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ON CERTAIN COMPLEX ANALYTIC COBORDISM BETWEEN SUBVARIETIES REALIZING CHERN CLASSES OF BUNDLES

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Introduction

Cobordism invariants have been applied to both real and complex categories. For example, the index of $4k$-dimensional manifolds was treated by Hirzebruch for the generalization of the Riemann-Roch theorem [3]. He also considered, in relation with this, virtual genus or virtual characteristics. But many invariants, such as virtual characteristics, have their origin in complex analytic category. In view of this, we consider certain complex analytic cobordism, i.e., quasilinear cobordism among quasilinear subvarieties in complex manifolds (see for the definition of quasilinear structure [4].) Quasilinear body has very simple type of singularities, as well as its quasilinear boundaries. Therefore, the theory of quasilinear cobordism can be reduced, through σ-processes, to that of non-singular cobordism theory.

In [4] we considered analytic subvarieties which realize Chern classes of holomorphic vector bundles over a complex manifold. We proved the existence of such subvarieties which have certain simple singularities and we called these subvarieties quasilinear subvarieties.

In the present paper we shall consider certain complex analytic cobordism between these quasilinear subvarieties. The definition of quasilinear cobordism is given in Definitions 1.6 and 1.7. We shall show in Theorem 3.2 that if ξ and ξ' are analytically equivalent holomorphic vector bundles over a complex manifold M which can be induced by some holomorphic maps from M into the complex Grassmann manifold, then quasilinear subvarieties V and V' given as above are quasilinearly cobordant.

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§1.

For positive integers $q, N, 1 \leq p \leq q$, let $G_{q,N+p}$ be the complex Grassmann manifold in q-dimensional linear subspaces in the complex Euclidean space $C^p \times C^{q+N}$. Let M be a complex manifold and f a holomorphic map from M into $G_{q,N+p}$. Then, f induces a holomorphic vector bundle $f^*(\gamma_{q,N+p}) = (E, \pi, M)$ over M from the universal vector bundle $\gamma_{q,N+p} = (E_{q,N+p}, \pi_{q,N+p}, G_{q,N+p})$, where $\gamma_{q,N+p}$ consists of pairs $(\tau, v) \in E_{q,N+p}$ of q-dimensional linear subspace τ of $C^p \times C^{q+N}$ and vectors v in τ. We denote by $\tilde{f}: E \to E_{q,N+p}$ the lifting of f and by $\varphi: E_{q,N+p} \to C^p \times C^{q+N}$ the map sending each (τ, v) to v. Notice that the composition $\varphi \circ \tilde{f}$ is a map from E into $C^p \times C^{q+N}$ which sends each fibre of E to q-dimensional linear subspaces of $C^p \times C^{q+N}$. Let $\pi_{q+N}: C^p \times C^{q+N} \to C^{q+N}$ denote the projection. We shall identify the complex Grassmann manifold $G_{q,N}$ with the space of q-dimensional linear subspaces in $\{0\} \times C^{q+N} \subset C^p \times C^{q+N}$.

DEFINITION 1.1. A holomorphic map f from M into $G_{q,N+p}$ is said to be reducible to $G_{q,N}$ if the map $\pi_{q+N} \circ \varphi \circ \tilde{f}$ from E into C^{q+N} sends each fibre of E to q-dimensional linear subspaces of C^{q+N}.

If $f: M \to G_{q,N+p}$ is reducible to $G_{q,N}$, then f induces a holomorphic map from M into $G_{q,N}$ in a natural way which will be denoted by $\bar{f}: M \to G_{q,N}$. $G_{q,N+p}$ contains Schubert varieties $F_1 \supset F_2 \supset \cdots \supset F_p$ each of which is defined by

$$F_r = \{q\text{-dimensional linear subspaces } \tau \text{ in } C^p \times C^{q+N};\ \dim(\pi_p|\tau|) \leq p - r\},$$

where $\pi_p: C^p \times C^{q+N} \to C^p$ is the projection and $|\tau|$ is the carrier of τ.

We shall fix q, N and p, and we consider, for any integer $R > 0$, the Grassmann manifold $G_{q,N+R+p}$. In $G_{q,N+R+p}$, Schubert varieties are defined in a similar way with respect to the decomposition $C^p \times C^{q+N+R}$ which will be denoted by $F_1^R \supset F_2^R \supset \cdots \supset F_p^R$.

Under the canonical inclusions $C^p \times C^{q+N} \cong C^p \times C^{q+N} \times \{0\} \subset C^p \times C^{q+N} \times C^R \cong C^p \times C^{q+N+R}$, we regard q-dimensional linear subspaces of $C^p \times C^{q+N}$ as those of $C^p \times C^{q+N+R}$. This gives rise to the following commutative diagram of inclusions:

$$\begin{array}{ccccccccc} G_{q,N+p} & \supset & F_1 & \supset & F_2 & \supset & \cdots & \supset & F_p \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ G_{q,N+R+p} & \supset & F_1^R & \supset & F_2^R & \supset & \cdots & \supset & F_p^R . \end{array}$$

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Under the inclusion $G_{q,N+p} \subset G_{q,N+R+p}$, any holomorphic maps from M into $G_{q,N+p}$ will be considered as holomorphic maps from M into $G_{q,N+R+p}$ for any positive integer R.

Let M be a complex manifold of dimension n.

DEFINITION 1.2. A holomorphic map f from M into $G_{q,N+p}$ is called a quasilinear map if the map f is reducible to $G_{q,N}$ and is transverse-regular to Schubert varieties $F_1, F_2, \cdots, F_p$ in $G_{q,N+p}$.

DEFINITION 1.3. Let f_1 and f_2 be holomorphic maps from M into $G_{q,N+p}$ which are quasilinear maps. The map f_1 is said to be strongly quasilinearly homotopic to f_2 if there exist an integer R and a holomorphic map $F: M \times C \to G_{q,N+R+p}$ such that

i) $F|_{M\times 0} = f_1$, $F|_{M\times 1} = f_2$

ii) F is transverse-regular to Schubert varieties $F_1^R F_2^R \cdots F_p^R$ in $G_{q,N+R+p}$.

DEFINITION 1.4. Let f and g be quasilinear maps from M into $G_{q,N+p}$. The map f is said to be quasilinearly homotopic to g if there exist quasilinear maps $f_0 = f, f_1, f_2, \cdots, f_{r-1}, f_r = g$ such that f_i is strongly quasilinearly homotopic to f_{i+1}, $i = 0, \cdots, r-1$.

Let M be a complex manifold of dimension n and V be a quasilinear subvariety of codimension k in M. For the definition and of properties of quasilinear subvarieties, see [4]. We assume that V is given by a pullback of the Schubert variety. In other words, we assume that there exists for some integer q a quasilinear map $f: M \to G_{q,N+p}$, $p = q - k + 1$ such that $V = f^{-1}(F_1)$. We say that V is associated to f. We shall fix an integer q and sufficiently large N.

DEFINITION 1.5. Let V_k, $k = 1, \cdots, q$ be quasilinear subvarieties of M. The sequence $(V_1, \cdots, V_q)$ is said to be a quasilinear sequence if each V_k is associated to some quasilinear map $f_k: M \to G_{q,N+p}$, $p = q - k + 1$.

Let V and V' be quasilinear subvarieties of codimension k in M associated to some quasilinear maps.

DEFINITION 1.6. V and V' are said to be quasilinearly cobordant in the strong sense if there exists a strong quasilinear homotopy $F: M \times C \to G_{q,N+R+p}$ for some R such that V and V' are associated to the restrictions $F|_{M\times 0}$ and $F|_{M\times 1}$ respectively. V and V' are said to be quasilinearly

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cobordant if there exists a sequence of quasilinear subvarieties $V_0 = V$, $V_1, \cdots, V_s = V'$ such that V_r is quasilinearly cobordant to V_{r+1}, $r = 0, \cdots, s-1$ in the strong sense.

DEFINITION 1.7. Quasilinear sequences $(V_1, \cdots, V_q)$ and $(V'_1, \cdots, V'_q)$ are said to be quasilinearly cobordant (in the strong sense) if each V_k and V'_k, $1 \leq k \leq q$ are quasilinearly cobordant (in the strong sense).

§ 2.

This section is devoted to prove our fundamental lemma:

LEMMA 2.1. *Let M be a complex manifold and let V^1 and V^2 be quasilinear subvarieties in M of codimension k. If there exist quasilinear maps $\Psi_i: M \to G_{q,N+p}$, $p = q - k + 1$, $i = 1, 2$ such that each V^i is associated to Ψ_i and such that the induced bundles $\Psi_i^*(\gamma_{q,N+p})$ from the universal bundle $\gamma_{q,N+p}$ over $G_{q,N+p}$ are analytically equivalent, then V^1 and V^2 are quasilinearly cobordant.*

We begin with the consideration of certain topological space obtained from total spaces of bundles under certain identifications.

Let $\xi = (E_\xi, \pi_\xi, M)$ be a holomorphic vector bundle over M. We denote by $\eta = (E_\eta, \pi_\eta, M \times \boldsymbol{C})$ a holomorphic vector bundle over $M \times \boldsymbol{C}$ induced from ξ by the projection $\pi_M: M \times \boldsymbol{C} \to M$. We consider r-copies of the bundle η and denote them by $\eta_i = (E_i, \pi_i, M \times \boldsymbol{C})$, $i = 1, \cdots, r$. Let $\alpha_i: \eta_i \to \eta$ denote the isomorphisms and $\tilde{\alpha}_i: E_i \to E_\eta$ their liftings. We put $E_i^0 = \pi_i^{-1}(M \times 0)$, $E_i^1 = \pi_i^{-1}(M \times 1)$ for each $i = 1, \cdots, r$.

Let $\bigcup_{i=1}^r E_i$ be a disjoint union of the total spaces E_i. From this union, we construct a topological space by the identifications $\alpha_{i,i+1}: E_i^1 \to E_{i+1}^0$, $i = 1, \cdots, r-1$, where $\alpha_{i,i+1}$ denote the restrictions of $\tilde{\alpha}_{i+1}^{-1} \circ \tilde{\alpha}_i$. We shall denote the space obtained in this way by $\mathscr{E} = \{E_i, \alpha_{j,j+1}\}_{1 \leq i \leq r, 1 \leq j \leq r-1}$.

DEFINITION 2.2. A complex valued function f on $\mathscr{E}$ is said to be holomorphic if each restriction $f|_{E_i}: E_i \to \boldsymbol{C}$ is holomorphic. A map $f = (f^1, \cdots, f^p): \mathscr{E} \to \boldsymbol{C}^p$ is said to be holomorphic if each f^i is holomorphic.

LEMMA 2.3. *Given a holomorphic function f on E_k for some integer $1 \leq k \leq r$, there exists a holomorphic function f on $\mathscr{E}$ such that*

i) $\tilde{f}|_{E_k} = f$

ii) $\tilde{f}|_{E_{k-1}^0} \equiv 0$, $\tilde{f}|_{E_{k+1}^1} \equiv 0$

iii) $\tilde{f}|_{E_i} \equiv 0$, $i \leq k-2$, $i \geq k+2$.

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Proof. Notice that $f \circ \alpha_{k-1,k}$ is a holomorphic function on E^1_{k-1}. Extending it on E_{k-1} by the canonical projection $E_{k-1} \to E^1_{k-1}$, we obtain a holomorphic function $g_{k-1}: E_{k-1} \to \boldsymbol{C}$ such that $g_{k-1} = f \circ \alpha_{k-1,k}$ on E^1_{k-1}. Let ρ_0 and ρ_1 be holomorphic functions on $\boldsymbol{C}$ such that $\rho_0(0) = \rho_1(1) = 0$, $\rho_0(1) = \rho_1(0) = 1$. Let $\tilde{\rho}_0$ denote the pullback of ρ_0 under the projections $E_{k-1} \to M \times \boldsymbol{C} \to \boldsymbol{C}$. If we put $\tilde{f}_{k-1} = \tilde{\rho}_0 \cdot g_{k-1}$, then the function $\tilde{f}_{k-1}$ satisfies $\tilde{f}_{k-1} = f \circ \alpha_{k-1,k}$ on E^1_{k-1} and $\tilde{f}_{k-1} \equiv 0$ on E^0_{k-1}. In a similar way, making use of ρ_1 and g_{k+1}, we obtain a holomorphic function $\tilde{f}_{k+1}$ on E_{k+1} which coincides with $f \circ (\alpha_{k,k+1})^{-1}$ on E^0_{k+1} and vanishes identically on E^1_{k+1}.

If we define $\tilde{f}_k = f$ and $\tilde{f}_i = 0$, $i \leq k-2$, $i \geq k+2$, then functions $\tilde{f}_1, \cdots, \tilde{f}_r$ define a holomorphic function $\tilde{f}$ on $\mathscr{E}$. It is easy to verify that the function $\tilde{f}$ satisfies i) ii) and iii). Q.E.D.

Let $\xi = (E_\xi, \pi_\xi, M)$ be a holomorphic vector bundle over M which is holomorphically equivalent to the induced bundles $\Psi_i^*(\gamma_{q,N+p}) = (E(\Psi_i), \pi(\Psi_i), M)$ with biholomorphic bundle maps $b_i: E_\xi \to E(\Psi_i)$ $i = 1, 2$. Let $\eta = (E_\eta, \pi_\eta, M \times \boldsymbol{C})$ be a holomorphic vector bundle over $M \times \boldsymbol{C}$ induced from ξ by the projection $M \times \boldsymbol{C} \to M$. Considering three copies of the bundle η, we obtain as in the beginning of this section a topological space

$$\mathscr{E} = \{E_i, \alpha_{j,j+1}\}_{1 \leq i \leq 3,\, 1 \leq j \leq 2}\,.$$

We shall extend to the space $\mathscr{E}$ the notions of singularities of holomorphic mappings on holomorphic vector bundles developped in [4].

DEFINITION 2.4. A holomorphic map f from $\mathscr{E}$ into $\boldsymbol{C}^m$ is said to be fibrewise regular if each restriction $f|_{E_i}$ is fibrewise regular on E_i, $i = 1, 2, 3$, i.e., the differential $d(f_i|_{\pi_i^{-1}(z)})$ of the restriction to any fibre $\pi_i^{-1}(z)$, $z \in M \times \boldsymbol{C}$ is injective at any point of the zero cross-section.

Let $E_{q,N+p}$ denote the total space of the universal vector bundle $\gamma_{q,N+p}$ over $G_{q,N+p}$. Under the notations in § 1, we denote by $\varphi_1: E_1 \to \boldsymbol{C}^p \times \boldsymbol{C}^{q+N}$ a fibrewise regular map obtained from the pullback of $\varphi_{q,N+p} \circ \tilde{\Psi}_1: E(\Psi_1) \to \boldsymbol{C}^p \times \boldsymbol{C}^{q+N}$ through the canonical projection $E_1 \to E(\Psi_1)$. We denote the restrictions $\varphi_1|_{E^0_1}$ by $\varphi^{(1)} = (\varphi^{(1)}_p, \varphi^{(1)}_{q+N})$, $\varphi^{(1)}_p: E^0_1 \to \boldsymbol{C}^p$, $\varphi^{(1)}_{q+N}: E^0_1 \to \boldsymbol{C}^{q+N}$. Notice that $\varphi^{(1)}$ induces a holomorphic map from M into $G_{q,N+p}$ which coincides with Ψ_1.

In a similar way we define a holomorphic map $\varphi_3: E_3 \to \boldsymbol{C}^p \times \boldsymbol{C}^{q+N}$ by the pullback of $\varphi_{q,N+p} \circ \tilde{\Psi}_2$ under the canonical projection $E_3 \to E(\Psi_2)$. We denote the restriction $\varphi_3 | E^1_3$ by $\varphi^{(2)} = (\varphi^{(2)}_p, \varphi^{(2)}_{q+N})$. Then $\varphi^{(2)}$ induces Ψ_2.

PROPOSITION 2.5. *For the above constructed topological space $\mathscr{E}$, there exists a holomorphic map $\Phi = (\Phi_p, \Phi_{q+N}, \Phi_R)\colon \mathscr{E} \to C^p \times C^{q+N} \times C^R$, $\Phi_p\colon \mathscr{E} \to C^p$, $\Phi_{q+N}\colon \mathscr{E} \to C^{q+N}$, $\Phi_R\colon \mathscr{E} \to C^R$ for some integer R such that*

i) $\Phi|_{E_1^0} = (\varphi_p^{(1)}, \varphi_{q+N}^{(1)}, 0)$
$\Phi|_{E_3^1} = (\varphi_p^{(2)}, \varphi_{q+N}^{(2)}, 0)$

ii) $(\Phi_{q+N}, \Phi_R)\colon \mathscr{E} \to C^{q+N} \times C^R$ *is fibrewise regular.*

Proof. From Lemma 2.3, there exist holomorphic maps $\tilde{\varphi}_1$, $\tilde{\varphi}_3$ from $\mathscr{E}$ into $C^p \times C^{q+N}$ such that

i) $\tilde{\varphi}_1 = \varphi_1$ on E_1, $\tilde{\varphi}_1 = 0$ on E_3

ii) $\tilde{\varphi}_3 = 0$ on E_1, $\tilde{\varphi}_3 = \varphi_3$ on E_3.

Since Ψ_1, Ψ_2 are reducible to $G_{q,N}$, $\pi_{q+N} \circ \varphi_1$ and $\pi_{q+N} \circ \varphi_3$ are also fibrewise regular for the projection $\pi_{q+N}\colon C^p \times C^{q+N} \to C^{q+N}$.

Since the bundle E_2 is induced by a holomorphic map from the composition of the projection and Ψ_1 (or Ψ_2) $M \times C \to M \to G_{q,N}$, E_2 has a fibrewise regular map $\varphi_2\colon E_2 \to C^R$, $R = q + N$. From Lemma 2.3, there is a map $\tilde{\varphi}_2\colon \mathscr{E} \to C^R$ such that

i) $\tilde{\varphi}_2 = \varphi_2$ on E_2

iii) $\tilde{\varphi}_2 | E_1^0 = 0$, $\tilde{\varphi}_2 | E_3^1 = 0$.

If we define

$$(\Phi_p, \Phi_{q+N}) = (\tilde{\varphi}_1 + \tilde{\varphi}_3)\,, \qquad \Phi_R = \tilde{\varphi}_2$$

then the map $\Phi = (\Phi_p, \Phi_{q+N}, \Phi_R)$ satisfies the required conditions. Q.E.D.

We shall extend the notion of general position defined in [4] to maps on $\mathscr{E}$. Each zero cross-section of E_i will be identified with $M \times C$ for $i = 1, 2, 3$.

DEFINITION 2.6. A holomorphic map f from $\mathscr{E}$ into C^p is said to be in general position if each restrictions $f|_{E_i}$, $i = 1, 2, 3$ is in general position on $M \times C$.

Let K_M and K_C be arbitrary compact subsets of M and C respectively such that $0, 1 \in K_C$. For each $i = 1, 2, 3$, $K_M \times K_C \subset M \times C$ can be regarded as a compact subset of E_i and will be denoted by K_i. We define a compact subset of $\mathscr{E}$ by $K = K_1 \cup K_2 \cup K_3$. Let L be an arbitrary compact subset of $\mathscr{E}$. For holomorphic maps f from $\mathscr{E}$ into C^p, norms $\|f\|_{L \cup K}$ can be defined by

$$\|f\|_{L\cup K} = \sum_{i=1}^{3} \|f|_{E_i}\|_{E_i\cap(L\cup K)} ,$$

where norms $\|f|_{E_i}\|_{E_i\cap(L\cup K)}$ are those defined in [4].

LEMMA 2.7. *Let $f\colon \mathscr{E} \to C^p$ be a holomorphic map such that $f|_{E_1^0}$ is in general position on $K_M \times 0 \hookrightarrow E_1^0$. Then, for any $\varepsilon > 0$, there exists a holomorphic map $g\colon \mathscr{E} \to C^p$ such that*

i) *$g = f$ on E_1^0 and E_3^1*
ii) *$\|f - g\|_{L\cup K} < \varepsilon$*
iii) *g is in general position on K_1*
iv) *The restriction $g|_{E_1^1}$ is in general position on $K_M \times 1 \hookrightarrow E_1^1$.*

Proof. Since $f|_{E_1}$ is in general position on $K_M \times 0 \subset E_1^0$, f is in general position on some compact neighbourhood A of $K_M \times 0$ in $K_M \times K_C \subset E_1$. From the stability of general position, there is $\delta_A > 0$ such that if $\|f - g\|_A$ then g and $g|_{E_1^0}$ are in general position on A and $K_M \times 0 \subset E_1^0$ respectively.

Let $\tilde{\varphi}_1\colon \mathscr{E} \to C^p \times C^{q+N}$ be the holomorphic map defined in the proof of Proposition 2.5. Denote $\tilde{\varphi}_1 = (\alpha^1, \cdots, \alpha^p, \beta^1, \cdots, \beta^{q+N})$. We define a holomorphic function ρ on E_1 by composing projections $E_1 \to M \times C \to C$. Then $\rho \equiv 0$ on E_0^1, $\rho \equiv 1$ on E_1^1 and does not vanish on $E_1 - E_1^0$. Define holomorphic functions $\bar{\beta}^j\colon \mathscr{E} \to C$, $j = 1, 2, \cdots, q + N$ by

$$\begin{aligned} \bar{\beta}^j &= \rho\cdot\beta^j && \text{on } E_1 \\ \bar{\beta}^j &= \beta^j && \text{on } E_2, E_3 . \end{aligned}$$

We shall deform the mapping $f\colon \mathscr{E} \to C^p$ into the following form, for some constants ε_j^i,

$$g = (g^1, g^2, \cdots, g^p)\colon \mathscr{E} \longrightarrow C^p$$

$$g^i = f^i + \sum_{j=1}^{q+N} \varepsilon_j^i \bar{\beta}^j , \qquad j = 1, 2, \cdots, p .$$

Since the maps $\bar{\beta}^1, \cdots, \bar{\beta}^{q+N}$ identically vanish on E_1^0 and E_3^1, the map g coincides with f on E_1^0 and E_3^1. Because ρ does not vanish on $E_1 - E_1^0$, the map $(\bar{\beta}^1, \bar{\beta}^2, \cdots, \bar{\beta}^r)\colon \mathscr{E} \to C^r$, $r = q + N$ is fibrewise regular on $E_1 - E_1^0$.

Let B be the closure of $K_M \times K_C - A$ in $K_M \times K_C \subset E_1$. Since $(\bar{\beta}^1, \cdots, \bar{\beta}^r)$ is fibrewise regular on $\pi_1^{-1}(B)$, $\pi_1\colon E_1 \to M \times C$, we can apply the approximation method developped in [4] making use of $\bar{\beta}^i$, $i = 1, \cdots, r$. Hence, we come to see that there are ε_j^i, $1 \le i \le p$, $1 \le j \le r$ such that the corresponding g is in general position on B and satisfies

$$\|f-g\|_{L\cup K}<\min\left\{\frac{\varepsilon}{2}, \frac{\delta_A}{2}\right\}.$$

In fact, at each stage of approximations, we only add terms of the form $\varepsilon^i_{kj}\bar{\beta}^j$ for some constants ε^i_{kj}. Therefore, the final form becomes

$$g^i = f^i + \sum_k \sum_j \varepsilon^i_{kj}\bar{\beta}^j .$$

See for the details [4]. We denote this map g by g'.

From the stability of general position, there is $\delta_B > 0$ such that if $\|g'-g\|_B < \delta_B$ then g is in general position on B.

Since the restriction of $(\bar{\beta}^1, \cdots, \bar{\beta}^r)$ to E^1_1 is also fibrewise regular, we obtain, in a similar way, a holomorphic map $g'': \mathscr{E} \to C^p$ the restriction of which to E^1_1 is in general position on $K_M \times 1 \subset E^1_1$ and which satisfies

$$\|g'-g''\|_{L\cup K}<\min\left\{\frac{\varepsilon}{2}, \frac{\delta_A}{2}, \delta_B\right\}.$$

If we set $g = g''$, then g satisfies the required conditions. Q.E.D.

Applying the same method, we have the approximation lemma with respect to K_3 and $K_M \times 0$:

LEMMA 2.8. *Let $f: \mathscr{E} \to C^p$ be a holomorphic map such that $f|_{E^1}$ is in general position on $K_M \times 1 \subset E^1_3$. Then, for any $\varepsilon > 0$, there exists a holomorphic map $g: \mathscr{E} \to C^p$ such that*

i) *$f = g$ on E^0_1 and E^1_3*
ii) *$\|f-g\|_{L\cup K} < \varepsilon$*
iii) *g and $g|_{E^1_3}$ are in general position on K_3 and $K_M \times 0$ respectively.*

Finally with respect to $K_2 \subset E_2$, we prove the following approximation lemma.

LEMMA 2.9. *Let $f: \mathscr{E} \to C^p$ be a holomorphic map. Then, for any $\varepsilon > 0$, there exists a holomorphic map $g: \mathscr{E} \to C^p$ such that*

i) *$g = f$ on E^1_0 and E^1_3*
ii) *$\|f-g\|_{L\cup K} < \varepsilon$*
iii) *g is in general position on K_2.*

Proof. Let $\tilde{\varphi}^2: \mathscr{E} \to C^R$ be the holomorphic map defined in the proof of Proposition 2.5. We denote $\tilde{\varphi}_2 = (\gamma^1, \cdots, \gamma^R)$. We deform f into the form

$$g = (g^1, \cdots, g^p) : \mathscr{E} \longrightarrow C^p$$
$$g^i = f^i + \sum_{j=1}^{R} \varepsilon^i_j \gamma^j , \qquad i = 1, \cdots, p .$$

Since γ^j, $j = 1, \cdots, R$ vanish identically on E^1_0 and E^1_3, g coincides with f on E^0_1 and E^1_3.

Because the map $\tilde{\varphi}_2$ is fibrewise regular on E_2, there exist, by the approximation method in [4], $\{\varepsilon^i_j\}$ such that corresponding g satisfies

i) $\|f - g\|_{L \cup K} < \varepsilon$

ii) g is in general position on K_2. Q.E.D.

PROPOSITION 2.10. *Let $f: \mathscr{E} \to C^p$ be a holomorphic map such that $f|_{E^0_1}$ and $f|_{E^1_3}$ are in general position on $K_M \times 0 \subset E^0_1$ and $K_M \times 1 \subset E^1_3$ respectively. Then, for any $\varepsilon > 0$, there exists a holomorphic map $g: \mathscr{E} \to C^p$ such that*

i) *$g = f$ on E^0_1 and E^1_3*

ii) $\|f - g\|_{L \cup K} < \varepsilon$

iii) *g and $g|_{E^j_i}$, $i = 1, 2, 3$, $j = 0, 1$, are in general position on K and $K_M \times j \subset E^j_i$ respectively.*

Proof. Firstly, by Lemma 2.7, we deform f into g_1 so that $g_1, g_1|_{E^0_1}$ and $g_1|_{E^1_1}$ are in general position on K_1, $K_M \times 0 \subset E^0_1$ and $K_M \times 1 \subset E^1_1$ respectively. Secondarily, by Lemma 2.8, we deform g_1 into g_2 so that $g_2, g_2|_{E^0_3}$ and $g_2|_{E^1_3}$ are in general position on $K_3, K_M \times 0 \subset E^0_3$, $K_M \times 1 \subset E^1_3$. And finally, by Lemma 3.8, we deform g_2 into g_3 so that g_3 is in general position on K_2. Notice that in each process, approximation is carried out on $L \cup K$ and that f, g_1, g_2 and g_3 coincide on E^0_1 and E^1_3. From the stability of general position, it is easy to see that there is g_3 with condition iii). Q.E.D.

Let $\varphi^{(1)}_p : E^0_1 \to C^p$, $\varphi^{(2)}_p : E^1_3 \to C^p$ be those defined in Proposition 2.5.

PROPOSITION 2.11. *There exists a holomorphic map $f: \mathscr{E} \to C^p$ such that*

i) $f|_{E^0_1} = \varphi^{(1)}_p$, $f|_{E^1_3} = \varphi^{(2)}_p$

ii) *f is in general position on $M \times C \subset E_i$ for any $i = 1, 2, 3$.*

iii) *$f|_{E^j_i}$, $i = 1, 2, 3$, $j = 0, 1$ are in general position on $M \times j \subset E^j_i$.*

Proof. Let $\{K^n_M\}$, $\{K^n_C\}$, $n = 1, 2, 3, \cdots$, be compact coverings of M and C respectively such that $K^n_M \subset K^{n+1}_M$, $\{0, 1\} \subset K^n_C \subset K^{n+1}_C$ for any n. The set $K^n_M \times K^n_C$ can be regarded as compact subsets in zero cross sections

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$M \times C \subset E_i$ for each $i = 1, 2, 3$. We consider the union $K^n = K_1^n \cup K_2^n \cup K_3^n$ in $\mathscr{E}$. Let $\{L^n\}_{n=1,2,\dots}$, be a compact covering of $\mathscr{E}$ such that $K^n \subset L^n \subset L^{n+1}$ for any n.

Let $\Phi = (\Phi_p, \Phi_q, \Phi_R): \mathscr{E} \to C^p \times C^{q+N} \times C^R$ be the map given in Proposition 2.5. Put $f_0 = \Phi_p: \mathscr{E} \to C^p$. Then $f_0 | E_1^0 = \varphi_p^{(1)}$ and $f_0 | E_3^1 = \varphi_p^{(2)}$. Furthermore $f_0 | E_1^0$ and $f_0 | E_3^1$ are in general position on $M \times 0 \subset E_0^1$ and $M \times 1 \subset E_3^1$.

By successive application of Proposition 2.10 to (L^n, K^n) $n = 1, 2, \cdots$, we obtain, from the stability of general position, holomorphic maps $f_n: \mathscr{E} \to C^p$ and $\delta_n > 0$ such that for any n

i) $f_n = f_0$ on E_1^0 and E_3^1

ii) $f_n, f_n | E_i^j$, $1 \leq i \leq 3$, $j = 0, 1$ are in general position on $K^n, K_M^n \times j \subset E_i^j$ respectively.

iii) If $\|f_n - g\|_{K^n} < \delta_n$ then g and $g | E_i^j$, $1 \leq i \leq 3$, $j = 0, 1$ are in general position on $K^n, K_M^n \times j \subset E_i^j$.

iv) $\|f_{n-1} - f_n\|_{L^n} < \min\left\{\dfrac{1}{2^n}, \dfrac{\delta_1}{2^n}, \dfrac{\delta_2}{2^{n-1}}, \cdots, \dfrac{\delta_{n-1}}{2^2}\right\}$.

Define $f = \lim_{n\to\infty} f_n$. Then f is a holomorphic map from $\mathscr{E}$ into C^p which satisfies the required conditions. Q.E.D.

The results which we have proved so far in this section can be summed up as follows.

Let $\Phi = (\Phi_p, \Phi_{q+N}, \Phi_R): \mathscr{E} \to C^p \times C^{q+N} \times C^R$ be the map given by Proposition 2.5, and let $f: \mathscr{E} \to C^p$ be the map given by Proposition 2.11. We define $\tilde{\Phi} = (\tilde{\Phi}_p, \tilde{\Phi}_{q+N}, \tilde{\Phi}_R) = (f, \Phi_{q+N}, \Phi_R)$. From Proposition 2.5 and Proposition 2.11, we have

PROPOSITION 2.12. *There exists a holomorphic map $\tilde{\Phi}: \mathscr{E} \to C^p \times C^{q+N} \times C^R$ such that*

i) $\tilde{\Phi} | E_1^0 = (\varphi_p^{(1)}, \varphi_{q+N}^{(1)}, 0)$
$\tilde{\Phi} | E_3^1 = (\varphi_p^{(2)}, \varphi_{q+N}^{(2)}, 0)$

ii) $(\tilde{\Phi}_{q+N}, \tilde{\Phi}_R): \mathscr{E} \to C^{q+N} \times C^R$ *is fibrewise regular on each* $M \times C \subset E_i$, $i = 1, 2, 3$.

iii) $\tilde{\Phi}_p: \mathscr{E} \to C^p$ *and* $\tilde{\Phi}_p | E_i^j$ *are in general position on each* $M \times C \subset E_i$, $i = 1, 2, 3$ *and on each* $M \times j \subset E_i^j$ *respectively.*

We are now in a position to prove our fundamental lemma (Lemma 2.1). Since the map $(\tilde{\Phi}_{q+N}, \tilde{\Phi}_R): \mathscr{E} \to C^{q+N} \to C^R$ is, from ii) of Proposition

2.12, fibrewise regular on each E_i, $i = 1, 2, 3$, the map $\tilde{\Phi}$ carries each fibre of E_i to q-planes in $C^p \times C^{q+N} \times C^R$. Therefore $\tilde{\Phi}|E_i$ induce holomorphic maps

$$\tilde{\Psi}_i: M \times C \longrightarrow G_{q,N+R+p}\,, \qquad i = 1, 2, 3$$

such that $\tilde{\Psi}_1|_{M\times 1} = \tilde{\Psi}_2|_{M\times 0}$ and $\tilde{\Psi}_2|_{M\times 1} = \tilde{\Psi}_3|_{M\times 0}$. From the condition i) of Proposition 2.12, we have

$$\tilde{\Psi}_1|_{M\times 0} = \Psi_1: M \longrightarrow G_{q,N+p} \subset G_{q,N+R+p}$$
$$\tilde{\Psi}_3|_{M\times 1} = \Psi_2: M \longrightarrow G_{q,N+p} \subset G_{q,N+R+p}\,.$$

Let $F_1^R \supset F_2^R \supset \cdots \supset F_p^R$ be Schubert varieties in $G_{q,N+R+p}$ with respect to the decomposition $C^p \times (C^{q+N} \times C^R)$. From the condition iii) of Proposition 2.12, it follows that $\tilde{\Psi}_i$, $\tilde{\Psi}_i|_{M\times j}$, $i = 1, 2, 3$, $j = 0, 1$ are transverse-regular to $F_1^R, F_2^R, \cdots, F_p^R$. This completes the proof of Lemma 2.1.

§ 3.

In [4] we have considered the existence of quasilinear subvarieties which realize Chern classes of given vector bundles. The following theorem is an immediate consequence from the proof of Main Theorem of [4].

THEOREM 3.1. *Let M be a complex manifold and ξ be a holomorphic vector bundle of rank q over M which is assumed to be induced from the universal bundle $\gamma_{q,N}$ over the Grassmann manifold $G_{q,N}$ under some holomorphic map $f: M \to G_{q,N}$. Then there exists a quasilinear sequence $(V_1, V_2, \cdots, V_q)$ in M such that*

i) *V_k realizes the k-th Chern class $C_k(\xi)$*

ii) *V_k is associated to some quasilinear map $f_k: M \to G_{q,N+p}$, $p = q - k + 1$ such that ξ is analytically equivalent to the induced bundle $f_k^*(\gamma_{q,N+p})$.*

From our fundamental lemma we shall prove the following theorem which asserts that the quasilinear cobordism of the sequences $\{(V_1, \cdots, V_q)\}$ results from the analytic equivalence of bundles $\{\xi\}$.

THEOREM 3.2. *Let M be a complex manifold. Let ξ and ξ' be holomorphic vector bundles of rank q over M which satisfy the assumption of the above theorem. Let $(V_1, V_2, \cdots, V_q)$ and $(V_1', V_2', \cdots, V_q')$ be quasilinear sequences given by the above theorem with respect to bundles ξ and ξ' re-*

spectively. If ξ and ξ' are analytically equivalent, then $(V_1, \cdots, V_q)$ and $(V'_1, \cdots, V'_q)$ are quasilinearly cobordant.

Proof. From the condition ii) of Theorem 3.1, there exists a holomorphic map $f_k, f'_k: M \to G_{q,N+p}$, $p = q - k + 1$ such that V_k, V'_k are associated to f_k and f'_k respectively and such that the induced bundles $f^*_k(\gamma_{q,N+p})$, $f'^*_k(\gamma_{q,N+p})$ are analytically equivalent to ξ and ξ'. Since ξ and ξ' are analytically equivalent, $f^*_k(\gamma_{q,N+p})$ and $f'^*_k(\gamma_{q,N+p})$ are also analytically equivalent. Therefore, from our fundamental lemma, it follows that V_k and V'_k are quasilinearly cobordant for any $k = 1, 2, \cdots, q$. Q.E.D.

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