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A RESULT CONCERNING ADDITIVE MAPS ON THE SET OF QUATERNIONS AND AN APPLICATION

DAMJAN KOBAL AND PETER ŠEMRL

We determine all additive $F, G : \mathbf{H} \longrightarrow \mathbb{R}$ and multiplicative $M : \mathbf{H} \longrightarrow \mathbb{R}$ satisfying the functional equation $F(\lambda) + M(\lambda)G(\lambda^{-1}) = 0$. As an application we generalise Kurepa's solution of one of Halperin's problem concerning quadratic functionals.

The functional equation (FE) $F(x) + M(x)G(x^{-1}) = 0$, for additive F, G, and multiplicative M, and its special cases have been studied by many authors. Kurepa [5] came across the functional equation (FE) with $M(x) = x^2$ on the reals $\mathbb{R}$. Through this equation he obtained the general form of functionals Q on $\mathbb{R}$-vector spaces satisfying the parallelogram law and the homogeneity $Q(\lambda x) = \lambda^2 Q(x)$, thus answering a question raised by I. Halperin in 1963 in Paris. With this result over $\mathbb{R}$ he further solved [6] the problem on vector spaces over the field of complex numbers $\mathbb{C}$ or the skew field of quaternions $\mathbb{H}$ under the homogeneity $Q(\lambda x) = |\lambda|^2 Q(x)$. P. Vrbová [10] managed to solve (FE) with $F = G$ and $M(x) = |x|^2$ on $\mathbb{C}$ and gave a shorter proof of the result of S. Kurepa on complex spaces. These and further results of many other authors were unified and generalised in the work of Ng [9]. He determined the general solution of (FE) on a commutative field k of characteristic $\neq 2$ in order to generalise the results of various authors concerning the Halperin problem on quadratic functionals. The other motivation for the study of (FE) is its application in information theory [8]. The problem of characterising multiplicative-type recursive measures of information in n dimensions [3] leads to (FE) with $G = F$ and $M, F : \mathbb{R}^n \longrightarrow \mathbb{R}$ [1]. Working in this area Ebanks [2] extended the result of Ng [9] by solving the (FE) for additive $F, G : k^n \longrightarrow k$ and multiplicative $M : k^n \longrightarrow k$, where k is a commutative field of characteristic $\neq 2$. Some functional equations closely related to (FE) have been treated by Vukman on Banach algebras [11].

One can obtain further generalisations of results of Ng and Ebanks by studying all triples $F, G, M : R \longrightarrow k$ of additive F, G and multiplicative M satisfying (FE) on the subset of all invertible elements $R^* \subset R$, where k is a commutative field with

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characteristic 0 and R is a commutative ring with identity and regularity property. The regularity property is defined similarly as in [7]: A ring R with identity has regularity property if for any $x \in R$ there exists a positive integer n_x such that for any $m \in \mathbb{N}$, $m \geqslant n_x$, the element $(m + x)$ is invertible. For example, all Banach algebras with identity have this property. The regularity property implies that there are a lot of invertible elements in R. This seems to be a reason that the general form of F, G, and $M_{|R^*}$ in this more general setting can be described in almost the same way as in the special case treated by Ng [9]. We shall omit the details since the basic ideas are exactly the same as those of Ng.

In our note we shall determine all additive $F, G : \mathbb{H} \longrightarrow \mathbb{R}$ and multiplicative $M : \mathbb{H} \longrightarrow \mathbb{R}$ satisfying (FE). This result shows us that the solution of (FE) in a noncommutative case can be essentially different from that in the commutative case.

We shall apply this result in the theory of quadratic functionals. Let X be a quaternionic vector space. Let us recall that a mapping $Q : X \longrightarrow \mathbb{R}$ is called a quadratic functional if it satisfies the parallelogram law

$$Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y) \tag{i}$$

and the homogeneity

$$Q(\lambda x) = |\lambda|^2 Q(x). \tag{ii}$$

A mapping $B : X \times X \longrightarrow \mathbb{H}$ is called a sesquilinear functional if

$$B(\lambda_1 x_1 + \lambda_2 x_2, y) = \lambda_1 B(x_1, y) + \lambda_2 B(x_2, y), \tag{iii}$$

$$B(x, \mu_1 y_1 + \mu_2 y_2) = B(x, y_1)\overline{\mu_1} + B(x, y_2)\overline{\mu_2}, \tag{iv}$$

where $\overline{\mu}$ denotes the conjugate of μ. One of the questions posed by I. Halperin can be formulated as follows: Does a quadratic functional Q possess the property that

$$B(x, y) = m(x, y) + im(x, iy) + jm(x, jy) + km(x, ky)$$

where $$m(x, y) = (1/4)(Q(x + y) - Q(x - y))$$

is a sesquilinear functional. It has been proved by Kurepa [6] that the answer to this problem is in the affirmative. We shall generalise this result by allowing a very general notion of homogeneity on Q. More precisely, we shall replace (ii) by a weaker assumption that Q is functionally homogeneous, that is, for some scalar function $M : \mathbb{H} \longrightarrow \mathbb{R}$ the relation $Q(\lambda x) = M(\lambda)Q(x)$ holds for all $\lambda \in \mathbb{H}$ and all $x \in X$.

Let us recall that a derivation D on the reals is an additive mapping satisfying $D(ts) = sD(t) + tD(s)$. For an arbitrary quaternion $\lambda = t_1 + t_2 i + t_3 j + t_4 k$, the notations $\overline{\lambda}$ and $|\lambda|$ are used for usual conjugation and norm on $\mathbb{H}$. We denote the set of all nonzero quaternions by $\mathbb{H}^*$.

THEOREM 1. *Let additive $F, G : \mathbb{H} \longrightarrow \mathbb{R}$ and multiplicative $M : \mathbb{H} \longrightarrow \mathbb{R}$ be nonzero maps satisfying the equation $F(\lambda) + M(\lambda)G(\lambda^{-1}) = 0$ on $\mathbb{H}^*$. Then they are of the form:*

$$F(t_1 + t_2 i + t_3 j + t_4 k) = b_1 t_1 + b_2 t_2 + b_3 t_3 + b_4 t_4,$$
$$G(t_1 + t_2 i + t_3 j + t_4 k) = -b_1 t_1 + b_2 t_2 + b_3 t_3 + b_4 t_4,$$

and
$$M(\lambda) = |\lambda|^2,$$

where b_1, b_2, b_3, b_4 are real constants satisfying $(b_1, b_2, b_3, b_4) \neq (0,0,0,0)$. The converse is also true.

PROOF: Comparing the equations

$$F(\lambda) + M(\lambda)G(\lambda^{-1}) = 0 \text{ and } F(-\lambda) + M(-\lambda)G(-\lambda^{-1}) = 0$$

we obtain $G(\lambda^{-1})(M(\lambda) - M(-\lambda)) = 0$. As G is nonzero we have necessarily $M(\mu) = M(-\mu)$ for at least one $\mu \in \mathbb{H}^*$. Using the multiplicativity of M we get

$$M(\lambda) = M(\mu)M(\mu^{-1}\lambda) = M(-\mu)M(\mu^{-1}\lambda) = M(-\lambda)$$

for any $\lambda \in \mathbb{H}$. For every $\lambda \in \mathbb{H}$ we can find a quaternion μ such that $\lambda = \mu^2$. Thus, we have $M(\lambda) = M(\mu^2) \geqslant 0$ for any $\lambda \in \mathbb{H}$.

Since F and G are additive they can be written as

$$F(t_1 + t_2 i + t_3 j + t_4 k) = \sum_{i=1}^{4} f_i(t_i),$$
$$G(t_1 + t_2 i + t_3 j + t_4 k) = \sum_{i=1}^{4} g_i(t_i),$$

where $f_i, g_i : \mathbb{R} \longrightarrow \mathbb{R}$ are additive functions. Substituting t, ti, tj, and tk in (FE) we get

$$f_i(t) + s_i M(t) g_i(t^{-1}) = 0, \quad t \in \mathbb{R}^*,$$

where $s_1 = 1$, $s_2 = -M(i)$, $s_3 = -M(j)$, and $s_4 = -M(k)$. The restriction $M_{|\mathbb{R}}$ is not identically zero since otherwise all f_i would be zero functions which would contradict the fact that F is nonzero. The multiplicativity of M implies now that $M(t)$ is a nonzero real number for every nonzero real t. Moreover, we have $M(1) = M(-1) = M(i) = M(j) = M(k) = 1$. It follows that for every i, $i = 1,2,3,4$, additive functions f_i and $s_i g_i$ are either both nonzero or both identically equal to zero. We may now apply [9, Corollary 4.2] in order to derive from $f_i(t)+s_iM(t)g_i(t^{-1})=0$ that f_i, g_i, and the restriction of M to the field of reals are either of the form

$$f_i(t)=D_i(t)+b_it,\ \ g_i(t)=u_i(D_i(t)-b_it),\ \ M(t)=t^2,$$

where D_i, $i=1,2,3,4$, are derivations on the field of real numbers, b_i, $i=1,2,3,4$, are real constants, and $u_1=1$, while $u_2=u_3=u_4=-1$; or

$$f_i(t)=Im(a_i\phi(t))+Re(b_i\phi(t)),\ \ g_i(t)=u_i(Im(a_1\phi(t))-Re(b_1\phi(t))),\ \ M(t)=|\phi(t)|^2,$$

where $\phi:\mathbb{R}\longrightarrow\mathbb{C}$ is a nontrivial embedding, a_i, b_i, $i=1,2,3,4$, are real constants, $u_1=1$, $u_2=u_3=u_4=-1$, and $Im\phi$, $Re\phi$, denote respectively the imaginary part of ϕ, the real part of ϕ.

Let us first consider the case that $M(t)=t^2$ holds for all real numbers t. Let λ be a strictly imaginary quaternion, that is $\lambda=-\overline{\lambda}$. For such λ we have

$$|\lambda|^4=M\left(|\lambda|^2\right)=M(\lambda\overline{\lambda})=M(\lambda)M(-\lambda)=(M(\lambda))^2,$$

and consequently, $M(\lambda)=|\lambda|^2$. For an arbitrary $\lambda\in\mathbb{H}$ one can always find strictly imaginary quaternions λ_1 and λ_2 such that $\lambda=\lambda_1\lambda_2$. It follows that we have $M(\lambda)=|\lambda|^2$ for all $\lambda\in\mathbb{H}$. We can easily derive from (FE) that for an arbitrary quaternion $\lambda=t_1+t_2i+t_3j+t_4k$ with $|\lambda|=1$ the relation

$$D_1(t_1)+D_2(t_2)+D_3(t_3)+D_4(t_4)=0$$

is valid. It follows also that $D_1(t_1)+D_2(-t_2)+D_3(-t_3)+D_4(-t_4)=0$, and consequently, $D_1(t)=0$ for all t, $0\leqslant t\leqslant 1$. From additivity of D_1 we get that D_1 is identically equal to zero. Clearly, the same must be true for D_i, $i=2$, 3, 4. One can now easily verify that

$$F(t_1+t_2i+t_3j+t_4k)=\sum_{i=1}^{4}b_it_i,$$

$$G(t_1+t_2i+t_3j+t_4k)=-b_1t_1+\sum_{i=2}^{4}b_it_i,$$

and
$$M(\lambda)=|\lambda|^2$$

satisfy the (FE).

In the case that we have $M(t)=|\phi(t)|^2$ for all real numbers we get, in the same way as above, that $M(\lambda)=|\phi(|\lambda|)|^2$ holds for all quaternions λ. The same method as in the first case gives us

$$Im(a_1\phi(t_1)+\ldots+a_4\phi(t_4))=0,$$

and $a_1 Im\phi = \ldots = a_4 Im\phi = 0$. Since $Im\phi$ cannot be identically zero, we have $a_1 = \ldots = a_4 = 0$. Substituting quaternions $\lambda = 1 + t_2 i + t_3 j + t_4 k$ and $\overline{\lambda}$ in (FE), and comparing the results so obtained, we get

$$b_1\left(1 - \left|\phi\left(|\lambda|^2\right)\right| Re\phi\left(|\lambda|^{-2}\right)\right) = 0.$$

As ϕ is a nontrivial ring morphism, this implies $b_1 = 0$. Similarly, one can verify that $b_2 = b_3 = b_4 = 0$. This completes the proof. ∎

The following theorem is an extension of Kurepa's solution of Halperin's problem concerning quadratic forms on quaternionic vector space [**6**, Theorem 2].

THEOREM 2. *Let X be a vector space over the skew field of quaternions $\mathbb{H}$, M a real function on $\mathbb{H}$, and Q a nonzero real functional on X satisfying the parallelogram law and the homogeneity $Q(\lambda x) = M(\lambda)Q(x)$. Then $M(\lambda) = |\lambda|^2$, the functional $B : X \times X \longrightarrow \mathbb{H}$ given by*

$$B(x,y) = m(x,y) + im(x,iy) + jm(x,jy) + km(x,ky),$$

where $$m(x,y) = (1/4)(Q(x+y) - Q(x-y)),$$

is a sequilinear functional, and

$$B(x,x) = Q(x)$$

holds for all $x \in X$.

PROOF: It is well known that the functional $m(x,y)$ is biadditive [**5**, Lemma 1]. Obviously, we have $m(\lambda x, \lambda y) = M(\lambda)m(x,y)$. Assume that $m(x,y) = 0$ for all $x, y \in X$. Then we have $Q(x+y) = Q(x-y)$. Putting $x = y = z/2$ in this equation we get $Q(z) = 0$ for all $z \in X$ which is a contradiction with our assumption that Q is nonzero. Fix now $x_0, y_0 \in X$ such that $m(x_0, y_0) \neq 0$. Comparing relations

$$m(\lambda\mu x_0, \lambda\mu y_0) = M(\lambda\mu)m(x_0, y_0)$$

and $$m(\lambda\mu x_0, \lambda\mu y_0) = M(\lambda)m(\mu x_0, \mu y_0) = M(\lambda)M(\mu)m(x_0, y_0)$$

we see that M is multiplicative. For any nonzero $\lambda \in \mathbb{H}$ we have

$$m(\lambda x_0, y_0) - M(\lambda)m\left(x_0, \lambda^{-1} y_0\right) = 0.$$

Consequently, mappings $M : \mathbb{H} \longrightarrow \mathbb{R}$ and $F, G : \mathbb{H} \longrightarrow \mathbb{R}$ given by

$$F(\lambda) = m(\lambda x_0, y_0), \quad G(\lambda) = -m(x_0, \lambda y_0),$$

satisfy the (FE). Obviously, mappings F and G are additive and nonzero. According to Theorem 1 the mapping M is of the form $M(\lambda) = |\lambda|^2$. Using [**6**, Theorem 2] one can complete the proof. ∎

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Department of Mathematics
University of Ljubljana
Jadranska 19
61000 Ljubljana
Yugoslavia