R. We must show that $\mathrm{ad}_T(R) \subseteq C(H)$, and since R is the norm-closed linear span of its projections, it suffices to show that $\mathrm{ad}_T(p) \in C(H)$ for each projection p of R. Let p be such a projection, and let $\{a_\alpha\}$ be a bounded net in R with

$$\lim_\alpha a_\alpha = 0(\mathrm{WOT}).$$

Since

$$a_\alpha(I - p)Tp = -(\mathrm{ad}_T(a_\alpha(I - p)))p \text{ for each } \alpha,$$

we conclude by weak compactness of ad_T on R that

$$a_\alpha(I - p)Tp \to 0 \text{ weakly},$$

after perhaps passing to a cofinal subnet and reindexing. It follows that $(I - p)Tp$ is a (right) weakly compact multiplier of R, and so

$$(I - p)Tp \in C(H).$$

Applying the same reasoning with p replaced by $I - p$, we conclude that

$$\mathrm{ad}_T(p) = (I - p)Tp - pT(I - p) \in C(H).$$

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($\Leftarrow$). Let $E \in B(H)$ be a nontrivial weakly compact multiplier of R, which we may assume is a projection of infinite rank by Lemma 3.2. Suppose R has property JP. We will find a minimal projection in R which is central of infinite rank.

Let $T \in B(H)$. Then ad_{ET^*TE} is weakly compact on R, and thus for each $V \in B(H)$ ad_{EVE} is weakly compact on R. Since R has property JP,

(3.1) $\quad \mathrm{ad}_{EVE}(R) \subseteq C(H), \quad$ for all $V \in B(H)$.

Let $a \in R$. It follows from (3.1) that

$$r_{Ea(I-E)}(B(E(H))) \subseteq C(H) \quad \text{and}$$

$$\mathrm{ad}_{EaE}(B(E(H))) \subseteq C(E(H)),$$

and thus by Lemma 3.2 of [**10**], we deduce that $Ea(I - E) \in C(H)$ and that there is a scalar $\lambda = \lambda(a)$ such that

$$EaE - \lambda(a) \cdot E \in C(H).$$

From the fact that E has infinite rank hence follows the existence of a self-adjoint linear functional λ on R such that

(3.2) $\quad Ea - \lambda(a) \cdot E \in C(H), \quad$ for all $a \in R$.

We assert that $\ker(\lambda)$ is a WOT-closed, two-sided ideal in R. If $a, b \in R$, then by (3.2),

$$Eab - \lambda(a) \cdot Eb \in C(H)$$

$$\lambda(a) \cdot Eb - \lambda(a)\lambda(b) \cdot E \in C(H),$$

whence

$$Eab - \lambda(a)\lambda(b) \cdot E \in C(H).$$

Since E has infinite rank, it follows that

$$\lambda(ab) = \lambda(a)\lambda(b),$$

and λ is hence multiplicative, and therefore bounded. Thus $\ker(\lambda)$ is a norm-closed, two-sided ideal in R. Let $a \in R$ be a WOT-limit point of $\ker(\lambda)$. We may choose a bounded net $\{a_\alpha\} \subseteq \ker(\lambda)$ with

$$\text{WOT-}\lim_{\alpha} a_\alpha = a.$$

Since E is a weakly compact multiplier of R, we may assume upon passage to a cofinal subnet if necessary that $Ea_\alpha \to Ea$ weakly in $B(H)$. Let f be a singular linear functional on $B(H)$ with $f(E) \neq 0$. By (3.2),

$$f(E(a_\alpha - a)) = \lambda(a_\alpha - a)f(E) \text{ for each } \alpha,$$

and so

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$$\lim_\alpha \lambda(a_\alpha - a) = \lim_\alpha f(E)^{-1} f(E(a_\alpha - a)) = 0.$$

Thus $\lambda(a) = 0$, $a \in \ker(\lambda)$, and $\ker(\lambda)$ is WOT-closed.

Let $I - z =$ central support of $\ker(\lambda)$ in R. Then z is a minimal projection which is central in R, and z must have infinite rank since $E(I - z) \in C(H)$, while $E = Ez + E(I - z) \notin C(H)$.

3.4. LEMMA. *R has property* JP *if and only if either*

(1) *R has no nontrivial weakly compact multipliers, or*

(2) *R has a unique minimal projection p which is central of infinite rank, and $R(I - p)$ has no non-trivial weakly compact multipliers in $B((I - p)(H))$.*

Proof. ($\Rightarrow$). If R has at least two minimal projections which are central of infinite rank, then R cannot have property JP. If R has no such projections, then (1) holds by Lemma 3.3. Otherwise, R has a unique such projection, and thus $R(I - p)$ has no such projections, whence by Lemma 3.3 again, (2) must hold.

($\Leftarrow$). If (1) holds, R has property JP by Lemma 3.3. Suppose (2) holds. Let $T \in B(H)$ have ad_T weakly compact on R, and write

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$$

as an operator matrix relative to the decomposition

$$H = p(H) \oplus (I - p)(H).$$

Then ad_{T_4} is weakly compact on $R(I - p)$, and so by Lemma 3.3,

$$\mathrm{ad}_{T_4}(R(I - p)) \subseteq C((I - p)(H)).$$

T_2 and $T_3{}^*$ are weakly compact multipliers of $R(I - p)$, and thus by Lemma 3.1, they are compact. Since for each $a \in R$, there is a scalar λ for which

$$\mathrm{ad}_T(a) = \begin{pmatrix} 0 & T_2 a(I - p) - \lambda T_2 \\ \lambda T_3 - a(I - p)T_3 & \mathrm{ad}_{T_4}(a(I - p)) \end{pmatrix},$$

we conclude that $\mathrm{ad}_T(R) \subseteq C(H)$.

Our next task, as mandated by Lemma 3.4, is to determine the von Neumann subalgebras of $B(H)$ with no nontrivial weakly compact multipliers. It is easy to find von Neumann algebras with nontrivial weakly compact multipliers. A particular class of such examples which will be useful later is obtained as follows. Let κ be a fixed cardinal number. A

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von Neumann subalgebra R of $B(H)$ is said to have *multiplicity* κ if there is a Hilbert space $K \neq (0)$ and a von Neumann subalgebra N of $B(K)$ such that R is unitarily equivalent to $\kappa \cdot N$. If R has infinite multiplicity (i.e., if κ is infinite), then $\kappa \cdot T$ is a nontrivial weakly compact multiplier of R for any nonzero $T \in C(K)$. Since R has multiplicity κ if and only if there is a family of κ pairwise orthogonal, equivalent projections in the commutant R' of R with sum I, any von Neumann subalgebra of $B(H)$ with a properly infinite commutant has nontrivial weakly compact multipliers.

A von Neumann algebra with nontrivial weakly compact multipliers on the other extreme is provided by $L^{\infty}(0, 1)$, realized as a subalgebra of $B(L^2(0, 1))$ in the usual way. To see this, recall that a sequence $\{k_n\}$ of positive integers is said to be *lacunary* if there exists $q > 1$ such that $k_{n+1} \geqq qk_n$, for all n. Let $\{k_n\}$ be a lacunary sequence, and set S = closed linear span of $\{\exp(2\pi i k_n x)\}$ in $H = L^2(0, 1)$. Then by the proof of Lemma V. 6.5 of [**19**], there is a constant $C > 0$, depending only on q, such that for each $s \in \mathrm{Ball}(S)$ and measurable subset E of $[0, 1]$,

$$(*) \qquad \int_E |s|^2 dx \leqq |E| + C\sqrt{|E|},$$

where $|E|$ denotes the Lebesgue measure of E. If P denotes the projection of H onto S, then one easily checks from (*) and Corollary 2.2 that P is an infinite-rank projection which is a weakly compact multiplier of $L^{\infty}(0, 1)$.

More generally, if (X, μ) is any continuous finite measure space, then results of Segal [**15**] and Maharam [**12**] show that there is a cardinal number κ such that a direct summand of $L^{\infty}(X, \mu)$ acting on $L^2(X, \mu)$ is unitarily equivalent to $L^{\infty}([0, 1]^{\kappa}, \lambda^{\kappa})$ acting on $L^2([0, 1]^{\kappa}, \lambda^{\kappa})$, where $[0, 1]^{\kappa}$ is the Cartesian product of κ copies of $[0, 1]$ and λ^{κ} is product Lebesgue measure on $[0, 1]^{\kappa}$. We may hence use a lacunary sequence of exponentials as defined above and the arguments of Lemma V. 6.5 of [**19**] to construct an infinite-rank projection on $L^2(X, \mu)$ which is a weakly compact multiplier of $L^{\infty}(X, \mu)$.

We will now show that a von Neumann subalgebra R of $B(H)$ with no nontrivial weakly compact multipliers must be a direct sum of factors.

If κ is a cardinal number, we denote by $R \otimes M_{\kappa}$ the von Neumann subalgebra of $B(\kappa \cdot H)$ consisting of all $\kappa \times \kappa$ operator matrices with entries in R which act as bounded operators on $\kappa \cdot H$.

3.5. Lemma. *Let κ be a cardinal number, and let $T \in \mathrm{Re}(B(H))$. Let α_0 be a fixed coordinate of $K = \kappa \cdot H$, and let $\widetilde{T}$ denote the element of $B(K)$ with operator matrix having T in the (α_0, α_0) entry and zeros elsewhere. Then $\widetilde{T}$ is a weakly compact multiplier of $R \otimes M_{\kappa}$ if and only if T is a weakly compact multiplier of R.*

Proof. ($\Rightarrow$). This is clear.

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($\Leftarrow$). Let $\{p_n\}$ be a decreasing sequence of projections in $R \otimes M_\kappa$ with infimum zero, with respective α_0-th columns $(p^{(n)}_{\alpha\alpha_0})_\alpha$. Let $x \in H$, and set $\widetilde{x}$ = the vector in K with α_0-th coordinate x and all others 0. Then

$$\|p_n(\widetilde{x})\| \downarrow 0,$$

and letting x range over all of H, we deduce the following: if

$$a_n = \left(\sum_\alpha p^{(n)}_{\alpha\alpha_0}{}^* p^{(n)}_{\alpha\alpha_0}\right)^{1/2},$$

then $\{a_n\}$ is a decreasing sequence in $\mathrm{Ball}(R)_+$ with $\lim_n a_n = 0$ in the strong operator topology.

Now

$$\|\widetilde{T}p_n\|^2 = \|p_n\widetilde{T}\|^2 = \sup\left\{\sum_\alpha \|p^{(n)}_{\alpha\alpha_0}Tx\|^2 : x \in \mathrm{Ball}(H)\right\}$$
$$= \sup\{(a_n^2Tx, Tx) : x \in \mathrm{Ball}(H)\}$$
$$= \|Ta_n^2T\| \leqq \|T\|\,\|Ta_n\|,$$

and so by Corollary 2.2, we must show that

$$\lim_n \|Ta_n\| = 0,$$

and this follows from the proof of Proposition 2.1, Corollary II. 5 of [**1**], and the fact that T is a weakly compact multiplier of R.

3.6. LEMMA. *Let $\{H_\alpha : \alpha \in \mathscr{A}\}$ be a family of Hilbert spaces, set $H = \oplus_\alpha H_\alpha$, P_α = projection of H onto H_α, and*

$$P_\sigma = \oplus\{P_\alpha : \alpha \in \sigma\}$$

for each finite subset σ of $\mathscr{A}$. Let R_α be a von Neumann subalgebra of $B(H_\alpha)$, and let $T \in B(H)$. Then T is a weakly compact multiplier of $R = \oplus_\alpha R_\alpha$ if and only if the following conditions hold:

(i) $\lim_\sigma \|T - TP_\sigma\| = 0$.

(ii) *$P_\alpha T^*TP_\alpha$ is a weakly compact multiplier of R_α for each $\alpha \in \mathscr{A}$.*

Proof. ($\Rightarrow$). To see (i), notice that

$$\|T - TP_\sigma\| = \|l_T|_{R(I-P_\sigma)}\|,$$

and apply Lemma 2.4. As for (ii), fix α, and notice that $l_T = l_{TP_\alpha}$ on R_α, and so l_{TP_α} is weakly compact on R_α. Now apply Lemma 3.1.

($\Leftarrow$). By Lemma 3.1 and the first part of the proof, TP_α is a weakly compact multiplier of R for each α, and so therefore is TP_σ for each σ. Since

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$$0 = \lim_{\sigma} \|T - TP_\sigma\|,$$

T is a weakly compact multiplier of R.

3.7. LEMMA. *Let R be a von Neumann subalgebra of $B(H)$. Then R has no nontrivial weakly compact multipliers if and only if R is a direct sum of factors with no nontrivial weakly compact multipliers.*

Proof. ($\Leftarrow$). This is a straightforward consequence of Lemma 3.6.

($\Rightarrow$). Suppose the center of R has a nonzero continuous part. Then from Segal's classification of abelian von Neumann algebras [**14**], we deduce the existence of a cardinal number κ, a continuous finite measure space (X, μ), and a central projection p of R such that Rp is unitarily equivalent to a von Neumann subalgebra N of $B(\kappa \cdot L^2(X, \mu))$ with

$$N \cap N' = \kappa \cdot L^\infty(X, \mu).$$

Then N is contained in $L^\infty(X, \mu) \otimes M_\kappa$, and thus by Lemma 3.5 and the examples presented in the discussion before that lemma, N, and hence R, has a nontrivial weakly compact multiplier, contrary to assumption. Thus R has a purely atomic center, and is hence a direct sum of the desired type.

Now, suppose R is as in Lemma 3.7, with $R = \oplus_\alpha R_\alpha$ the direct sum decomposition of that lemma. We assert that each R_α must be type I. To see this, notice that if R_α is not type I, then any maximal abelian self-adjoint subalgebra A of R'_α must be continuous, and so there exists a continuous, finite measure space (X, μ), a cardinal number κ, and a projection $p \in A$ such that pA is unitarily equivalent to $\kappa \cdot L^\infty(X, \mu)$. Since $p \in R'_\alpha$, it follows that pR_α is unitarily equivalent to a subalgebra of $L^\infty(X, \mu) \otimes M_\kappa$, and so by Lemma 3.5 and the examples preceeding it, R_α has a nontrivial weakly compact multiplier (we are indebted to Chuck Akemann for this argument). It is easy to see that a type I factor has no nontrivial weakly compact multipliers if and only if it is of finite multiplicity (cf. Proposition 2.2 of [**4**] and the remarks before Lemma 3.5), and we may thus deduce the following theorem from Lemmas 3.4 and 3.7:

3.8. THEOREM. *Let R be a von Neumann subalgebra of $B(H)$. The following are equivalent*:

(1) *R has property* JP;

(2) *Each weakly compact derivation of R into $B(H)$ has range in $C(H)$*;

(3) *R is either a direct sum of type* I *factors each of finite multiplicity or R contains a unique minimal projection p which is central of infinite rank, and $R(I - p)$ is a direct sum of type* I *factors each of finite multiplicity.*

Proof. We have already seen that (1) and (3) are equivalent by Lemmas 3.4 and 3.7. It is clear that (2) implies (1). If (3) holds, then R is type I, and

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so by the results of [7], every derivation of R into $B(H)$ is generated by an element of $B(H)$. Thus by Lemmas 3.4, 3.7, and the remarks after Lemma 3.7, (2) holds.

4. Weakly compact derivations of type I von Neumann algebras. Let R be a type I von Neumann subalgebra of $B(H)$, $\delta:R \to B(H)$ a derivation. We know from the work of E. Christensen [6], [7] that there is a $T \in B(H)$ such that $\mathrm{ad}_T|_R = \delta$, and thus to determine when δ is weakly compact, we need to study the structure of those $T \in B(H)$ for which ad_T is weakly compact on R. Now the multiplicity theory of Segal [14] determines the structure of R as follows: there exists an indexing set $\mathscr{A}$, cardinal-number-valued functions $\alpha \to \kappa(\alpha)$, $\alpha \to \gamma(\alpha)$, $\alpha \in \mathscr{A}$, and (not necessarily distinct) finite measure spaces (X_α, μ_α) such that R is unitarily equivalent to

$$\oplus \{(\kappa(\alpha) \cdot L^\infty(X_\alpha, \mu_\alpha)) \otimes M_{\gamma(\alpha)}:\alpha \in \mathscr{A}\}.$$

These facts all motivate our first lemma.

Let $\{H_\alpha:\alpha \in \mathscr{A}\}$ be a family of Hilbert spaces, R_α a von Neumann subalgebra of $B(H_\alpha)$. Let $H = \oplus_\alpha H_\alpha$, and let P_α, P_σ be defined as in Lemma 3.6. We recall a definition from [16]: an operator $T \in B(H)$ is said to *almost commute with* $R = \oplus_\alpha R_\alpha$ if T commutes with $R_\sigma = R(I - P_\sigma)$ for some σ. T is said to *approximately commute with* R if T is the norm limit of operators which almost commute with R.

4.1. LEMMA. *Let R and H be as in the preceding paragraph, and suppose no R_α is type* II_1. *Let $T \in B(H)$. Then ad_T is weakly compact on R if and only if the following conditions hold*:

(i) *T approximately commutes with R.*

(ii) *For each $\alpha \in \mathscr{A}$, $\mathrm{ad}_{P_\alpha T P_\alpha}$ is weakly compact on R_α and $P_\alpha T(I - P_\alpha)T^*P_\alpha$ (resp., $P_\alpha T^*(I - P_\alpha)TP_\alpha$) is a right (resp., left) weakly compact multiplier of R_α.*

Proof. ($\Rightarrow$). For each finite subset σ of $\mathscr{A}$, R_σ is not of type II_1, and so by the comments following the proof of Theorem 2.4 of [5],

$$\mathrm{dist}(T, R'_\sigma) \leqq \frac{3}{2}\|\mathrm{ad}_T|_{R_\sigma}\|.$$

By Lemma 2.4,

$$0 = \lim_\sigma \|\mathrm{ad}_T|_{R_\sigma}\|,$$

and so (i) follows.

To see (ii), fix $\alpha \in \mathscr{A}$, and write

$$T = \begin{pmatrix} T_{1\alpha} & T_{2\alpha} \\ T_{3\alpha} & T_{4\alpha} \end{pmatrix}$$

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as an operator matrix relative to the decomposition

$$H = P_\alpha(H) \oplus (I - P_\alpha)(H).$$

Then for each $a \in R_\alpha$,

$$\mathrm{ad}_T(a) = \begin{pmatrix} \mathrm{ad}_{T_{1\alpha}}(a) & -aT_{2\alpha} \\ T_{3\alpha}a & 0 \end{pmatrix},$$

and (ii) hence follows by Lemma 3.1.

($\Leftarrow$). By reversing the reasoning of the second part of the previous implication, the mapping $\phi_\alpha : a \to \mathrm{ad}_T(a)$, $a \in R_\alpha$, is weakly compact for each $\alpha \in \mathscr{A}$. We have

$$\lim_n \|T - T_n\| = 0,$$

where for each n, $T_n \in R'_{\sigma_n}$ for some finite subset σ_n of $\mathscr{A}$. It follows that

$$\mathrm{ad}_T|_R : R \to B(H)$$

is the norm limit of finite sums of mappings ϕ_α, and is hence weakly compact.

Thus by Lemma 4.1 and the comments preceding it, we may reduce to the following case: let κ and γ be fixed cardinal numbers, and let (X, μ) be a fixed finite measure space. Let

$$K = \kappa \cdot L^2(X, \mu),\ H = \gamma \cdot K,\ R = (\kappa \cdot L^\infty(X, \mu)) \otimes M_\gamma.$$

We must determine all weakly compact multipliers of R and all $T \in B(H)$ for which ad_T is weakly compact on R.

We turn first to the weakly compact multipliers. For each measurable subset E of X, we let P_E denote the projection in $\kappa \cdot L^\infty(X, \mu) = \kappa \cdot L^\infty$ given by $\kappa \cdot M_E$, where M_E is multiplication by the characteristic function of E on $L^2(X, \mu)$. If N is a von Neumann algebra and $\phi : \kappa \cdot L^\infty \to N$ is a bounded, σ-continuous linear map, then by Theorem I. 2.1 of **[8]**, Corollary 2.2, and the finiteness of (X, μ), we deduce:

(4.1) *ϕ is weakly compact if and only if for each $\epsilon > 0$, there exists a $\delta > 0$ such that for each measurable subset E of X with $\mu(E) < \delta$, $\|\phi(P_E)\| < \epsilon$.*

An analogue of (4.1) in fact holds for a general von Neumann algebra M. If $\phi : M \to N$ is a σ-continuous linear map, then Theorem II. 3 of **[1]** implies that ϕ is weakly compact if and only if there exists a normal positive linear functional ρ on M, depending in general on ϕ, with the following property: for each $\epsilon > 0$, there is a $\delta > 0$ such that for any projection p in M with $\rho(p) < \delta$, $\|\phi(p)\| < \epsilon$. The content of (4.1) is that for $M = \kappa \cdot L^\infty$, a single specific functional, namely

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$$\kappa \cdot T \to \int_X (\text{symbol of } T)\, d\mu, \quad T \in L^\infty,$$

controls weak compactness in this sense for any σ-continuous linear map of $\kappa \cdot L^\infty$ into N.

Following [**11**], we say that a subset S of K has *uniformly absolutely continuous norm* if for each $\epsilon > 0$, there is a $\delta > 0$ such that for any measurable subset E of X with $\mu(E) < \delta$,

$$\sup \{ \|P_E(s)\| : s \in S\} < \epsilon.$$

It follows easily from (4.1) that the projection of K onto a subspace M is a weakly compact multiplier of $\kappa \cdot L^\infty$ if and only if Ball(M) has uniformly absolutely continuous norm. Applying Lemma 3.2, we hence deduce:

4.2. PROPOSITION. *Let $T \in B(K)$, and let $E(\lambda)$ denote the spectral measure of T^*T. Then T is a weakly compact multiplier of $\kappa \cdot L^\infty(X, \mu)$ if and only if for each $\epsilon > 0$,* Ball($E([\epsilon, +\infty))(K)$) *has uniformly absolutely continuous norm.*

Remark. When combined with Lemma 3.6, Proposition 4.2 determines the structure of the weakly compact multipliers of an arbitrary abelian von Neumann algebra.

Now, let Q_α = projection of $H = \gamma \cdot K$ onto its α-th coordinate, and for each finite subset σ of coordinates, set

$$Q_\sigma = \oplus \{Q_\alpha : \alpha \in \sigma\}$$

4.3. THEOREM. *Let $R = (\kappa \cdot L^\infty(X, \mu)) \otimes M_\gamma$, and let $T \in B(H)$. Then T is a (left) weakly compact multiplier of R if and only if it satisfies the following conditions*:

(i) $\lim_\sigma \|T^*T - Q_\sigma T^*TQ_\sigma\| = 0.$

(ii) *If $(T_{\alpha\beta})$ is the operator matrix for T^*T relative to $H = \gamma \cdot K$ and $E_{\alpha\beta}$ is the special measure of $T^*_{\alpha\beta}T_{\alpha\beta}$, then for each α, β, and $\epsilon > 0$,* Ball($E_{\alpha\beta}([\epsilon, +\infty))(K)$) *has uniformly absolutely continuous norm.*

Proof. ($\Rightarrow$). Let D denote the *diagonal subalgebra* of R, i.e., the abelian subalgebra of R formed by the direct sum of γ copies of $\kappa \cdot L^\infty$ acting on H. Then $I - Q_\sigma \in D$ for each σ, and, with $D_\sigma = D(I - Q_\sigma)$, we have

$$\|(I - Q_\sigma)T^*T\| = \|T^*T(I - Q_\sigma)\| = \|l_{T^*T}|_{D_\sigma}\|.$$

But by Lemma 2.4,

$$\lim_\sigma \|l_{T^*T}|_{D_\sigma}\| = 0,$$

and (i) hence follows.

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As for (ii), fix α and β, notice that

$$T_{\alpha\beta} = Q_\alpha T^* T Q_\beta \in B(K)$$

is a weakly compact multiplier of $\kappa \cdot L^\infty$, and apply Proposition 4.2.

($\Leftarrow$). For each fixed α and β, let $\widetilde{T}_{\alpha\beta}$ denote the operator on H whose operator matrix has $T_{\alpha\beta}$ in its (α, β)-entry and zeros elsewhere. Then $\widetilde{T}^*_{\alpha\beta}\widetilde{T}_{\alpha\beta}$ has $T^*_{\alpha\beta}T_{\alpha\beta}$ in its (β, β)-entry and zeros elsewhere, and so by (ii), Proposition 4.2, and Lemma 3.5, $\widetilde{T}^*_{\alpha\beta}\widetilde{T}_{\alpha\beta}$, and hence $\widetilde{T}_{\alpha\beta}$, is a weakly compact multiplier of R. Since $Q_\sigma T^* T Q_\sigma$ is a finite sum of such operators for each σ, it is a weakly compact multiplier of R for each σ, and so therefore by (i) is T^*T, and hence T.

Remark. The proof of Theorem 4.3 shows that $T \in B(H)$ is a weakly compact multiplier of R if and only if T is one for the diagonal subalgebra D of R.

We turn next to the structure of those $T \in B(H)$ for which ad_T is weakly compact on R.

4.4. *Definition.* An operator $T \in B(K)$ is said to *locally approximately commute with* $\kappa \cdot L^\infty$ if for each $\epsilon > 0$, there exists a $\delta > 0$ such that for each measurable subset E of X with $\mu(E) < \delta$,

$$\mathrm{dist}(T, N_E') < \epsilon, \quad \text{where } N_E = (\kappa \cdot L^\infty)P_E.$$

4.5. PROPOSITION. *Let* $T \in B(K)$. *Then* ad_T *is weakly compact on* $\kappa \cdot L^\infty(X, \mu)$ *if and only if* T *locally approximately commutes with* $\kappa \cdot L^\infty(X, \mu)$.

Proof. By (4.1), ad_T is weakly compact on $M = \kappa \cdot L^\infty$ if and only if

(4.2) for each $\epsilon > 0$, there exists $\delta > 0$ such that for any measurable subset E of X with $\mu(E) < \delta$,

$$\|\mathrm{ad}_T(P_E)\| < \epsilon.$$

We will show that (4.2) holds if and only if T locally approximately commutes with M.

Assume that (4.2) holds, and fix $\epsilon > 0$. Choose $\delta > 0$ so that (4.2) holds for δ and $\epsilon/4$. Let E be any measurable subset of X with $\mu(E) < \delta$. Set $M_E = MP_E$. If p is a projection in M_E, then $p = P_F$ for some measurable set F with $\mu(F) < \delta$, and so

$$\|\mathrm{ad}_T(P_F)\| < \frac{\epsilon}{4},$$

whence

$$\|\mathrm{ad}_T|_{M_E}\|_{++} \leqq \frac{\epsilon}{4}.$$

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Thus by Theorem 2.3 of [5] and Lemma 2.3,

$$\mathrm{dist}(T, M'_E) \leqq 4\|\mathrm{ad}_T|_{M_E}\|_{++} \leqq \epsilon.$$

The converse follows easily from the inequality

$$\|\mathrm{ad}_T|_{M_E}\| \leqq 2\ \mathrm{dist}(T, M'_E).$$

Remark. Let N be an arbitrary abelian von Neumann subalgebra of $B(H)$. When combined with Lemma 4.1, Proposition 4.5 determines the structure of those $T \in B(H)$ for which ad_T is weakly compact on N.

4.6. *Definition.* Let $(A, B) \in B(H) \times B(H)$. The mapping

$$\mathrm{ad}_{(A,B)}: B(H) \to B(H)$$

defined by $T \to AT - TB$, $T \in B(H)$, will be called a *generalized derivation* of $B(H)$ *with generator* (A, B).

4.7. LEMMA. *Let R be a von Neumann subalgebra of $B(H)$. Let*

$$(T_1, T_2) \in \mathrm{Re}\, B(H) \times \mathrm{Re}\, B(H).$$

Then $\mathrm{ad}_{(T_1,T_2)}$ is weakly compact on R if and only if the following conditions hold:

(i) ad_{T_1} *and* ad_{T_2} *are weakly compact on R.*

(ii) *For any sequences $\{p_n\}$ of pairwise orthogonal projections in R (resp., any decreasing sequence $\{p_n\}$ of projections in R with infimum zero),*

$$\lim_n \|p_n(T_1 - T_2)p_n\| = 0.$$

Proof. Let $\{p_n\}$ be a sequence of projections in R. For each n, we have relative to the decomposition $H = p_n(H) \oplus (I - p_n)(H)$:

$$\mathrm{ad}_{T_i}(p_n) = \begin{pmatrix} 0 & -p_nT_i(I - p_n) \\ (I - p_n)T_ip_n & 0 \end{pmatrix}, \quad i = 1, 2; \tag{4.3}$$

$$\mathrm{ad}_{(T_1,T_2)}(p_n) = \begin{pmatrix} p_n(T_1 - T_2)p_n & -p_nT_2(I - p_n) \\ (I - p_n)T_1p_n & 0 \end{pmatrix}. \tag{4.4}$$

By (4.4),

$$\lim_n \|\mathrm{ad}_{(T_1,T_2)}(p_n)\| = 0$$

if and only if

$$\lim_n \|p_n(T_1 - T_2)p_n\| = \lim_n \|p_nT_2(I - p_n)\| = \lim_n \|(I - p_n)T_1p_n\| = 0. \tag{4.5}$$

But

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$$\|p_n T_i(I - p_n)\| = \|(I - p_n)T_i p_n\| \text{ for each } n$$

since $T_i \in \operatorname{Re} B(H)$, $i = 1, 2$, and so by (4.3) and (4.5)

$$\lim_n \|\operatorname{ad}_{(T_1,T_2)}(p_n)\| = 0$$

if and only if

$$\lim_n \|p_n(T_1 - T_2)p_n\| = 0 = \lim_n \|\operatorname{ad}_{T_i}(p_n)\|, \; i = 1, 2.$$

The lemma now follows from Corollary 2.2.

From (4.1), Proposition 4.5, and Lemma 4.7, we deduce

4.8. COROLLARY. *Let* $(T_1, T_2) \in \operatorname{Re} B(K) \times \operatorname{Re} B(K)$. *Then* $\operatorname{ad}_{(T_1,T_2)}$ *is weakly compact on* $\kappa \cdot L^\infty(X, \mu)$ *if and only if the following conditions hold*:

(i) T_1 *and* T_2 *locally approximately commute with* $\kappa \cdot L^\infty(X, \mu)$.

(ii) *For* $\epsilon > 0$, *there is a* $\delta > 0$ *such that for any measurable set* E *with* $\mu(E) < \delta$,

$$\|P_E(T_1 - T_2)P_E\| < \epsilon.$$

4.9. LEMMA. *Let* k *be a fixed positive integer, and let* $H = k \cdot K$. *Let* $T \in \operatorname{Re} B(H)$, *with operator matrix* (T_{mn}). *Then* ad_T *is weakly compact on*

$$R = (\kappa \cdot L^\infty(X, \mu)) \otimes M_k$$

if and only if the following conditions hold:

(i) *For* $m, n = 1, \ldots, k$, $m \neq n$, T_{mn} *is a right and left weakly compact multiplier of* $\kappa \cdot L^\infty(X, \mu)$.

(ii) *For* $n = 1, \ldots, k$, T_{nn} *locally approximately commutes with* $\kappa \cdot L^\infty(X, \mu)$.

(iii) *For* $m, n = 1, \ldots, k$, $m \neq n$, *each pair* (T_{mm}, T_{nn}) *satisfies condition* (ii) *of Corollary* 4.8.

Proof. For $n = 1, \ldots, k$, let P_n = projection of H onto its n-th coordinate, and set $R_{mn} = P_m R P_n$. Since

$$R = \sum_{m,n} R_{mn}$$

(vector space sum), ad_T is weakly compact on R if and only if

(4.6) ad_T is weakly compact on R_{mn} for each m and n.

Evaluating ad_T on each subspace R_{mn} and examining the resulting operator matrices, we see that (4.6) holds if and only if (i) and the following two conditions hold:

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(ii)′ For $n = 1, \ldots, k$, $\mathrm{ad}_{T_{nn}}$ is weakly compact on $\kappa \cdot L^\infty$.

(iii)′ For $m, n = 1, \ldots, k$, $m \neq n$, $\mathrm{ad}_{(T_{mm}, T_{nn})}$ is weakly compact on $\kappa \cdot L^\infty$.

But by Proposition 4.5 and Corollary 4.8, conditions (i), (ii)′, and (iii)′ hold if and only (i), (ii), and (iii) of the lemma hold.

In the following theorem, we take $H = \gamma \cdot K$ and $T \in \mathrm{Re}\, B(H)$ to shorten an already fairly lengthy statement. This is no loss of generality, because one can easily obtain a statement for arbitrary $T \in B(H)$, or observe that ad_T is weakly compact on R if and only if $\mathrm{ad}_{\mathrm{Re}\,T}$ and $\mathrm{ad}_{\mathrm{Im}\,T}$ are both weakly compact on R.

4.10. THEOREM. *Let $H = \gamma \cdot K$, $R = (\kappa \cdot L^\infty(X, \mu)) \otimes M_\gamma$, and let Q_α and Q_σ be defined as in Theorem* 4.3. *Let $T \in$* Re $B(H)$, *with operator matrix $(T_{\alpha\beta})$ relative to $H = \gamma \cdot K$. For each σ, let*

$$\begin{pmatrix} T_{1\sigma} & T_{2\sigma} \\ T_{3\sigma} & T_{4\sigma} \end{pmatrix}$$

be the operator matrix of T relative to the decomposition

$$H = Q_\sigma(H) \oplus (I - Q_\sigma)(H).$$

Then ad_T *is weakly compact on R if and only if the following conditions hold*:

$$\text{(i) } \lim_\sigma \|T - (T_{1\sigma} \oplus T_{4\sigma})\| = 0.$$

$$\text{(ii) } \lim_\sigma \operatorname{dist}(T_{4\sigma}, (I - Q_\sigma)R(I - Q_\sigma)') = 0.$$

(iii) *The map $a \to T_{1\sigma}a - aT_{4\sigma}$, $a \in Q_\sigma R(I - Q_\sigma)$, is weakly compact for each σ.*

(iv) *For each α and β with $\alpha \neq \beta$, $T_{\alpha\beta}$ is a right and left weakly compact multiplier of $\kappa \cdot L^\infty(X, \mu)$.*

(v) *For each α, $T_{\alpha\alpha}$ locally approximately commutes with $\kappa \cdot L^\infty(X, \mu)$.*

(vi) *For each α and β with $\alpha \neq \beta$, the pair $(T_{\alpha\alpha}, T_{\beta\beta})$ satisfies the following condition: for each $\epsilon > 0$, there exists $\delta > 0$ such that for any measurable subset E of X with $\mu(E) < \delta$,*

$$\|P_E(T_{\alpha\alpha} - T_{\beta\beta})P_E\| < \epsilon.$$

Proof. ($\Rightarrow$). We have for each σ,

$$\mathrm{ad}_T(I - Q_\sigma) = \begin{pmatrix} 0 & T_{2\sigma} \\ -T_{3\sigma} & 0 \end{pmatrix},$$

and applying the diagonal subalgebra argument used in the proof of Theorem 4.3, we conclude that

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$$\lim_{\sigma} \|\mathrm{ad}_T(I - Q_\sigma)\| = 0,$$

and so

$$0 = \lim_{\sigma} \|T_{2\sigma}\| = \lim_{\sigma} \|T_{3\sigma}\|,$$

whence (i) follows.

To verify (ii), let

$$R_\sigma = (I - Q_\sigma)R(I - Q_\sigma).$$

We claim that

$$(4.7) \qquad 0 = \lim_{\sigma} \|\mathrm{ad}_T|_{R_\sigma}\|.$$

Suppose not. Then by Lemma 2.3, there exists $\delta > 0$, sequences $\{a_k\} \subseteq \mathrm{Ball}(R)_+$, and $\{\sigma_k\}$ such that

$$(4.8) \qquad \sigma_{k-1} \subset \sigma_k, \text{ for all } k,$$

$$(4.9) \qquad a_{k-1} \in (Q_{\sigma_{k+1}} - Q_{\sigma_k})R(Q_{\sigma_{k+1}} - Q_{\sigma_k}), \text{ for all } k, \text{ and}$$

$$(4.10) \quad \|\mathrm{ad}_T(a_k)\| \geqq \delta, \text{ for all } k.$$

From Theorem II.3 of [**1**] follows the existence of a normal state $\hat{\rho}$ on R with the following property: for $\epsilon > 0$, there is a $\delta' > 0$ such that for any $a \in \mathrm{Ball}(R)$ with $\hat{\rho}(a^*a + aa^*) < \delta'$,

$$\|\mathrm{ad}_T(a)\| < \epsilon.$$

By (4.8) and (4.9),

$$\lim_{k} a_k = 0$$

in the strong operator topology, and so

$$\lim_{k} \hat{\rho}(a_k^2) = 0.$$

We hence conclude that for k sufficiently large,

$$\|\mathrm{ad}_T(a_k)\| < \delta,$$

which contradicts (4.10). This verifies (4.7).

If $T_\sigma = T_{1\sigma} \oplus T_{4\sigma}$, then

$$0 = \lim_{\sigma} \|T - T_\sigma\|,$$

and so by Theorem 2.4 of [**5**] and (4.7), we can write

$$0 = \lim_{\sigma} \|\mathrm{ad}_T|_{R_\sigma}\| = \lim_{\sigma} \|\mathrm{ad}_{T_\sigma}|_{R_\sigma}\|$$
$$= \lim_{\sigma} \|\mathrm{ad}_{T_{4\sigma}}|_{R_\sigma}\|$$

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$$\geqq \frac{3}{2} \overline{\lim_{\sigma}} \operatorname{dist}(T_{4\sigma}, R'_\sigma).$$

This yields (ii).

To see (iii), notice that for each σ, ad_T is weakly compact on $Q_\sigma R(I - Q_\sigma)$, and for $a \in Q_\sigma R(I - Q_\sigma)$,

$$\mathrm{ad}_T(a) = \begin{pmatrix} -aT_{3\sigma} & T_{1\sigma}a - aT_{4\sigma} \\ 0 & T_{3\sigma}a \end{pmatrix}.$$

ad_T is also weakly compact on $Q_\sigma R Q_\sigma$, and for $a \in Q_\sigma R Q_\sigma$,

$$\mathrm{ad}_T(a) = \begin{pmatrix} \mathrm{ad}_{T_{1\sigma}}(a) & -aT_{2\sigma} \\ T_{3\sigma}a & 0 \end{pmatrix}.$$

We now apply Lemma 4.9 to deduce (iv), (v), and (vi).

($\Leftarrow$). For each σ, define linear maps $\varphi_\sigma, \psi_\sigma, \xi_\sigma : R \to B(H)$ as follows: for $a \in R$ with operator matrix

$$\begin{pmatrix} a_{1\sigma} & a_{2\sigma} \\ a_{3\sigma} & a_{4\sigma} \end{pmatrix}$$

relative to $H = Q_\sigma(H) \oplus (I - Q_\sigma)(H)$, set

$$\varphi_\sigma(a) = \begin{pmatrix} \mathrm{ad}_{T_{1\sigma}}(a_{1\sigma}) & 0 \\ 0 & 0 \end{pmatrix}$$

$$\psi_\sigma(a) = \begin{pmatrix} 0 & T_{1\sigma}a_{2\sigma} - a_{2\sigma}T_{4\sigma} \\ 0 & 0 \end{pmatrix},$$

$$\xi_\sigma(a) = \begin{pmatrix} 0 & 0 \\ T_{4\sigma}a_{3\sigma} - a_{3\sigma}T_{1\sigma} & 0 \end{pmatrix}.$$

By (iv), (v), (vi), and Lemma 4.9, φ_σ is weakly compact, and ψ_σ is weakly compact by (iii). If $a \in \mathrm{Re}\, R$, then

$$\xi_\sigma(a) = -\psi_\sigma(a)^*,$$

and so ξ_σ is weakly compact on $\mathrm{Re}\, R$, and hence on $\mathrm{Re}\, R + i\mathrm{Re}\, R = R$.

By (ii) and the inequality

$$\|\mathrm{ad}_{T_{4\sigma}}|_{R_\sigma}\| \leqq 2 \operatorname{dist}(T_{4\sigma}, R'_\sigma),$$

we have

$$0 = \lim_{\sigma} \|\mathrm{ad}_{T_{4\sigma}}|_{R_\sigma}\|.$$

By (i),

$$0 = \lim_{\sigma} \|\mathrm{ad}_T - \mathrm{ad}_{T_\sigma}\|,$$

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and since

$$||\mathrm{ad}_{T_\sigma}|_R - (\varphi_\sigma + \psi_\sigma + \xi_\sigma)|| = ||\mathrm{ad}_{T_{4\sigma}}|_{R_\sigma}||,$$

we conclude that

$$0 = \lim_\sigma ||\mathrm{ad}_T|_R - (\varphi_\sigma + \psi_\sigma + \xi_\sigma)||.$$

Thus $\mathrm{ad}_T|_R$ is the norm-limit of maps weakly compact on R, and is hence weakly compact on R.

Remark. Condition (iii) of Theorem 4.10 is not completely satisfactory. One would ideally want a condition guaranteeing the desired weak compactness which involves only conditions intrinsic to $\kappa \cdot L^\infty(X, \mu)$ and the operators $T_{1\sigma}$ and $T_{4\sigma}$. We have not been able to obtain such conditions. However, since $T_{1\sigma}$ represents a matrix with only finitely many nonzero entries and (by (ii)) $T_{4\sigma}$ is "essentially" a diagonal matrix with constant diagonal, condition (iii) does make verification of the desired weak compactness easier in most cases.

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