ON CERTAIN SIMULTANEOUS FUNCTIONAL EQUATIONS

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1. Introduction. G. de Rham [1] has shown that certain remarkable curves, defined by limits of sequences of inscribed polygons, are solutions of functional equations of the form

$$\begin{aligned} \phi(\tfrac{x}{2}) &= F_1[\phi(x)] \\ \phi(\tfrac{1+x}{2}) &= F_2[\phi(x)] \end{aligned} \qquad x \in [0,1] .$$

If the functions $F_p(z)$ $(p = 1, 2)$ are contraction mappings of the complex plane into itself, i.e. for fixed $r < 1$

$$|F_p(z_1) - F_p(z_2)| < r\,|z_1 - z_2| ,$$

then $F_p(z)$ is continuous and has a single fixed point w_p, $F_p(w_p) = w_p$. De Rham has shown that if in addition $F_1(w_2) = F_2(w_1)$, the above simultaneous functional equations have a unique bounded solution ϕ and this solution is continuous.

In this paper we are concerned with the solution of the simultaneous functional equations

(1) $$\phi(x) = F_1[x, \phi(\tfrac{x}{2})] ,$$

(2) $$\phi(x) = F_2[x, \phi(\tfrac{1+x}{2})] .$$

We note that if $y = 1 - x$, $\phi(x) = \psi(y)$, then

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$$\phi(\frac{x}{2}) = \psi(\frac{1+y}{2}), \quad \phi(\frac{1+x}{2}) = \psi(\frac{y}{2})$$

and $x \in [0,1]$ if and only if $y \in [0,1]$. Consequently, any emphasis placed on one of the above equations could equally well be placed on the other.

In the following C is the complex plane; $\bar{C}$ is the extended complex plane; D is some subset of $\bar{C}$; $\theta(x)$ is defined for $x \in (\frac{1}{2}, 1]$ by $\theta(x) = x$, and successively on $(\frac{1}{4}, \frac{1}{2}]$, $(\frac{1}{8}, \frac{1}{4}]$, ..., by $\theta(x) = \theta(2x)$; for $x \in (\frac{1}{2}, 1]$ $p(x)$ and its iterates are defined by $p(x) = \theta(2x-1)$, $p^{o}(x) = x$, and $p^{n+1}(x) = p^{n}(p(x))$, $n = 0, 1, 2, \ldots$.

2. Reduction to one equation. The algebraic solution of simple simultaneous equations in two variables is often accomplished by the elimination of one variable. An analogous method applicable to some simultaneous functional equations is developed in the next theorem; we reduce the dependence of solutions ϕ of (1) and (2) on the period functions $\frac{x}{2}$ and $\frac{1+x}{2}$ to dependence on the period function $p(x)$.

THEOREM 1. Let $x \in (0, 1]$. For each x let $F_1(x, z)$ be a one-one map of D onto itself, $F_2(x, z)$ map D into itself, and $g(x, z)$ be a one-one map of D onto itself which satisfies the functional equation

$$(3) \qquad g(\frac{x}{2}, z) = g[x, F_1(x, z)], \qquad z \in D.$$

Then the function ϕ with range in D is a solution of (1) and (2) if and only if it satisfies (1) at 0, (2) on $[0, \frac{1}{2}]$ and

$$(4) \qquad g\{p(x), \phi[p(x)]\} = g\{2x-1, F_2[2x-1, \phi(x)]\}, \quad x \in (\frac{1}{2}, 1].$$

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Proof. The existence of such a function $g(x, z)$ is guaranteed by the properties of $F_1(x, z)$. We may for example define $g(x, z) = z$, $x \in (\frac{1}{2}, 1]$, $z \in D$, and extend the domain of x to $(0, 1]$ by using (3).

Let ϕ be a solution of (1) and (2) with range in D and

$$u(x) = g[x, \phi(x)], \quad x \in (0, 1].$$

Then if $x \in (0, 1]$

$$u(\frac{x}{2}) = g[\frac{x}{2}, \phi(\frac{x}{2})] = g\{x, F_1[x, \phi(\frac{x}{2})]\}$$

$$= g[x, \phi(x)] = u(x),$$

and there exists a positive integer n such that $2^n x \in (\frac{1}{2}, 1]$; hence

$$u[\theta(x)] = u[\theta(2^n x)] = u(2^n x) = u(x),$$

$$u[p(x)] = u(2x-1), \quad x \in (\frac{1}{2}, 1].$$

The latter equation together with (2) gives (4). The remainder of the assertion is trivial.

Assume the converse; then for $x \in (0, 1]$

$$g\{x, F_2[x, \phi(\frac{1+x}{2})]\} = g\{\theta(x), \phi[\theta(x)]\} = g\{\theta(\frac{x}{2}), \phi[\theta(\frac{x}{2})]\}$$

$$= g\{\frac{x}{2}, F_2[\frac{x}{2}, \phi(\frac{1}{2}+\frac{x}{4})]\} = g\{\frac{x}{2}, \phi(\frac{x}{2})\}$$

$$= g\{x, F_1[x, \phi(\frac{x}{2})]\}$$

$$= g\{x, \phi(x)\}, \quad x \in (\frac{1}{2}, 1].$$

Hence

$$F_2[x, \phi(\frac{1+x}{2})] = F_1[x, \phi(\frac{x}{2})], \qquad x \in (0, 1]$$

$$= \phi(x), \qquad x \in (\frac{1}{2}, 1].$$

The last equality also holds for $x \in [0, \frac{1}{2}]$.

EXAMPLE 1. If both of the equations (1) and (2) are linear and soluble, $g(x, z)$ can be found so that (4) is also linear. Consider the equations [2]

$$\phi(\frac{x}{2}) = a\,\phi(x) \tag{5}$$

$$x \in [0, 1]$$

$$\phi(\frac{1+x}{2}) = a + (I - a)\,\phi(x) \tag{6}$$

where $a \in C$ such that $|a| < 1$, $|1 - a| < 1$. We may employ Theorem 1 with $D = C$, $F_1(x, z) = z/a$, $F_2(x, z) = (z - a)/(1 - a)$, and (with the principal value of the logarithm)

$$g(x, z) = z\,x^{\log_2 a}.$$

Then (5) and (6) reduce on $(\frac{1}{2}, 1]$ to

$$\phi(x) = a + c(x)\,\phi[p(x)], \tag{7}$$

where

$$c(x) = (1 - a)\,[(2x - 1)/p(x)]^{-\log_2 a}.$$

Since $|c(x)| \leqq |1 - a| < 1$, (7) has the unique bounded solution

$$\phi(x) = a + a \sum_{n=1}^{\infty} \prod_{m=0}^{n-1} c[p^m(x)].$$

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But if ϕ is a bounded solution of (5) and (6) so must the four functions obtained from ϕ by taking left and right upper and lower limits also be bounded solutions. Hence all five solutions are equal and ϕ is continuous.

EXAMPLE 2. Let ϕ be a solution of Cauchy's functional equation

$$\phi(x + y) = \phi(x) + \phi(y) .$$

Then certainly ϕ is a solution of the equations

$$\phi(x) = 2\,\phi(\frac{x}{2}) = 2\,\phi(\frac{1+x}{2}) - \phi(1) , \qquad x \in [0, 1] .$$

Using the methods of Example 2 we see that ϕ is also a solution of the equation

$$\phi(x) = \frac{1}{2}\phi(1) + \frac{1}{2}\{(2x - 1)/p(x)\}\,\phi[p(x)] , \qquad x \in (\frac{1}{2}, 1] ,$$

which has at most one bounded solution ϕ passing through the point $x = 1$, $z = K$; such a solution is

$$\phi(x) = K x , \qquad x \in (\frac{1}{2}, 1] .$$

Obviously we may replace boundedness on $(\frac{1}{2}, 1]$ by the same on any interval to establish the well known fact that the functions Kx, $K \in C$, are the only solutions of Cauchy's equation bounded on some interval.

3. <u>The number of continuous solutions.</u> We prove the next theorem with the help of Theorem 1.

THEOREM 2. Let $F_1(x, z)$, $F_2(x, z)$, be one-one maps of D onto itself for each $x \in (0, 1]$. Then the number of solutions of (1) and (2) with range in D and continuous on $(0, 1]$ is not greater than the number of finite fixed points of $F_2(1, z)$ in D.

Proof. We may replace (4) by an equivalent equation of the form

$$(8) \qquad \phi(x) = F\{x, \phi[p(x)]\}, \qquad x \in (\tfrac{1}{2}, 1].$$

If y is a fixed finite dyad in $(\frac{1}{2}, 1]$ then there exists an integer n such that $p^n(y) = 1$ and hence by iteration of the equation (8) $n-1$ times we obtain $\phi(y)$ as a function of $\phi(1)$ alone. The possible values of $\phi(1)$ are the fixed points of $F_2(1, z)$ in D. Therefore, since the finite dyads are everywhere dense in $(\frac{1}{2}, 1]$, the number of continuous solutions of (8) in D cannot be more than the number of finite fixed points of $F_2(1, z)$ in D.

EXAMPLE 3. The simultaneous functional equations

$$\phi(x) = \frac{3\phi(\frac{x}{2})+1}{\phi(\frac{x}{2})+3} = \frac{2\phi(\frac{1+x}{2})}{1+\phi(\frac{1+x}{2})}$$

have the continuous solutions $\phi(x) = 1$, $(x-1)/(x+1)$, $x \in [0, 1]$; they are the only continuous solutions since the function $2z/(1+z)$ has only two fixed points, 0 and 1, in C.

REFERENCES

1. G. de Rham, Sur quelques courbes definies par des equations fonctionnelles, Univ. e Politec, Torino. Rend. Sem. Mat. 16 (1956-7), 101-113.

2. ________, Sur une courbe plane, Journal de Math. Pures et Appl., 35 (1956), 25-42.

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