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Decomposition of Splitting Invariants in Split Real Groups

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Abstract. For a maximal torus in a quasi-split semi-simple simply-connected group over a local field of characteristic 0, Langlands and Shelstad constructed a cohomological invariant called the splitting invariant, which is an important component of their endoscopic transfer factors. We study this invariant in the case of a split real group and prove a decomposition theorem which expresses this invariant for a general torus as a product of the corresponding invariants for simple tori. We also show how this reduction formula allows for the comparison of splitting invariants between different tori in the given real group.

In applications of harmonic analysis and representation theory of reductive groups over local fields to questions in number theory, a central role is played by the theory of endoscopy. This theory associates a given connected reductive group G over a local field F with a collection of connected reductive groups over F, often denoted by H, which have smaller dimension (except when $H = G$), but are usually not subgroups of G. The geometric side of the theory is then concerned with transferring functions on $G(F)$ to functions on $H(F)$ in such a way that suitable linear combinations of their orbital integrals are comparable, while the spectral side is concerned with transferring "packets" of representations on $H(F)$ to "packets" of representations on $G(F)$ in such a way that suitable linear combinations of their characters are comparable. In both cases, the comparison involves certain normalizing factors, called geometric or spectral transfer factors.

Over the real numbers, the theory of endoscopy was developed by Diana Shelstad in a series of profound papers [8–11] (but see also [13–15] for a more modern point of view and additional results), in which she defined geometric and spectral transfer factors and proved that these factors indeed give a comparison of orbital integrals and character formulas between G and H. A very subtle and complicated feature of the transfer factors was the need to assign a $\pm$-sign to each maximal torus in G in a coherent manner, and Shelstad was able to prove that this is possible. A uniform and explicit definition of geometric transfer factors for all local fields was given in [5]. An explicit construction of spectral transfer factors over the real numbers was given in [14], while over the p-adic numbers their existence is still conjectural (see however [3] for a proof of the spectral transfer in a special case). The structure of transfer factors is quite complex — both the geometric and the real spectral ones

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are a product of multiple terms of group-theoretic or Galois-cohomological nature. There are numerous choices involved in the construction of each individual term, but the product is independent of most choices. One term that is common to both the geometric and the real spectral transfer factors is called Δ_I. It is regarded as the most subtle and is the one that makes explicit the choice of coherent collection of signs in Shelstad's earlier work. At its heart is a Galois-cohomological object, called the splitting invariant. The splitting invariant is an element of $H^1(F, T)$ associated with any maximal torus T of a quasi-split semi-simple simply-connected group G, whose construction occupies the first half of [5, §2]. It depends on the choice of a splitting $(T_0, B_0, \{X_\alpha\}_{\alpha\in\Delta})$ of G as well as a-data $\{a_\beta\}_{\beta\in R(T,G)}$.

This paper addresses the following question. If one has two maximal tori in a given real group which originate from the same endoscopic group, how can one compare their splitting invariants? While there will in general be no direct relation between $H^1(F, T_1)$ and $H^1(F, T_2)$ for two maximal tori T_1 and T_2 of G, if both those tori originate from H, then there are certain natural quotients of their cohomology groups which are comparable, and it is the image of the splitting invariant in those quotients that is relevant to the construction of Δ_I. An example of a situation where this problem arises is the stabilization of the topological trace formula of Goresky–MacPherson. One is led to consider characters of virtual representations which occur as sums indexed over tori in G that originate from the same endoscopic group H, and each summand carries a Δ_I-factor associated with the corresponding torus.

To describe the results of this paper, we take G to be a split simply-connected real group and $(T_0, B_0, \{X_\alpha\}_{\alpha\in\Delta})$ to be a fixed splitting. For a set A consisting of roots of T_0 in G which are pairwise strongly orthogonal, let S_A denote the element of the Weyl group of T_0 given by the product of the reflections associated with the elements of A (the order in which the product is taken is irrelevant). We show that associated with A there is a canonical maximal torus T_A of G and a set of isomorphisms of real tori $T_0^{S_A} \to T_A$, where $T_0^{S_A}$ is the twist by S_A of T_0. Any maximal torus in G is $G(\mathbb{R})$-conjugate to one of the T_A, so it is enough to study the tori T_A. We give an expression in purely root-theoretic terms for a certain 1-cocycle in $Z^1(\mathbb{R}, T_0^{S_A})$. This cocycle has the property that its image in $Z^1(\mathbb{R}, T_A)$ under any of the isomorphisms $T^{S_A} \to T_A$ above is the same, and the class in $H^1(\mathbb{R}, T_A)$ of that image is the splitting invariant of T_A (associated with a specific choice of a-data). Moreover, we prove a reduction theorem which shows that this cocycle is a product over $\alpha \in A$ of the cocycles associated with the canonical tori $T_{\{\alpha\}}$, thereby reducing the study of the splitting invariant of T_A to those of the various $T_{\{\alpha\}}$. This product decomposition takes place inside the group $Z^1(\mathbb{R}, T_0^{S_A})$, that is, we show that the elements of $Z^1(\mathbb{R}, T_0^{s_\alpha})$ associated with the various $T_{\{\alpha\}}$ with $\alpha \in A$ also lie in $Z^1(\mathbb{R}, T_0^{S_A})$ and that their product is the element associated with T_A. Finally we show that if $A' \subset A$ and the tori $T_{A'}$ and T_A originate from the same endoscopic group, then the endoscopic characters on the cohomology groups $H^1(\mathbb{R}, T_{A'})$ and $H^1(\mathbb{R}, T_A)$ factor through certain explicitly given quotients of these groups, and the quotient of $H^1(\mathbb{R}, T_{A'})$ is canonically embedded into that of $H^1(\mathbb{R}, T_A)$. This, together with the reduction theorem, allows for a direct comparison of the values that the endoscopic characters associate to the splitting invariants for $T_{A'}$ and T_A.

Our techniques rely heavily on the study of sets of strongly orthogonal roots in root systems and the fact that each element of order 2 in the Weyl group of a root system is of the form S_A for some set A consisting of strongly orthogonal roots. In a split real group the Galois action on any maximal torus is realized by such an element. This is the reason why we restrict our attention to such groups. It may be possible to use our techniques also in the case of non-split quasi-split groups which possess an anisotropic maximal torus, for then the Galois-action on any maximal torus is of the form $-S_A$, but we have not pursued this line of thought here.

The paper is organized as follows: Section 1 contains a few basic facts and serves mainly to fix notation for the rest of the paper. Section 2 contains proofs of general facts about subsets of strongly orthogonal roots in reduced root systems, which are needed as a preparation for the reduction theorem mentioned above. The study of the splitting invariants takes place in Section 3, where first the splitting invariant for the tori $T_{\{\alpha\}}$ is computed, and after that the results of Section 2 are used to reduce the case of T_A to that of $T_{\{\alpha\}}$. While the statement of the reduction theorem appears natural and clear, the proof contains some subtle points. First, one has to choose the Borel B_0 in the splitting of G with care according to the strongly orthogonal set A. As remarked in Section 3, this choice does not affect the splitting invariant, but it significantly affects its computation. Moreover, the root system G_2 exhibits a singular behaviour among all reduced root systems as far as pairs of strongly-orthogonal roots are concerned. Section 4 contains explicit computations of the splitting invariants of the tori $T_{\{\alpha\}}$ for all split almost-simple classical groups. In Section 5 we construct the aforementioned quotients of the cohomology groups and the embedding between them. Moreover we show that the endoscopic character factors through these quotients and is compatible with the constructed embedding.

1 Notation and Preliminaries

Throughout this paper G will stand for a split semi-simple simply-connected group over $\mathbb{R}$ and $(B_0, T_0, \{X_\alpha\})$ will be a splitting of G. We write $R = R(T_0, G)$ for the set of roots of T_0 in G, set $\alpha > 0$ if $\alpha \in R(T_0, B_0)$, denote by Δ the set of simple roots in $R(T_0, B_0)$ and by Ω the Weyl-group of R, which is identified with $N(T_0)/T_0$. Moreover, we put $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ and denote by σ both the non-trivial element in that group, as well as its action on T_0. The notation $g \in G$ will be shorthand for $g \in G(\mathbb{C})$, and $\mathrm{Int}(g)h = ghg^{-1}$.

1.1 sl_2-triples

For any $\alpha \in R(T_0, B_0)$ we have the coroot $\alpha^\vee \colon \mathbb{G}_m \to T_0$ and its differential $d\alpha^\vee \colon \mathbb{G}_a \to \mathrm{Lie}(T_0)$. We put $H_\alpha := d\alpha^\vee(1) \in \mathrm{Lie}(T_0)$. Given $X_\alpha \in \mathrm{Lie}(G)_\alpha$ nonzero, there exists a unique $X_{-\alpha} \in \mathrm{Lie}(G)_{-\alpha}$ so that $[H_\alpha, X_\alpha, X_{-\alpha}]$ is an sl_2-triple. The map

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto H_\alpha, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto X_\alpha, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto X_{-\alpha}$$

gives a homomorphism $sl_2 \to \mathrm{Lie}(G)$ which integrates to a homomorphism $\mathrm{SL}_2 \to G$ and one has

$$\begin{array}{ccc} sl_2 & \longrightarrow & \mathrm{Lie}(G) \\ \Big\downarrow{\scriptstyle \exp} & & \Big\downarrow{\scriptstyle \exp} \\ \mathrm{SL}_2 & \longrightarrow & G \end{array}$$

The image of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2$ under this homomorphism will be called $\left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\right)_{X_\alpha}$. Notice that $\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)_{X_\alpha} = \alpha^\vee(t)$.

Fact 1.1 *Let $\alpha, \beta \in R$ be such that $\alpha + \beta \notin R$ and $\alpha - \beta \notin R$. For any non-zero elements $X_\alpha \in \mathrm{Lie}(G)_\alpha$ and $X_\beta \in \mathrm{Lie}(G)_\beta$, the homomorphisms $\varphi_{X_\alpha}, \varphi_{X_\beta} \colon \mathrm{SL}_2 \to G$ given by X_α and X_β commute.*

Proof Since for any field k, $\mathrm{SL}_2(k)$ is generated by its two subgroups

$$\left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \;\middle|\; u \in k \right\}, \qquad \left\{ \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \;\middle|\; u \in k \right\},$$

it is enough to show that for any $u, v \in \mathbb{C}$, each of $\exp(uX_\alpha)$ and $\exp(uX_{-\alpha})$ commutes with each of $\exp(vX_\beta)$ and $\exp(vX_{-\beta})$. This follows from [16, 10.1.4] and our assumption on α, β. ■

1.2 Chevalley Bases

For $\alpha \in \Delta$ let $n_\alpha = \exp(X_\alpha)\exp(-X_{-\alpha})\exp(X_\alpha) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)_{X_\alpha}$. Given $\mu \in \Omega$ we have the lift $n(\mu) \in N(T_0)$ given by $n(\mu) = n_{\alpha_1} \cdots n_{\alpha_q}$, where $s_{\alpha_1} \cdots s_{\alpha_q} = \mu$ is any reduced expression (by [16, 11.2.9] this lift is independent of the choice of reduced expression). Notice $n(\mu) \in N(T_0)(\mathbb{R})$ since T_0 is split. Put $X_{\mu|\alpha} := \mathrm{Int}(n(\mu)) \cdot X_\alpha$. Then $X_{\mu|\alpha} \in \mathrm{Lie}(G)_{\mu\alpha}$ is a non-zero element.

Lemma 1.2 *If $\alpha, \alpha' \in \Delta$ and $\mu, \mu' \in \Omega$ are such that $\mu\alpha = \mu'\alpha'$, then we have in $\mathrm{Lie}(G)_{\mu\alpha}$ the equality*

$$X_{\mu'|\alpha'} = \prod_{\substack{\beta>0 \\ (\mu')^{-1}\beta<0 \\ \mu^{-1}\beta>0}} (-1)^{\langle \beta^\vee, \mu\alpha \rangle} \cdot X_{\mu|\alpha}.$$

Proof By [16, 11.2.11] the relation $(\mu')^{-1} \cdot \mu\alpha = \alpha'$ implies

$$X_{\alpha'} = \mathrm{Int}\big[n\big((\mu')^{-1} \cdot \mu \big) \big] X_\alpha$$

The claim now follows from [5, 2.1.A] and the following computation:

$$\begin{aligned} X_{\mu'|\alpha'} &= \mathrm{Int}\big(n(\mu') \big) X_{\alpha'} = \mathrm{Int}\big[n(\mu') n\big((\mu')^{-1}\mu \big) \big] X_\alpha \\ &= \mathrm{Int}\big[t\big(\mu', (\mu')^{-1}\mu \big) \cdot n(\mu) \big] X_\alpha = \mathrm{Int}\big[t\big(\mu', (\mu')^{-1}\mu \big) \big] X_{\mu|\alpha} \\ &= (\mu\alpha)\big(t(\mu', (\mu')^{-1}\mu) \big) \cdot X_{\mu|\alpha}. \end{aligned}$$

■

We see that while the "absolute value" of $X_{\mu|\alpha}$ only depends on the root $\mu \cdot \alpha$, its "sign" does depend on both μ and α.

Definition 1.3 For $\gamma \in R$, $\mu, \mu' \in \Omega$ put

$$\epsilon(\mu', \gamma, \mu) := \prod_{\substack{\beta>0 \\ (\mu')^{-1}\beta<0 \\ \mu^{-1}\beta>0}} (-1)^{\langle \beta^\vee, \gamma \rangle}.$$

With this definition we can reformulate the above lemma as follows.

Corollary 1.4 *If $\gamma \in R$ and $\mu, \mu' \in \Omega$ are such that $\mu^{-1}\gamma, (\mu')^{-1}\gamma \in \Delta$, then*

$$X_{\mu'|(\mu')^{-1}\gamma} = \epsilon(\mu', \gamma, \mu) \cdot X_{\mu|\mu^{-1}\gamma}.$$

If for each $\gamma \in R$ we choose $\mu_\gamma \in \Omega$ so that $\mu_\gamma^{-1}\gamma \in \Delta$, then $\{X_{\mu_\gamma|\mu_\gamma^{-1}\gamma}\}_{\gamma \in R}$ is a Chevalley system in the sense of [7, exp XXIII §6].

1.3 Cayley-Transforms

Let $\alpha \in R(T_0, B_0)$ and choose $X_\alpha \in \mathrm{Lie}(G)_\alpha(\mathbb{R}) - \{0\}$. Put

$$g_\alpha := \exp\Big(\frac{i\pi}{4}(X_\alpha + X_{-\alpha})\Big).$$

Then

$$\sigma(g_\alpha) = \exp\Big(-\frac{i\pi}{4}(X_\alpha + X_{-\alpha})\Big) = g_\alpha^{-1},$$
$$\sigma(g_\alpha)^{-1} \cdot g_\alpha = g_\alpha^2 = \exp\Big(\frac{i\pi}{2}(X_\alpha + X_{-\alpha})\Big).$$

We have

$$g_\alpha = \left[\frac{\sqrt{2}}{2}\begin{pmatrix}1 & i \\ i & 1\end{pmatrix}\right]_{X_\alpha}, \quad g_\alpha^2 = \begin{pmatrix}0 & i \\ i & 0\end{pmatrix}_{X_\alpha}, \quad g_\alpha^4 = \begin{pmatrix}-1 & 0 \\ 0 & -1\end{pmatrix}_{X_\alpha} = \alpha^\vee(-1).$$

Fact 1.5 *The images of T_0 under $\mathrm{Int}(g_\alpha)$ and $\mathrm{Int}(g_\alpha^{-1})$ are the same. They are a torus T defined over $\mathbb{R}$ and the transports of the Γ-action on T to T_0 via $\mathrm{Int}(g_\alpha^{-1})$ and $\mathrm{Int}(g_\alpha)$ both equal $s_\alpha \rtimes \sigma$.*

Proof

$$\mathrm{Int}(g_\alpha)T_0 = \mathrm{Int}(g_\alpha^{-1})\,\mathrm{Int}(g_\alpha^2)T_0 = \mathrm{Int}(g_\alpha^{-1})s_\alpha T_0 = \mathrm{Int}(g_\alpha^{-1})T_0,$$
$$\sigma(\mathrm{Int}(g_\alpha)T_0) = \mathrm{Int}(\sigma(g_\alpha))T_0 = \mathrm{Int}(g_\alpha^{-1})T_0,$$
$$\mathrm{Int}(\sigma(g_\alpha)^{-1}g_\alpha) = \mathrm{Int}(g_\alpha^2) = s_\alpha = \mathrm{Int}(g_\alpha^{-2}) = \mathrm{Int}(\sigma(g_\alpha)g_\alpha^{-1}).$$
■

Different choices of X_α will lead to different (yet conjugate) tori T. However, since we have fixed a splitting, there is up to a sign a canonical X_α. Changing the sign of X_α changes g_α to g_α^{-1}, hence T does not change. Thus we conclude that the the choice of a splitting gives for each $\alpha \in R(T_0, B_0)$ the following canonical data:

(i) a pair $\{X, X'\} \subseteq \mathrm{Lie}(G)_\alpha(\mathbb{R}) - \{0\}$ with $X' = -X$,
(ii) a torus T_α on which Γ acts via $s_\alpha \rtimes \sigma$,
(iii) a pair φ, φ' of isomorphisms $T_0^{s_\alpha} \to T_\alpha$ such that $\varphi' = \varphi \circ s_\alpha$, given by the Cayley-transforms with respect to X, X'.

Corollary 1.6 *For $\alpha \in R(T_0, B_0)$, let T_α be the canonically given torus as above. For $\mu, \mu' \in \Omega$ such that $\mu^{-1}\alpha, (\mu')^{-1}\alpha \in \Delta$, let $\varphi, \varphi' : T_0^{s_\alpha} \to T_\alpha$ be the isomorphisms given by* $\mathrm{Int}(g_{X_{\mu|\mu^{-1}\alpha}})$ *and* $\mathrm{Int}(g_{X_{\mu'|(\mu')^{-1}\alpha}})$. *Then*

$$\varphi' = \begin{cases} \varphi, & \epsilon(\mu', \alpha, \mu) = 1, \\ \varphi \circ s_\alpha, & \epsilon(\mu', \alpha, \mu) = -1. \end{cases}$$

Proof Clear. ■

From now on we will write $g_{\mu,\alpha}$ instead of $g_{X_{\mu|\mu^{-1}\alpha}}$. This notation will only be employed in the case that $\alpha \in R(T_0, B_0)$ and $\mu^{-1}\alpha \in \Delta$.

2 Strongly Orthogonal Subsets of Root Systems

In this section, a few technical facts about strongly orthogonal subsets of root systems are proved.

Definition 2.1

- $\alpha, \beta \in R$ are called *strongly orthogonal* if $\alpha + \beta \notin R$ and $\alpha - \beta \notin R$.
- $A \subset R$ is called a *strongly orthogonal subset* (SOS) if it consists of pairwise strongly orthogonal roots.
- $A \subset R$ is called a *maximal strongly orthogonal subset* (MSOS) if it is an SOS and is not properly contained in an SOS.

A classification of the Weyl group orbits of MSOS in irreducible root systems was given in [1]. In some cases, there exists more than one orbit. To handle these cases, we will use the following definition and lemma.

Definition 2.2 Let A_1,A_2 be SOS in R. Then A_2 will be called *adapted to* A_1 if $\mathrm{span}(A_2) \subset \mathrm{span}(A_1)$ and for all distinct $\alpha, \beta \in A_2$

$$\{a \in A_1 : (a, \alpha) \neq 0\} \cap \{a \in A_1 : (a, \beta) \neq 0\} = \varnothing,$$

where $(\cdot\,,\cdot)$ is any Ω-invariant scalar product on the real vector space spanned by R.

Note that any A is adapted to itself.

Lemma 2.3 *There exist representatives $A_1, \dots, A_k$ of the Weyl group orbits of MSOS such that A_1 has maximal length and $A_2, \dots, A_k$ are adapted to A_1.*

Proof This follows from the explicit classification in [1]. ■

If A is an SOS, then all reflections with respect to elements in A commute. Their product will be denoted by S_A.

Definition 2.4 For a root system R, a choice of positive roots $>$, and a subset A of R, let $\#(R, >, A)$ be the statement

$$\forall \alpha_1, \alpha_2 \in A,\ \forall \beta > 0, \quad \alpha_1 \neq \alpha_2\ \wedge s_{\alpha_1}(\beta) < 0 \Longrightarrow s_{\alpha_2}(\beta) > 0$$

and let $\#\#(R, >, A)$ be the statement

$$\begin{aligned} &\forall A_1, A_2 \subset A,\ \forall \beta > 0, \\ &\quad A_1 \cap A_2 = \varnothing\ \wedge\ S_{A_1}(\beta) < 0 \Longrightarrow S_{A_2}(\beta) > 0\ \wedge\ S_{A_1} S_{A_2}(\beta) < 0 \end{aligned}$$

We will soon show that these statements are equivalent. Moreover we will show that for any SOS $A \subset R$ we can choose $>$ so that the triple $(R, >, A)$ verifies these statements. For this it is more convenient to work with #. For the applications however, we need ##.

Lemma 2.5 *Let R be a reduced root system and $A \subset R$ an SOS. There exists a choice of positive roots $>$ such that $\#(R, >, A')$ holds for any A' adapted to A.*

Proof Let V denote the real vector space spanned by R, and let $(\cdot\,,\cdot)$ be an Ω-invariant scalar product on V. The elements of A are orthogonal with respect to $(\cdot\,,\cdot)$. Extend A to an orthogonal basis $(a_1, \dots, a_n)$ of V. Define the following notion of positivity on R

$$\alpha > 0 \Longleftrightarrow (\alpha, a_{i_0}) > 0 \text{ for } i_0 = \min\{i : (\alpha, a_i) \neq 0\}.$$

It is clear from the construction that with this notion $\#(R, >, A')$ is satisfied for any A' adapted to A. We just need to check that > 0 defines a choice of positive roots, which we will now do.

It is clear that for each $\alpha \in R$ precisely one of $\alpha > 0$ or $-\alpha > 0$ is true. We will construct $p \in V$ such that for all $\alpha \in R$, $\alpha > 0 \Longleftrightarrow (\alpha, p) > 0$. Let

$$\begin{aligned} m &= \min\{|(\alpha, a_i)| : \alpha \in R, 1 \leq i \leq n, (\alpha, a_i) \neq 0\}, \\ M &= \max\{|(\alpha, a_i)| : \alpha \in R, 1 \leq i \leq n\}. \end{aligned}$$

Construct recursively real numbers $p_1, \dots, p_n$ such that

$$p_n = 1, \quad p_i > \frac{M}{m} \sum_{k>i} p_k,$$

and put $p = \sum p_i a_i$. If $\alpha \in R$ is such that $\alpha > 0$ and i_0 is the smallest i such that $(\alpha, a_i) \neq 0$, then

$$(\alpha, p) = \sum_{i=i_0}^{n} p_i(\alpha, a_i) > mp_{i_0} - M \sum_{k>i_0} p_k > 0.$$

Thus $\alpha > 0 \Longrightarrow (\alpha, p) > 0$. The converse implication follows formally:

$$\neg(\alpha > 0) \Leftrightarrow -\alpha > 0 \Rightarrow (-\alpha, p) > 0 \Rightarrow \neg((\alpha, p) > 0). \qquad \blacksquare$$

The truth value of the statement $\#(R, >, A)$ and the notion of being adapted to A are unchanged if one replaces elements of A by their negatives. Thus we can always assume that the elements of A are positive.

It is necessary to choose the set of positive roots based on A in order for $\#(R, >, A)$ to be true. An example that $\#(R, >, A)$ may be false is provided by $V = \mathbb{R}^3$, $R = D_3$ with positive roots

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

and

$$A = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad \beta = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Fact 2.6 *Let $R = G_2$ and $>$ be any choice of positive roots. All MSOS A of R lie in the same Weyl-orbit and moreover automatically satisfy $\#(R, >, A)$. Some of these A contain simple roots.*

Proof This is an immediate observation. $\blacksquare$

Proposition 2.7 *Let $A \subset R$ be an SOS and let $>$ be a choice of positive roots. Then the statements $\#(R, >, A)$ and $\#\#(R, >, A)$ are equivalent.*

Proof First, we show that # implies the following statement, to be called $\#_1$:

$$(\#_1) \qquad \forall \alpha_1, \alpha_2 \in A,\ \forall \beta > 0,$$
$$\alpha_1 \neq \alpha_2 \ \wedge\ s_{\alpha_1}(\beta) < 0 \Longrightarrow s_{\alpha_2}(\beta) > 0 \ \wedge\ s_{\alpha_1} s_{\alpha_2}(\beta) < 0.$$

Let $\alpha_1, \alpha_2 \in A$ and $\beta > 0$ be such that $s_{\alpha_1}(\beta) < 0$. Put $\beta' = -s_{\alpha_1}(\beta)$. Then $\beta' > 0$ and $s_{\alpha_1}(\beta') = -\beta < 0$. Then # implies that $s_{\alpha_2} s_{\alpha_1}(\beta) = -s_{\alpha_2}(\beta') < 0$.

Next we show that $\#_1$ implies the following statement, to be called $\#_2$:

$$(\#_2) \qquad \forall \alpha_1 \in A,\ \forall A_2 \subset A,\ \forall \beta > 0,$$
$$\alpha_1 \notin A_2 \ \wedge\ s_{\alpha_1}(\beta) < 0 \Longrightarrow S_{A_2}(\beta) > 0 \ \wedge\ s_{\alpha_1} S_{A_2}(\beta) < 0.$$

We do this by induction of the cardinality of A_2, the case of A_2 singleton being precisely $\#_1$. Now let $\alpha_1 \in A, A_2 \subset A \setminus \{\alpha_1\}$, and $\beta > 0$ be such that $s_{\alpha_1}(\beta) < 0$. Choose $\alpha_2 \in A_2$ and put $\beta' := s_{\alpha_2}(\beta)$. Then by $\#_1$ we have $\beta' > 0$ and $s_{\alpha_1}(\beta') < 0$. Applying the inductive hypothesis, we obtain $S_{A_2}(\beta) = S_{A_2\setminus\{\alpha_2\}}(\beta') > 0$ and $s_{\alpha_1} S_{A_2}(\beta) = s_{\alpha_1} S_{A_2\setminus\{\alpha_2\}}(\beta') < 0$.

Now we show that $\#_2$ implies the following statement, to be called $\flat$:

($\flat$) If $A_2 \subset A$ and $\beta > 0$ are such that $S_{A_2}(\beta) < 0$, then there exists $\alpha_2 \in A_2$ such that $s_{\alpha_2}(\beta) < 0$.

To see this, let $A_3 \subset A_2$ be a subset of minimal size such that $S_{A_3}(\beta) < 0$. Take $\alpha_3 \in A_3$ and put $\beta' = S_{A_3\setminus\{\alpha_3\}}(\beta)$. By minimality of A_3 we have $\beta' > 0$, and moreover $s_{\alpha_3}(\beta') = S_{A_3}(\beta) < 0$. Then $\#_2$ implies that $s_{\alpha_3}(\beta) = s_{\alpha_3} S_{A_3\setminus\{\alpha_3\}}(\beta') < 0$.

Finally we show that $\#_2$ implies the statement ##. Take $A_1, A_2 \subset A$ such that $A_1 \cap A_2 = \varnothing$ and $\beta > 0$ such that $S_{A_1}(\beta) < 0$. By $\flat$ there exists $\alpha_1 \in A_1$ such that $s_{\alpha_1}(\beta) < 0$. Since $\alpha_1 \notin A_2$, we get from $\#_2$ that $S_{A_2}(\beta) > 0$ and $S_{A_1} S_{A_2}(\beta) = s_{\alpha_1} S_{A_2} S_{A_1\setminus\{\alpha_1\}}(\beta) < 0$.

This shows that # implies ##. The converse implication is trivial. ■

Proposition 2.8 *For an SOS $A \subset R$, and a choice $>$ of positive roots, let*

$$R_A^+ = \{\beta \in R : \beta > 0 \ \wedge \ S_A\beta < 0\}.$$

Assume that $>$ is chosen so that ##$(R, >, A)$ is true. Then if $A', A'' \subset A$ are disjoint, so are $R_{A'}^+$ and $R_{A''}^+$, and $R_{A'\cup A''}^+ = R_{A'}^+ \cup R_{A''}^+$. Moreover, the action of $S_{A'}$ on R preserves $R_{A''}^+$.

Proof This follows immediately. ■

Corollary 2.9 *If A is an SOS and $>$ is chosen so that #$(R, >, A)$ is true, then*

$$R_A^+ = \coprod_{\alpha\in A} R_\alpha^+.$$

Proof Clear. ■

Lemma 2.10 *Let R be a root system, V the real vector space spanned by it, $Q \subset V$ the root lattice, and $(\,\cdot\,,\,\cdot\,)$ a Weyl-invariant scalar product on V. If $v \in Q$ is such that*

$$|v| \le \min\{|\alpha| : \ \alpha \in R\},$$

where $|\cdot|$ is the Euclidean norm arising from $(\,\cdot\,,\,\cdot\,)$, then $v \in R$ and the above inequality is an equality.

Proof Choose a presentation $v = \sum_{\alpha\in R} n_\alpha \alpha$, $n_\alpha \in \mathbb{Z}_{\ge 0}$ such that $\sum_\alpha n_\alpha$ is minimal. First we claim that if $\alpha, \beta \in R$ contribute to this sum, then $(\alpha, \beta) \ge 0$. If that were not the case, then by [2, Chapter VI,§1,no.3,Theorem 1] we have that $\gamma := \alpha + \beta \in R \cup \{0\}$ and we can replace the contribution $\alpha + \beta$ in the sum by γ, contradicting its minimality. Now, if $\gamma \in R$ is any root contributing to the sum, we get

$$|v|^2 = \sum_{\alpha,\beta\in R} n_\alpha n_\beta(\alpha,\beta) \geq n_\gamma^2(\gamma,\gamma) \geq (\gamma,\gamma) = |\gamma|^2$$

with equality precisely when $v = \gamma$. ■

Lemma 2.11 *Let R be a reduced root system and $\alpha, \beta \in R$ two strongly orthogonal roots. If $\alpha^\vee + \beta^\vee \in 2Q^\vee$, then α, β belong to the same copy of G_2.*

Proof Let V denote the real vector space spanned by R. Choose a Weyl-invariant scalar product $(\cdot,\cdot)$ and use it to identify V with its dual and regard $R^\vee$ as a root system in V.

Assume now that $\alpha^\vee + \beta^\vee \in 2Q^\vee$. Note that $\alpha^\vee$ and $\beta^\vee$ are orthogonal (but may not be strongly orthogonal elements of $R^\vee$).

First we show that then α, β belong to the same irreducible piece of R. To that end, assume that R decomposes as $R = R_1 \sqcup R_2$ and V decomposes accordingly as $V_1 \oplus V_2$. If $\alpha \in R_1$ and $\beta \in R_2$, then $\alpha^\vee \in V_1$ and $\beta^\vee \in V_2$. Then $\frac{1}{2}(\alpha^\vee + \beta^\vee) \in Q^\vee$ implies $\frac{1}{2}\alpha^\vee \in Q_1^\vee, \frac{1}{2}\beta^\vee \in Q_2^\vee$ (project orthogonally onto V_1, resp. V_2). This however contradicts the above lemma, because $\frac{1}{2}\alpha^\vee$ has length strictly less than the shortest elements in $R_1^\vee$.

Knowing that α, β lie in the same irreducible piece, we can now assume, without loss of generality, that R is irreducible. Normalize $(\cdot,\cdot)$ so that the short roots in R have length 1. We have the following cases:

- All elements of R have length 1. Then all elements of $R^\vee$ have length 2. The length of $\frac{1}{2}(\alpha^\vee + \beta^\vee)$ is $\sqrt{2}$, which by the above lemma is not a length of an element in $Q^\vee$.
- R contains elements of lengths 1 and $\sqrt{2}$. Then $R^\vee$ contains elements of lengths $\sqrt{2}$ and 2.
  - If both $\alpha^\vee, \beta^\vee$ have length $\sqrt{2}$, then $\frac{1}{2}(\alpha^\vee + \beta^\vee)$ has length 1, so is not in $Q^\vee$.
  - If $\alpha^\vee$ has length $\sqrt{2}$ and $\beta^\vee$ has length 2, then $\frac{1}{2}(\alpha^\vee + \beta^\vee)$ has length $\frac{\sqrt{6}}{2}$, so again is not in $Q^\vee$.
  - If both $\alpha^\vee, \beta^\vee$ have length 2, then $\frac{1}{2}(\alpha^\vee + \beta^\vee)$ has length $\sqrt{2}$ and thus could potentially be in $Q^\vee$. If it is, then by the above lemma it is also in $R^\vee$, so $\frac{1}{2}(\alpha^\vee + \beta^\vee)^\vee$ must be an element of R. One immediately computes that $[\frac{1}{2}(\alpha^\vee + \beta^\vee)]^\vee = \alpha + \beta$, but the latter is not an element of R because α, β are strongly orthogonal.
- R has elements of lengths 1 and $\sqrt{3}$. Then R is G_2 and $R^\vee$ is also G_2. As one sees immediately, up to the action of its Weyl-group, G_2 has a unique pair of orthogonal roots which are then automatically strongly orthogonal and half their sum is also a root. ■

3 Splitting Invariants

Recall that we have fixed a split semi-simple and simply-connected group G over $\mathbb{R}$ and a splitting $(T_0, B_0, \{X_\alpha\})$ of it. Given a maximal torus T, an element $h \in G$ such that $\mathrm{Int}(h)T_0 = T$, and a-data $\{a_\beta\}$ for $R(T,G)$, Langlands and Shelstad constructed [5, 2.3] a certain element of $Z^1(\Gamma, T)$, whose image in $H^1(\Gamma, T)$ they call $\lambda(T)$, the "splitting invariant" of T. They show that this image is independent of the choice of h. The introduction of this invariant was motivated by a (general) calculation in Langlands' paper [4], which was the basis for [5, 5.4]. It was also used by Shelstad to give an explicit formula for regular unipotent germs of p-adic groups [12].

In this section we want to study this splitting invariant in such a way that enables us to see how it varies when the torus varies. It turns out that a certain type of a-data is very well suited for this. This a-data is determined by a Borel subgroup $B \supset T$ as follows:

$$\alpha_\beta = \begin{cases} i, & \beta > 0 \wedge \sigma_T(\beta) < 0, \\ -i, & \beta < 0 \wedge \sigma_T(\beta) > 0, \\ 1, & \beta > 0 \wedge \sigma_T(\beta) > 0, \\ -1, & \beta < 0 \wedge \sigma_T(\beta) < 0, \end{cases}$$

where σ_T denotes the Galois-action on $X^*(T)$ and $\beta > 0$ means $\beta \in R(T,B)$. We will call this B-a-data. We would like to alert the reader to a similar, yet inequivalent, terminology — that of *based* a-data — which was introduced by Shelstad and is also specified by a choice of a Borel subgroup. For based a-data the positive imaginary roots are assigned i, while all other positive roots are assigned 1; for B-a-data, any positive root whose Galois-conjugate is negative is assigned i. Therefore, a splitting invariant computed using based a-data will, in general, be different from one computed using B-a-data. The precise difference is given by [5, 2.3.2]. It is, however, more important to note that according to [5, Lemma 3.2.C], this difference disappears once the splitting invariant has been paired with an endoscopic character. Thus, as far as applications to transfer factors are concerned, based a-data and B-a-data give the same result.

In view of the reduction theorem which we will prove in section 3.2, it will be helpful to consider not just the cohomology class, but also the actual cocycle constructed in [5, 2.3]. We will denote this cocycle by $\lambda(T,B,h)$ to record its dependence on the B-a-data and the element h, while the splitting $(T_0, B_0, \{X_\alpha\})$ is not present in the notation because it is assumed fixed. Since we are working over $\mathbb{R}$, we will identify a 1-cocycle and its value at $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$, and hence we will view $\lambda(T,B,h)$ as an element of T. Given h, there is an obvious choice for B, namely $\mathrm{Int}(h)B_0$. We will write $\lambda(T,h)$ for $\lambda(T, \mathrm{Int}(h)B_0, h)$. Note that in this notation, T is clearly redundant, because it equals $\mathrm{Int}(h)T_0$. However, we keep it so that the notation is close to that in [5]. We would like to alert the reader to one potential confusion: while the cohomology class of $\lambda(T,B,h)$ is independent of the choice of h, that of $\lambda(T,h)$ is not, because in the latter h influences not only the identification of T_0 with T, but also the choice of B-a-data for T.

3.1 The Splitting Invariant for T_α

Recall from Section 1.3 that for each $\alpha \in R(T_0, B_0)$ there is a canonical maximal torus T_α and a pair of isomorphisms $T_0^{s_\alpha} \to T_\alpha$. To fix one of the two, fix $\mu \in \Omega$ such that $\mu^{-1}\alpha \in \Delta$. Then $\mathrm{Int}(g_{\mu,\alpha})$ is one of the two isomorphisms $T_0^{s_\alpha} \to T_\alpha$. The goal of this section is to compute $\lambda(T_\alpha, B, g_{\mu,\alpha})$ for a given Borel $B \supset T_\alpha$, and in particular $\lambda(T_\alpha, g_{\mu,\alpha})$. We will give a formula for the latter in purely root-theoretic terms.

Lemma 3.1 *With $g := g_{\mu,\alpha}$ we have*

$$\lambda(T_\alpha, B, g) = \mathrm{Int}(g)\big(\alpha^\vee(i \cdot a_{\alpha\circ \mathrm{Int}(g^{-1})}) \cdot s_\alpha(\sigma(\delta))\delta^{-1}\big),$$

where

$$\delta = \prod_{\substack{\beta>0 \\ \mu^{-1}\beta<0}} \beta^\vee(a_{\beta\circ\mathrm{Int}(g^{-1})})^{-1},$$

and σ denotes complex conjugation on T_0.

Proof Put $u = n(\mu)$. We will first compute the cocycle $\lambda(T_\alpha, B, gu)$. The notation will be as in [5, 2.3]. The pullback of the Γ-action on T_α to T_0 via gu differs from σ by

$$\begin{aligned}\omega_{T_\alpha}(\sigma) &:= \mathrm{Int}((gu)^{-1}\sigma(gu)) = \mathrm{Int}\big(n(\mu)^{-1}g^{-1}\sigma(g)n(\mu)\big) = \mu^{-1}s_\alpha\mu \\ &= s_{\mu^{-1}\alpha}.\end{aligned}$$

Using that $\mu^{-1}\alpha$ is simple, we compute the three factors of $\mathrm{Int}(gu)^{-1}\lambda(T_\alpha, B, gu)$:

$$\begin{aligned}x(\sigma) &= \prod_{\substack{\beta>0 \\ \omega_{T_\alpha}(\sigma)\beta<0}} \beta^\vee(a_{\beta\circ\mathrm{Int}(gu)^{-1}}) \\ &= (\mu^{-1}\alpha)^\vee(a_{\mu^{-1}\alpha\circ\mathrm{Int}(u^{-1}g^{-1})}) \\ &= \mathrm{Int}(u^{-1})(\alpha^\vee(a_{\alpha\circ\mathrm{Int}(g^{-1})})) \\ n(\omega_{T_\alpha}(\sigma)) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{X_{\mu^{-1}\alpha}} = \mathrm{Int}(u^{-1})\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{X_{\mu|\mu^{-1}\alpha}} \\ \sigma(gu)^{-1}(gu) &= \mathrm{Int}(u^{-1})g^2 = \mathrm{Int}(u^{-1})\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}_{X_{\mu|\mu^{-1}\alpha}}\end{aligned}$$

Thus, $\lambda(T_\alpha, B, gu) = \mathrm{Int}(gu)\,\mathrm{Int}(u^{-1})(\alpha^\vee(a_{\alpha\circ\mathrm{Int}(g^{-1})})\alpha^\vee(i))$. From the proofs of [5, 2.3.A] and [5, 2.3.B] one sees that

$$\lambda(T_\alpha, B, gu) = \mathrm{Int}(g)(\delta\sigma_{T_\alpha}(\delta)^{-1}) \cdot \lambda(T_\alpha, B, g),$$

where σ_{T_α} is the transport of the action of complex conjugation on T_α to T_0 via g. This action is $s_\alpha \rtimes \sigma$. Notice that the term $\lambda^{-1}\sigma_T(\lambda)$ appearing in the proof of [5, 2.3.A] is trivial, since for us $u = n(\mu)$ and hence $\lambda = 1$. The claim now follows. ∎

Before we turn to the computation of $s_\alpha(\sigma(\delta))\delta^{-1}$ we will need to take a closer look at the following set.

Definition 3.2 For $\alpha > 0$ put $R_\alpha^+ = \{\beta \in R \mid \beta > 0 \wedge s_\alpha(\beta) < 0\}$.

Lemma 3.3 *Let $\alpha > 0$ and $\mu \in \Omega$ be such that $\mu^{-1}\alpha \in \Delta$. Then the sets*

$$\{\beta \in R \mid \beta > 0 \wedge s_\alpha(\beta) < 0 \wedge \mu^{-1}\beta < 0\},$$
$$\{\beta \in R \mid \beta > 0 \wedge s_\alpha(\beta) < 0 \wedge \mu^{-1}\beta > 0 \wedge \beta \neq \alpha\},$$

are disjoint and their union is $R_\alpha^+ - \{\alpha\}$. The map $\beta \mapsto -s_\alpha(\beta)$ is an involution on $R_\alpha^+ - \{\alpha\}$ which interchanges the above two sets.

Proof Every β in the first set satisfies $\beta \neq \alpha$ because $\mu^{-1}\alpha$ is positive. Hence the first set lies in $R_\alpha^+ - \{\alpha\}$ and clearly the second does also. The fact that the two are disjoint and cover $R_\alpha^+ - \{\alpha\}$ is obvious. Now to the bijection. Let β be an element in the first set, and consider $\tilde{\beta} = -s_\alpha(\beta)$. We have

$$\beta \neq \alpha \Rightarrow \tilde{\beta} \neq \alpha, \quad s_\alpha(\beta) < 0 \Rightarrow \tilde{\beta} > 0, \quad \beta > 0 \Rightarrow s_\alpha\tilde{\beta} = -\beta < 0,$$
$$\mu^{-1}\beta < 0 \Rightarrow \mu^{-1}\tilde{\beta} = \mu^{-1}s_\alpha(-\beta) = s_{\mu^{-1}\alpha}(-\mu^{-1}\beta) > 0,$$

where the last inequality holds because $\mu^{-1}\alpha$ is simple and

$$\beta > 0 \Rightarrow \beta \neq -\alpha \Rightarrow -\mu^{-1}\beta \neq \mu^{-1}\alpha$$ ∎

A similar observation appears in [6, §4.3].

Lemma 3.4 *We have*

$$s_\alpha(\sigma(\delta))\delta^{-1} = \prod_{\substack{\beta \in R_\alpha^+ \\ \mu^{-1}\beta < 0}} \left[\beta^\vee\left(a_{\beta\circ\mathrm{Int}(g^{-1})}\right)\ s_\alpha\beta^\vee\left(a_{s_\alpha\beta\circ\mathrm{Int}(g^{-1})}\right)^{-1}\right].$$

Proof According to the proof of [5, 2.3.B part (a)], the contributions to $\delta s_\alpha(\sigma(\delta))^{-1}$ are as follows:

$D_1 = \{\beta \mid \beta > 0 \wedge \mu^{-1}\beta < 0 \wedge s_\alpha\beta < 0\} : \beta^\vee(a_{\beta\circ\mathrm{Int}(g^{-1})})^{-1}$,
$D_2 = \{\beta \mid \beta < 0 \wedge \mu^{-1}\beta < 0 \wedge \mu^{-1}s_\alpha\beta < 0 \wedge s_\alpha\beta > 0\} : \beta^\vee(a_{\beta\circ\mathrm{Int}(g^{-1})})$,
$D_3 = \{\beta \mid \beta > 0 \wedge \mu^{-1}\beta < 0 \wedge s_\alpha\beta > 0 \wedge \mu^{-1}s_\alpha\beta > 0\} : \beta^\vee(a_{\beta\circ\mathrm{Int}(g^{-1})})^{-1}$,
$D_4 = \{\beta \mid \mu^{-1}\beta > 0 \wedge s_\alpha\beta > 0 \wedge \mu^{-1}s_\alpha\beta < 0\} : \beta^\vee(a_{\beta\circ\mathrm{Int}(g^{-1})})$.

We will use $\mu^{-1}s_\alpha(\beta) = s_{\mu^{-1}\alpha}(\mu^{-1}\beta)$ and the fact that $\mu^{-1}\alpha$ is simple to show that the last two sets are empty. In set D_3, the conditions $\mu^{-1}\beta < 0$ and $\mu^{-1}s_\alpha\beta > 0$ imply $\mu^{-1}\beta = -\mu^{-1}\alpha$, *i.e.*, $\beta = -\alpha$, which contradicts $\beta > 0$. In set D_4, the conditions $\mu^{-1}\beta > 0$ and $\mu^{-1}s_\alpha\beta < 0$ imply $\beta = \alpha$. Since $\alpha > 0$, this contradicts $s_\alpha\beta > 0$.

Next we claim $D_2 = s_\alpha(D_1)$. We have

$$\mu^{-1}\beta < 0 \wedge \mu^{-1}s_\alpha\beta < 0 \Leftrightarrow \mu^{-1}\beta < 0 \wedge \mu^{-1}\beta \neq -\mu^{-1}\alpha$$

from which we get

$$D_2 = \{-\beta | \beta > 0 \wedge s_\alpha\beta < 0 \wedge \mu^{-1}\beta > 0 \wedge \beta \neq \alpha\}$$

Now $D_2 = s_\alpha(D_1)$ follows from Lemma 3.3.

From these considerations it follows that

$$\delta s_\alpha(\sigma(\delta))^{-1} = \prod_{\beta\in D_1} \beta^\vee(a_{\beta\circ \mathrm{Int}(g^{-1})})^{-1} \prod_{\beta\in D_2} \beta^\vee(a_{\beta\circ\mathrm{Int}(g^{-1})})$$
$$= \prod_{\beta\in D_1} [\beta^\vee(a_{\beta\circ\mathrm{Int}(g^{-1})})^{-1}\, s_\alpha\beta^\vee(a_{s_\alpha\beta\circ\mathrm{Int}(g^{-1})})]. \qquad \blacksquare$$

Let us recall our notation: $\alpha \in R$ is any positive root, $\mu \in \Omega$ is such that $\mu^{-1}\alpha \in \Delta$, and $g = g_{\mu,\alpha}$ is the Cayley-transform corresponding to $X_{\mu|\mu^{-1}\alpha}$.

From Lemmas 3.1 and 3.4 we immediately get the following.

Corollary 3.5

$$\lambda(T_\alpha, B, g) =$$
$$\mathrm{Int}(g)\Big(\alpha^\vee(ia_{\alpha\circ\mathrm{Int}(g^{-1})})\cdot \prod_{\substack{\beta\in R_\alpha^+\\ \mu^{-1}\beta<0}} \left[\beta^\vee\left(a_{\beta\circ\mathrm{Int}(g^{-1})}\right)\, s_\alpha\beta^\vee\left(a_{s_\alpha\beta\circ\mathrm{Int}(g^{-1})}\right)^{-1}\right]\Big).$$

In the case $B = \mathrm{Int}(g)B_0$ this formula becomes simpler.

Corollary 3.6

$$\lambda(T_\alpha, g_{\mu,\alpha}) = \mathrm{Int}(g_{\mu,\alpha})\Big(\alpha^\vee(-1)\cdot \prod_{\substack{\beta\in R_\alpha^+\\ \mu^{-1}\beta<0}} (\beta^\vee\cdot s_\alpha\beta^\vee)(i)\Big).$$

Definition 3.7 Put

$$\rho(\mu,\alpha) := \alpha^\vee(-1)\cdot \prod_{\substack{\beta\in R_\alpha^+\\ \mu^{-1}\beta<0}} (\beta^\vee\cdot s_\alpha\beta^\vee)(i) \quad \in T_0.$$

By Corollary 3.6 and the work of [5, 2.3] we know that $\rho(\mu,\alpha) \in Z^1(\Gamma, T_0^{s_\alpha})$ and $\mathrm{Int}(g_{\mu,\alpha})\rho(\mu,\alpha) = \lambda(T_\alpha, g_{\mu,\alpha})$.

Proposition 3.8

(i) $\rho(\mu,\alpha) = \prod_{\beta\in R_\alpha^+} \beta^\vee(i) n(s_\alpha) g_{\mu,\alpha}^2$.

(ii) $s_\alpha\rho(\mu,\alpha) = \rho(\mu,\alpha)$, $\sigma(\rho(\mu,\alpha)) = \rho(\mu,\alpha)^{-1}$.

(iii) *The image of $\rho(\mu,\alpha)$ under the two canonical isomorphisms $T_0^{s_\alpha} \to T_\alpha$ is the same.*

(iv) *If $\mu' \in \Omega$ is another Weyl-element such that $(\mu')^{-1}\alpha \in \Delta$, then*

$$\rho(\mu',\alpha) = \alpha^\vee(\epsilon(\mu',\alpha,\mu))\rho(\mu,\alpha).$$

In particular,

$$\lambda(T_\alpha, g_{\mu',\alpha}) = \lambda(T_\alpha, g_{\mu,\alpha})\cdot \mathrm{Int}(g_{\mu,\alpha})[\alpha^\vee(\epsilon(\mu',\alpha,\mu))].$$

Proof The first point follows from Corollary 3.6, because the right-hand side is by construction $\mathrm{Int}(g_{\mu,\alpha})\lambda(T_\alpha, g_{\mu,\alpha})$. The second point is evident from the structure of ρ. The third point is now clear because, as remarked in Section 1.3, the two canonical isomorphisms differ by precomposition with s_α.

For the last point, $\rho(\mu',\alpha) = \prod_{\beta\in R^+_\alpha} \beta^\vee(i)n(s_\alpha)g^2_{\mu',\alpha}$. If $\epsilon(\mu',\alpha,\mu) = +1$, then $g_{\mu',\alpha} = g_{\mu,\alpha}$ and the statement is clear. Assume now that $\epsilon(\mu',\alpha,\mu) = -1$. Then $g_{\mu',\alpha} = g^{-1}_{\mu,\alpha}$. We see

$$\rho(\mu',\alpha) = \prod_{\beta\in R^+_\alpha} \beta^\vee(i)n(s_\alpha)g^2_{\mu,\alpha}g^{-4}_{\mu,\alpha}.$$

But $g^{-4}_{\mu,\alpha} = \alpha^\vee(-1)$, hence the claim. ■

3.2 The Splitting Invariant for T_A

Fact 3.9 *Let A be an SOS in R. Consider the set of automorphisms of G given by*

$$\left\{ \mathrm{Int}(g) \mid g = \prod_{\alpha\in A} g_{\mu_\alpha,\alpha}, \mu_\alpha \in \Omega, \mu_\alpha^{-1}\alpha \in \Delta \right\}.$$

The image of T_0 under any element of that set is the same; call it T_A. Then any element of that set induces an isomorphism of real tori $T_0^{S_A} \to T_A$.

Proof Let $\mathrm{Int}(g_1), \mathrm{Int}(g_2)$ be elements of the above set, with $g_i = \prod_{\alpha\in A} g_{\mu^i_\alpha,\alpha}$, and let $A' \subset A$ be the subset of those α such that $g_{\mu^1_\alpha,\alpha} \neq g_{\mu^2_\alpha,\alpha}$. For those α we then have $g_{\mu^1_\alpha,\alpha} = g^{-1}_{\mu^2_\alpha,\alpha}$, hence $\mathrm{Int}(g_1g_2^{-1})|_{T_0} = \prod_{\alpha\in A'} g^2_{\mu^1_\alpha,\alpha}|_{T_0} = S_{A'}$ which normalizes T_0. This shows that the images of T_0 under these two automorphisms are the same. Moreover, the transport of the Γ-action on T_A to T_0 via $\mathrm{Int}(g_1^{-1})$ differs from σ by $\mathrm{Int}(\sigma(g_1^{-1})g_1)|_{T_0} = \mathrm{Int}(g_1^2)|_{T_0} = S_A$. ■

Definition 3.10 For an SOS $A \subset R$, we will call the set

$$\left\{ \mathrm{Int}(g)|_{T_0} \mid g = \prod_{\alpha\in A} g_{\mu_\alpha,\alpha}, \mu_\alpha \in \Omega, \mu_\alpha^{-1}\alpha \in \Delta \right\}$$

the canonical set of isomorphisms $T_0^{S_A} \to T_A$. More generally, if $A' \subset A$, we will call the set

$$\left\{ \mathrm{Int}(g)|_{T_{A'}} \mid g = \prod_{\alpha\in A\setminus A'} g_{\mu_\alpha,\alpha}, \mu_\alpha \in \Omega, \mu_\alpha^{-1}\alpha \in \Delta \right\}$$

the canonical set of isomorphisms $T_{A'}^{S_{A\setminus A'}} \to T_A$.

Fact 3.11 *Any maximal torus $T \subset G$ is $G(\mathbb{R})$-conjugate to one of the T_A.*

Proof Choose $g \in G$ such that $\mathrm{Int}(g)T_0 = T$. The transport of the Γ-action on T to T_0 via $\mathrm{Int}(g^{-1})$ differs from σ by an element of $Z^1(\Gamma,\Omega) = \mathrm{Hom}(\Gamma,\Omega)$ and this element sends complex conjugation to an element of Ω of order 2. By [2, Chapter VI.Ex §1(15)] there exists an SOS A such that this element equals S_A. If $\mathrm{Int}(g_A)$ is one of the canonical isomorphisms $T_0^{S_A} \to T_A$, then $\mathrm{Int}(g_Ag^{-1})\colon T \to T_A$ is an isomorphism of real tori. By [8, Theorem. 2.1] there exists $g' \in G(\mathbb{R})$ such that $\mathrm{Int}(g')T = T_A$. ■

If we conjugate A by Ω to an A', then the tori T_A and $T_{A'}$ are also conjugate by $G(\mathbb{R})$. Thus we may fix representatives $A_1, \dots, A_k$ for the Ω-orbits of MSOS in R and study the tori T_A for A inside one of the A_i. We assume that the fixed splitting $(T_0, B_0, \{X_\alpha\})$ is compatible with the choice of representatives in the following sense:

- ##$(R, >, A_i)$ holds for all A_i.
- If $\alpha, \beta \in A_i$ lie in the same G_2-factor, then one of them is simple

This can always be arranged, as Lemmas 2.3, 2.5, Fact 2.6, and Proposition 2.7 show. Notice that this condition does not reduce generality; it is only a condition on B_0, but all Borels containing T_0 are conjugate under $N_{T_0}(\mathbb{R})$ and thus by [5, 2.3.1] the splitting invariants are independent of the choice of B_0.

Lemma 3.12 *If A', A'' are disjoint subsets of some A_i, then*

$$n(S_{A'})n(S_{A''}) = n(S_{A' \cup A''}).$$

In particular, $n(S_{A'})$ and $n(S_{A''})$ commute.

Proof This follows immediately from [5, Lemma 2.1.A], because by ## the set

$$\{\beta \in R : \ \beta > 0 \wedge S_{A'}(\beta) < 0 \wedge S_{A'}S_{A''}(\beta) > 0\}$$

is empty. ∎

Proposition 3.13 *Let α, γ be distinct elements of one of the A_i, and $\mu \in \Omega$ be such that $\mu^{-1}\alpha \in \Delta$. Then $\rho(\mu, \alpha)$ is fixed by s_γ.*

Proof We first show that

$$s_\gamma \rho(\mu, \alpha) = \rho(\mu, \alpha)\alpha^\vee(\epsilon(s_\gamma \mu, \alpha, \mu)). \tag{3.1}$$

We have

$$s_\gamma(\rho(\mu, \alpha)) = s_\gamma\Big(\alpha^\vee(-1) \prod_{\substack{\beta \in R_\alpha^+ \\ \mu^{-1}\beta < 0}} \beta^\vee(i) s_\alpha \beta^\vee(i)\Big).$$

Now s_γ preserves $\alpha^\vee$, commutes with s_α, and, by Proposition 2.8, also preserves the set R_α^+. Hence the last expression equals

$$\begin{aligned}
\alpha^\vee(-1) \prod_{\substack{\beta \in R_\alpha^+ \\ \mu^{-1}\beta < 0}} s_\gamma \beta^\vee(i) s_\alpha s_\gamma \beta^\vee(i) &= \alpha^\vee(-1) \prod_{\substack{s_\gamma\beta \in R_\alpha^+ \\ \mu^{-1}s_\gamma\beta < 0}} \beta^\vee(i) s_\alpha \beta^\vee(i) \\
&= \alpha^\vee(-1) \prod_{\substack{\beta \in R_\alpha^+ \\ \mu^{-1}s_\gamma\beta < 0}} \beta^\vee(i) s_\alpha \beta^\vee(i) \\
&= \rho(s_\gamma \mu, \alpha) \\
&= \rho(\mu, \alpha) \cdot \alpha^\vee(\epsilon(s_\gamma \mu, \alpha, \mu)),
\end{aligned}$$

the last equality coming from Proposition 3.8. This establishes equation (3.1).

We want to show $\alpha^\vee(\epsilon(s_\gamma\mu,\alpha,\mu)) = 1$. Choose $\nu \in \Omega$ such that $\nu^{-1}\gamma \in \Delta$. We will derive and compare two expressions for

$$\prod_{\beta\in R^+_{\{\alpha,\gamma\}}} \beta^\vee(i)n(s_\alpha s_\gamma)g^2_{\mu,\alpha}g^2_{\nu,\gamma}. \tag{3.2}$$

By Corollary 2.9 we have

$$\prod_{\beta\in R^+_{\{\alpha,\gamma\}}} \beta^\vee(i) = \prod_{\beta\in R^+_\alpha} \beta^\vee(i) \prod_{\beta\in R^+_\gamma} \beta^\vee(i).$$

By Proposition 2.8 s_α is a permutation of the set R^+_γ. Hence

$$n(s_\alpha)\prod_{\beta\in R^+_\gamma} \beta^\vee(i)n(s_\alpha)^{-1} = \prod_{\beta\in R^+_\gamma} \beta^\vee(i).$$

By Lemma 3.12, the elements $n(s_\alpha)$ and $n(s_\gamma)$ of $N(T_0)$ commute. Moreover, by Fact 1.1 the elements $g^2_{\mu,\alpha}$ and $g^2_{\nu,\gamma}$ commute. Thus we get on the one hand

$$\begin{aligned}(3.2) &= \prod_{\beta\in R^+_\gamma} \beta^\vee(i)n(s_\gamma)\prod_{\beta\in R^+_\alpha} \beta^\vee(i)n(s_\alpha)g^2_{\mu,\alpha}g^2_{\nu,\gamma}\\ &= \rho(\nu,\gamma)\,\mathrm{Int}(g^{-2}_{\nu,\gamma})[\rho(\mu,\alpha)]\\ &= \rho(\nu,\gamma)\rho(\mu,\alpha)\alpha^\vee(\epsilon(s_\gamma\mu,\alpha,\mu)),\end{aligned}$$

where the last equality follows from (3.1). Analogously, we obtain on the other hand

$$\begin{aligned}(3.2) &= \prod_{\beta\in R^+_\alpha} \beta^\vee(i)n(s_\alpha)\prod_{\beta\in R^+_\gamma} \beta^\vee(i)n(s_\gamma)g^2_{\nu,\gamma}g^2_{\mu,\alpha}\\ &= \rho(\mu,\alpha)\,\mathrm{Int}(g^{-2}_{\mu,\alpha})[\rho(\nu,\gamma)]\\ &= \rho(\mu,\alpha)\rho(\nu,\gamma)\gamma^\vee(\epsilon(s_\alpha\nu,\gamma,\nu)).\end{aligned}$$

We conclude that $\alpha^\vee(\epsilon(s_\gamma\mu,\alpha,\mu)) = \gamma^\vee(\epsilon(s_\alpha\nu,\gamma,\nu))$. We claim that both sides of this equality are trivial. Assume by way of contradiction that this is not the case. Then we have

$$\begin{aligned}\alpha^\vee(-1) = \gamma^\vee(-1) \Leftrightarrow 1 = (-1)^{(\alpha^\vee-\gamma^\vee)} = (-1)^{\alpha^\vee+\gamma^\vee} \in \mathbb{C}^\times \otimes X_*(T_0) = \mathbb{C}^\times \otimes Q^\vee\\ \Leftrightarrow \alpha^\vee + \gamma^\vee \in 2Q^\vee,\end{aligned}$$

where $Q^\vee$ is the coroot-lattice of T_0, which coincides with $X_*(T_0)$ since G is simply-connected. By Lemma 2.11 α,γ must lie in the same G_2-factor of R. In this case $\{\alpha,\gamma\}$ is an MSOS for that G_2-factor and, by our assumption from the beginning of this section, one of α,γ must be simple. Say, without loss of generality, that α is simple. By Proposition 3.8

$$\rho(\mu,\alpha) = \alpha^\vee(\epsilon(\mu,\alpha,1))\rho(1,\alpha) = \alpha^\vee(-\epsilon(\mu,\alpha,1)),$$

which is clearly fixed by s_γ. Thus we see

$$1 = s_\gamma(\rho(\mu,\alpha))\rho(\mu,\alpha)^{-1} = \alpha^\vee(\epsilon(s_\gamma\mu,\alpha,\mu)) = \gamma^\vee(\epsilon(s_\alpha\nu,\gamma,\nu)). \qquad \blacksquare$$

Corollary 3.14 *Let A be a subset of some A_i, $\alpha \in A$ and $\mu \in \Omega$ such that $\mu^{-1}\alpha \in \Delta$. Then $\rho(\mu,\alpha) \in Z^1(\Gamma, T_0^{S_A})$ and its image in T_A under any canonical isomorphism $T_0^{S_A} \to T_A$ is the same.*

Proof By Propositions 3.8 and 3.13 $\rho = \rho(\mu,\alpha)$ is fixed by s_γ for any $\gamma \in A$. The first statement now follows from $\rho S_A \sigma(\rho) = \rho\sigma(\rho) = 1$ showing $\rho \in Z^1(\Gamma, T_0^{S_A})$. The second holds because any two canonical isomorphisms $T_0^{S_A} \to T_A$ differ by pre-composition with $S_{A'}$ for some $A' \subset A$. ∎

Corollary 3.15 *Let A be a subset of some A_i. Choose, for each $\alpha \in A$, $\mu_\alpha \in \Omega$ such that $\mu_\alpha^{-1}\alpha \in \Delta$. Put $\rho(\{\mu_\alpha\}_{\alpha\in A}, A) = \prod_{\alpha\in A}\rho(\mu_\alpha,\alpha)$. Then*

(i) *$\rho(\{\mu_\alpha\}_{\alpha\in A}, A)$ is fixed by s_γ for all $\gamma \in A$ (even all $\gamma \in A_i$).*
(ii) *The image of $\rho(\{\mu_\alpha\}_{\alpha\in A}, A)$ under any of the canonical isomorphisms $T_0^{S_A} \to T_A$ is the same.*

Proof Clear by the preceding Corollary. ∎

Proposition 3.16 *Let A be a subset of some A_i. For each $\alpha \in A$ choose $\mu_\alpha \in \Omega$ such that $\mu_\alpha^{-1}\alpha \in \Delta$ and put $g_A = \prod_{\alpha\in A} g_{\mu_\alpha,\alpha}$. Then $\lambda(T_A, g_A)$ is the common image of $\rho(\{\mu_\alpha\}_{\alpha\in A}, A)$ under the canonical isomorphisms $T_0^{S_A} \to T_A$. In particular,*

$$\lambda(T_A, g_A) = \prod_{\alpha\in A} \mathrm{Int}(g_{A-\{\alpha\}})\lambda(T_\alpha, g_{\mu_\alpha,\alpha})$$

is a decomposition of $\lambda(T_A, g_A)$ as a product of elements of $Z^1(\Gamma, T_A)$.

Proof The factors of the cocycle $\mathrm{Int}(g_A^{-1})\lambda(T_A, g_A) \in Z^1(\Gamma, T_0^{S_A})$ associated with these choices are as follows:

$$x(\sigma_T) = \prod_{\beta\in R_A^+} \beta^\vee(i) = \prod_{\alpha\in A}\prod_{\beta\in R_\alpha^+} \beta^\vee(i),$$

where the second equality is due to Corollary 2.9,

$$n(\omega_T(\sigma)) = n(S_A) = \prod_{\alpha\in A} n(s_\alpha),$$

where the second equality is due to Lemma 3.12, and

$$\sigma(g_A)^{-1}g_A = \prod_{\alpha\in A}\sigma(g_\alpha)^{-1}g_\alpha = \prod_{\alpha\in A} g_\alpha^2.$$

Their product, which equals $\mathrm{Int}(g_A^{-1})\lambda(T_A, g_A)$, is thus

$$x(\sigma_T)n(\omega_T(\sigma))\sigma(g_A)^{-1}g_A = \prod_{\alpha\in A}\prod_{\beta\in R_\alpha^+}\beta^\vee(i)\prod_{\alpha\in A} n(s_\alpha)\prod_{\alpha\in A} g_\alpha^2.$$

Just as in the proof of Proposition 3.13 we can rewrite this product as

$$\prod_{\alpha\in A}\left[\prod_{\beta\in R_\alpha^+}\beta^\vee(i)n(s_\alpha)\right]\prod_{\alpha\in A} g_\alpha^2.$$

Now we induct on the size of A, with $|A| = 1$ being clear. Choose $\alpha_1 \in A$. Then

$$\begin{aligned}
&\prod_{\alpha\in A}\Big[\prod_{\beta\in R_\alpha^+}\beta^\vee(i)n(s_\alpha)\Big]\prod_{\alpha\in A} g_\alpha^2\\
&= \prod_{\alpha\in A\setminus\{\alpha_1\}}\Big[\prod_{\beta\in R_\alpha^+}\beta^\vee(i)n(s_\alpha)\Big]\Big\{\prod_{\beta\in R_{\alpha_1}^+}\beta^\vee(i)n(s_{\alpha_1})g_{\alpha_1}^2\Big\}\prod_{\alpha\in A\setminus\{\alpha_1\}} g_\alpha^2\\
&= \prod_{\alpha\in A\setminus\{\alpha_1\}}\Big[\prod_{\beta\in R_\alpha^+}\beta^\vee(i)n(s_\alpha)\Big]\rho(\mu_{\alpha_1},\alpha_1)\prod_{\alpha\in A\setminus\{\alpha_1\}} g_\alpha^2\\
&= \prod_{\alpha\in A\setminus\{\alpha_1\}}\Big[\prod_{\beta\in R_\alpha^+}\beta^\vee(i)n(s_\alpha)\Big]\prod_{\alpha\in A\setminus\{\alpha_1\}} g_\alpha^2\cdot\Big(\prod_{\alpha\in A\setminus\{\alpha_1\}}^{k} s_\alpha\Big)(\rho(\mu_{\alpha_1},\alpha_1))\\
&= \prod_{\alpha\in A\setminus\{\alpha_1\}}\rho(\mu_\alpha,\alpha)\cdot\rho(\mu_{\alpha_1},\alpha_1),
\end{aligned}$$

where the last equality follows from Proposition 3.13 and the inductive hypothesis. This shows that $\mathrm{Int}(g_A)\rho(\{\mu_\alpha\}_{\alpha\in A},A) = \lambda(T_A,g_A)$ and the result follows. ■

4 Explicit Computations

In this section we are going to use the classification of MSOS given in [1] to explicitly compute $\lambda(T_A,g_A)$ for the split simply-connected semi-simple groups associated with the classical irreducible root systems. By Propositions 3.8 and 3.16 it is enough to compute the cocycles $\rho(\mu,\alpha)$ for each $\alpha \in A_i$, and some $\mu \in \Omega$ with $\mu^{-1}\alpha \in \Delta$, where $A_1,\dots,A_k$ is a set of representatives for the Ω-classes of MSOS. We will use the notation from [1], which is also the notation used in the Plates of [2, Chapter VI]. There is only one cosmetic difference: in [2] the standard basis of $\mathbb{R}^k$ is denoted by (ϵ_i), in [1] by (λ_i), and we are using (e_i). The dual basis will be denoted by (e_i^*). One checks easily in each case that the choices of positive roots given in the Plates of [2, Chapter VI] and the MSOS given in [1] satisfy condition # of Section 2.

4.1 Case A_n

There is only one Ω-equivalence class of MSOS, and the representative given in [1] is $A_1 = \{e_{2i-1} - e_{2i} \mid 1 \le i \le [(n+1)/2]\}$. All elements of this MSOS are simple roots and for each of them we can choose $\mu = 1$. Then for any $\alpha \in A_1$ we have

$$\rho(1,\alpha) = \alpha^\vee(-1).$$

4.2 Case B_n

If $n = 2k+1$, then there is a unique equivalence class of MSOS, represented by

$$A_1 = \{e_{2i-1} \pm e_{2i} | \ 1 \le i \le k\} \cup \{e_n\}.$$

If $n = 2k$, then there are two equivalence classes of MSOS, represented by

$$A_1 = \{e_{2i-1} \pm e_{2i} \mid 1 \le i \le k-1\} \cup \{e_n\},$$
$$A_2 = \{e_{2i-1} \pm e_{2i} \mid 1 \le i \le k\}.$$

If $\alpha = e_{2i-1} - e_{2i}$ or $\alpha = e_n$, then α is simple; we can choose $\mu = 1$ and have

$$\rho(1, \alpha) = \alpha^\vee(-1).$$

If $\alpha = e_{2i-1} + e_{2i}$, then we take $\mu = s_{e_{2i}}$ and have $\mu^{-1}\alpha = e_{2i-1} - e_{2i} \in \Delta$. To compute $\rho(\mu, \alpha)$, we first observe

$$\begin{aligned}\{\beta \in R \mid \beta > 0 \wedge \mu^{-1}\beta < 0 \wedge s_\alpha\beta < 0\} &= \{\beta \in R \mid \beta > 0 \wedge \mu^{-1}\beta < 0\} \\ &= \{e_{2i}\} \cup \{e_{2i} \pm e_j \mid 2i < j\}.\end{aligned}$$

Hence

$$\begin{aligned}&\sum_{\substack{\beta\in R_\alpha^+ \\ \mu^{-1}\beta<0}} (\beta^\vee + s_\alpha\beta^\vee) \\ &\quad= 2e_{2i}^* + s_\alpha(2e_{2i}^*) + \sum_{j=2i+1}^{n} (e_{2i}^* - e_j^* + s_\alpha(e_{2i}^* - e_j^*) + e_{2i}^* + e_j^* + s_\alpha(e_{2i}^* + e_j^*)) \\ &\quad= -2(e_{2i-1}^* - e_{2i}^*) - 2(e_{2i-1}^* - e_{2i}^*)(n-2i) \\ &\quad= -2(n+1-2i)(e_{2i-1}^* - e_{2i}^*) \\ &\quad= -2(n+1-2i)\alpha^\vee.\end{aligned}$$

We get

$$\rho(\mu, \alpha) = \alpha^\vee\left((-1)^n\right).$$

4.3 Case C_n

The root system family C_n is the only family for which the number of equivalence classes of MSOS grows when n grows. Representatives for the equivalence classes of MSOS are given by

$$A_s = \{e_{2i-1} - e_{2i} \mid 1 \le i \le s\} \cup \{2e_i \mid 2s+1 \le i \le n\} \quad 0 \le s \le [n/2].$$

If $\alpha = e_{2i-1} - e_{2i}$, then α is simple and $\rho(1, \alpha) = \alpha^\vee(-1)$.

If $\alpha = 2e_i$ and we take $\mu = s_{e_i - e_n}$, we have $\mu^{-1}\alpha = 2e_n \in \Delta$. Again we first observe

$$\begin{aligned}\{\beta \in R \mid \beta > 0 \wedge \mu^{-1}\beta < 0 \wedge s_\alpha\beta < 0\} &= \{\beta \in R \mid \beta > 0 \wedge \mu^{-1}\beta < 0\} \\ &= \{e_i - e_j \mid i < j\}.\end{aligned}$$

Hence

$$\sum_{\substack{\beta\in R_\alpha^+\\ \mu^{-1}\beta<0}} (\beta^\vee + s_\alpha\beta^\vee) = \sum_{j=i+1}^{n} (e_i^* - e_j^*) + (-e_i^* - e_j^*) \;= -2\sum_{j=i+1}^{n} e_j^*.$$

We get

$$\rho(\mu,\alpha) = \prod_{j=i}^{n} e_i^*(-1).$$

4.4 Case D_n

There is a unique equivalence class of MSOS represented by

$$A_1 = \{e_{2i-1} \pm e_{2i} \mid 1 \le i \le [n/2]\}.$$

If $\alpha = e_{2i-1} - e_{2i}$ or $\alpha = e_{n-1} + e_n$, then α is simple and $\rho(1,\alpha) = \alpha^\vee(-1)$.

If $\alpha = e_{2i-1} + e_{2i}$ with $2i \neq n$, then we take $\mu = s_{e_{2i-1}-e_{n-1}} \circ s_{e_{2i}-e_n}$ and have $\mu^{-1}\alpha = e_{n-1} - e_n \in \Delta$. Then

$$\begin{aligned}\{\beta \in R \mid \beta > 0 \wedge \mu^{-1}\beta < 0 \wedge s_\alpha\beta < 0\}&\\ &= \{\beta \in R \mid \beta > 0 \wedge \mu^{-1}\beta < 0\}\\ &= \{e_{2i-1} - e_j \mid 2i < j\} \cup \{e_{2i} - e_j \mid 2i < j\}.\end{aligned}$$

Hence

$$\sum_{\substack{\beta\in R_\alpha^+\\ \mu^{-1}\beta<0}} (\beta^\vee + s_\alpha\beta^\vee) = -2\sum_{j=2i+1}^{n} 2e_j^* \in \begin{cases} 4Q^\vee, & 2|n,\\ 4Q^\vee + 4e_n^*, & 2|n+1.\end{cases}$$

Notice that $2e_n^* \in Q^\vee$, while $e_n^* \notin Q^\vee$. We get

$$\rho(\mu,\alpha) = \alpha^\vee(-1)\cdot 2e_n^*\big((-1)^n\big).$$

5 Comparison of Splitting Invariants

Now we would like to employ the results from the previous sections to compare splitting invariants of different tori. To describe our goal more precisely, let us recall briefly a few notions from the theory of endoscopy. An endoscopic triple (H, s, η) for G consists of a quasi-split real group H, an embedding $\eta\colon \widehat{H} \to \widehat{G}$ of the complex dual group of H to that of G, and an element $s \in Z(\widehat{H})^\Gamma$. On this triple one imposes the conditions that η identifies $\widehat{H}$ with the connected centralizer of $\eta(s)$ in $\widehat{G}$ and that the $\widehat{G}$-conjugacy class of η is fixed by Γ. This is part of the definition given in [13, §5], and for our purposes we will not need the finer structure brought by the object $\mathcal{H}$ discussed there. The conditions on η imply that it induces (by duality) an isomorphism η_0 of complex tori from the most split maximal torus in H to the most split maximal torus in G (the latter is in our case completely split). An isomorphism from some maximal torus in H to some maximal torus in G is then called admissible if it is of the form $\mathrm{Ad}(g) \circ \eta_0 \circ \mathrm{Ad}(h)$ for some $g \in G$, $h \in H$. If $T^H \to T$ is such an isomorphism defined over $\mathbb{R}$, then we say that T originates from H, and, using its dual isomorphism and the canonical embedding $Z(\widehat{H}) \to \widehat{T^H}$, we obtain from s an element of $\widehat{T}^\Gamma$, which by Tate–Nakayama-duality defines a character on $H^1(\Gamma, T)$.

Now we can state our goal. Given two maximal tori T_1, T_2 of G that originate from H, we would like to compare the results of pairing the endoscopic datum s against the splitting invariants for T_1 and T_2. In order for this to make sense, we need to compare the groups in which these invariants live. To that end, we will show that the endoscopic characters on $H^1(\Gamma, T_1)$ and $H^1(\Gamma, T_2)$ arising from s factor through certain quotients of these groups and that those quotients can be related.

For a maximal torus T in G put

$$X_*(T)_{-1} = \{\lambda \in X_*(T) \mid \sigma_T(\lambda) = -\lambda\},$$
$$IX_*(T) = \{\lambda - \sigma_T(\lambda) \mid \lambda \in X_*(T)\},$$

where σ_T is the action of σ on T. Recall the Tate–Nakayama isomorphism

$$\frac{X_*(T)_{-1}}{IX_*(T)} = H_T^{-1}(\Gamma, X_*(T)) \to H^1(\Gamma, T),$$

given by taking cup-product with the canonical class in $H^2(\Gamma, \mathbb{C}^\times)$. Via this isomorphism, the canonical pairing $\widehat{T} \times X_*(T) \to \mathbb{C}^\times$ induces a pairing

$$\widehat{T} \times H^1(\Gamma, T) \to \mathbb{C}^\times.$$

The splitting invariant enters into the construction of the Langlands–Shelstad transfer factors via this pairing.

Lemma 5.1 *Let $A' \subset A$ be SOS in R. Then each element in the canonical set of isomorphisms $T_{A'}^{S_{A \setminus A'}} \to T_A$ (Definition 3.10) induces the same embedding*

$$i_{A',A} \colon X_*(T_{A'})_{-1} \hookrightarrow X_*(T_A)_{-1}.$$

Proof For an element $\omega \in \Omega$ put $X_*(T_0)_{\omega=-1} = \{\lambda \in X_*(T_0) \mid \omega(\lambda) = -\lambda\}$. For any SOS B, $X_*(T_0)_{S_B=-1} = \mathrm{span}_{\mathbb{Q}}(B) \cap X_*(T_0)$ and any canonical isomorphism $T_0^{S_B} \to T_B$ identifies $X_*(T_0)_{S_B=-1}$ with $X_*(T_B)_{-1}$ (this identification will of course depend on the chosen isomorphism).

Fix one canonical isomorphism $T_{A'}^{S_{A \setminus A'}} \to T_A$. It is the composition of canonical isomorphisms

$$T_{A'}^{S_{A \setminus A'}} \xrightarrow{\varphi^{-1}} T_0^{S_A} \xrightarrow{\psi} T_A$$

and hence induces an inclusion as claimed, because $X_*(T_0)_{S_{A'}=-1} \subset X_*(T_0)_{S_A=-1}$. Moreover, any other canonical isomorphism $T_{A'}^{S_{A\setminus A'}} \to T_A$ is given by

$$T_{A'}^{S_{A\setminus A'}} \xrightarrow{\varphi^{-1}} T_0^{S_A} \xrightarrow{S_{A''}} T_0^{S_A} \xrightarrow{\psi} T_A$$

for $A'' \subset A \setminus A'$ and clearly $S_{A''}$ acts trivially on $X_*(T_0)_{S_{A'}=-1}$. ■

The embedding $i_{A',A}$ induces an embedding

$$\bar{i}_{A',A}\colon \frac{X_*(T_{A'})_{-1}}{IX_*(T_{A'}) + i_{A',A}^{-1}(IX_*(T_A))} \hookrightarrow \frac{X_*(T_A)_{-1}}{i_{A',A}(IX_*(T_{A'})) + IX_*(T_A)},$$

and via the Tate–Nakayama isomorphism these quotients correspond to quotients of $H^1(\Gamma, T_{A'})$ and $H^1(\Gamma, T_A)$, respectively. We will argue that if the tori $T_{A'}$ and T_A originate in an endoscopic group H, then the endoscopic character factors through these quotients. This provides a means of comparing the values of the endoscopic character on the cohomology of both tori.

Proposition 5.2 *Let (H, s, η) be an endoscopic triple for G and assume that $T_{A'}$ and T_A ($A' \subset A$) originate from H, that is, there exist tori $T_1, T_2 \subset H$ and admissible isomorphisms $T_1 \to T_{A'}$ and $T_2 \to T_A$. Write $s_{T_{A'}} \in \widehat{T}_{A'}$ and $s_{T_A} \in \widehat{T}_A$ for the images of s under the duals of these isomorphisms. Assume that there exists a canonical isomorphism $j\colon T_{A'}^{S_{A\setminus A'}} \to T_A$ such that $\widehat{j}(s_{T_A}) = s_{T_{A'}}$ (this can always be arranged). Then the characters $s_{T_{A'}}$ and s_{T_A} on $H^1(\Gamma, T_{A'})$, resp. $H^1(\Gamma, T_A)$, factor through the above quotients, and the pull-back of the character s_{T_A} via $\bar{i}_{A',A}$ equals $s_{T_{A'}}$.*

Proof We identify $H^1(\Gamma, -)$ with $H_T^{-1}(\Gamma, X_*(-))$ via the Tate–Nakayama isomorphism. Because the element $s_{T_A} \in X^*(T_A) \otimes \mathbb{C}^\times$ is Γ-invariant, its action on $X_*(T_A)$ annihilates the submodule $IX_*(T_A)$. Thus, the action of $j^*(s_{T_A}) \in X^*(T_{A'}) \otimes \mathbb{C}^\times$ on $X_*(T_{A'})$ annihilates the submodule $j_*^{-1}(IX_*(T_A))$. But we have arranged things so that $j^*(s_{T_A}) = s_{T_{A'}}$ and we see that the action of $s_{T_{A'}}$ on $X_*(T_{A'})$ via the standard pairing annihilates the submodule $IX_*(T_{A'}) + j_*^{-1}(IX_*(T_A))$. By the same argument, the action of s_{T_A} on $X_*(T_A)$ annihilates the submodule $IX_*(T_A) + j_*(IX_*(T_{A'}))$. Finally notice that by Lemma 5.1 the restriction of j_* to $X_*(T_{A'})_{-1}$ coincides with $i_{A',A}$. ■

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