ON CONTINUOUS LINEAR TRANSFORMATIONS OF INTEGRAL TYPE

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Let $(X \times Y, S \times T, \mu \times \nu)$ denote the completion of the Cartesian product of the σ-finite and complete measure spaces (X, S, μ) and (Y, T, ν) [3]. Let λ_x and λ_y denote arbitrary length functions defined on (X, S, μ) and (Y, T, ν) respectively, λ_x^*, λ_y^* the conjugate length functions [2]. We suppose that

$$(1) \qquad \lambda_{xy}(f) = \lambda_{xy}(|f|) = \lambda_x[\lambda_y(f)]$$

is defined for every $f(x, y)$ measurable $(S \times T)$. The Fubini theorem implies that $f(x, y)$ is measurable (T) for almost all x. Thus $\lambda_{xy}(f)$ will be defined when $\lambda_y(f)$ is measurable (S). If $L^{\lambda_y} = L^p$, $1 \leqslant p < \infty$, this is implied by the Fubini theorem. General conditions ensuring that $\lambda_y(f)$ is measurable (S) are given in [1, Theorem 3.2]. When $\lambda_{xy}(f)$ is defined for every $f(x, y)$ measurable $(S \times T)$, it is a length function and $L^{\lambda_{xy}}$ is a Banach space [1, Theorem 3.1].

We note that if $1 \leqslant p < \infty$ and

$$\lambda_x[g(x)] = \left(\int_X |g|^p \, d\mu\right)^{1/p}, \quad \lambda_y[g(y)] = \left(\int_Y |g|^p \, d\nu\right)^{1/p},$$

and $h(x, y)$ is measurable $(S \times T)$, then

$$\lambda_{xy}(h) = \lambda_x[\lambda_y(h)] = \left(\int_X \left(\int_Y |h|^p \, d\nu\right) d\mu\right)^{1/p}$$
$$= \left[\int_{X \times Y} |h(x, y)|^p \, d(\mu \times \nu)\right]^{1/p}$$

by the Fubini theorem. Thus if λ_x and λ_y correspond to the p-norms in (X, S, μ) and (Y, T, ν), λ_{xy} corresponds to the

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p-norm in the product space. The theorem below is no doubt well known when $L^{\lambda x}$ and $L^{\lambda y}$ are L^p spaces for two independent values of p, $1 \leqslant p < \infty$. (For the case $L^{\lambda x} = L^{\lambda y} = L^2$, see [4].)

If K(x, y) is measurable (S×T) and $g(y) \in L^{\lambda^* y}$, K(x, y)g(y) is measurable (S×T) and, since X and Y are σ-finite, the Fubini theorem implies that

$$Kg(x) = \int_Y K(x,y)g(y)d\nu \tag{2}$$

is measurable (S).

THEOREM. (i) Each element K(x, y) in $L^{\lambda xy}$ is the kernel of a linear transformation K of $L^{\lambda^* y}$ into $L^{\lambda x}$ defined by (2) with $\|K\| \leqslant \lambda_{xy}(K)$.

(ii) Suppose that $K_i(x,y) \in L^{\lambda xy}$, i = 1, 2, ... and that

$$\sum_1^\infty \lambda_{xy}(K_i) < \infty . \tag{3}$$

Then $\{\sum_1^n K_i(x,y)\}$ is a Cauchy sequence in $L^{\lambda xy}$ and, if K(x, y) is a limit in norm of this sequence, the sequence of bounded linear transformations $\{\sum_1^n K_i\}$ converges in norm to the bounded linear transformation K. Thus, for every $g \in L^{\lambda^* y}$, $\sum_1^n K_i g(x)$ converges strongly to Kg(x) in $L^{\lambda x}$ [5, p. 150]. Furthermore, given $g \in L^{\lambda^* y}$, there exists a set $X_0 \subset X$ with $\lambda_x(X - X_0) = 0$ such that $\sum_1^n K_i g(x)$ converges pointwise to Kg(x) in X_0. In particular if for $e \in S$, $\lambda_x(e) = 0$ implies that $\mu(e) = 0$, as is the case for the L^p spaces, $1 \leqslant p \leqslant \infty$, then $\sum_1^n K_i g(x)$ converges pointwise to Kg(x) almost everywhere.

Proof. (i) The assumption that $K(x,y) \in L^{\lambda xy}$ implies that K(x, y) is measurable (S×T) and that $\lambda_{xy}(K) < \infty$. The definition of $\lambda_x[\lambda_y(K)]$ then implies that $\lambda_y(K) \in L^{\lambda x}$. Thus the set $E = [x \in X: \lambda_y(K) = \infty]$ is λ_x-null (i.e. $\lambda_x(E_\infty) = 0$) [2, p. 579]. Thus $\lambda_y(K)$ is defined and finite in a set X_0 with $\lambda_x(X - X_0) = 0$.

If $g(y) \in L^{\lambda^* y}$, $K(x,y)g(y) \in L^1(Y)$ for $x \in X_0$ and

$$\int_Y |K(x,y)g(y)|\, d\nu \leqslant \lambda_y(K)\, \lambda^*_y(g) . \tag{4}$$

Thus, using (L 2) and (L 4) for λ_x[2], (1) and (4) above,

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(5) $\lambda_x(Kg) \leq \lambda_x[\lambda_y(K)\ \lambda_y^*(g)] = \lambda_y^*(g)\ \lambda_{xy}(K) < \infty$.

Thus $Kg(x) \in L^{\lambda_x}$ and (2) defines a transformation K of $L^{\lambda_y^*}$ into L^{λ_x}. That K is linear is easily verified and (4) implies that $\|K\| \leq \lambda_{xy}(K)$ so that K is bounded.

(ii) The part concerning pointwise convergence requires proof. We set

$$\bar{f}(x) = \lim_{n\to\infty} \sum_1^n |K_i g(x)|$$

where this limit is defined, and = 0 elsewhere. Then $\bar{f}$ is measurable (S) and, using (5),

$$\lambda_x(\bar{f}) = \sup_n \lambda_x(\sum_1^n |K_i g|) \leq \sup_n \sum_1^n \lambda_x(K_i g)$$
$$\leq \lambda_y^*(g) \sum_1^\infty \lambda_{xy}(K_i) < \infty .$$

Thus $\bar{f} \in L^{\lambda_x}$ and is finite in a set X_0 with $X - X_0$ λ_x-null. For $x \in X_0$, $\sum_1^n K_i g(x)$, n = 1, 2, ... , is a Cauchy sequence in R and defines a limit

$$f(x) = \sum_1^\infty K_i g(x).$$

We define f(x) = 0 in $X - X_0$. If K corresponds to a kernel K(x, y) which is a limit in norm of $\sum_1^n K_i(x, y)$,

$$\lambda_x(f - Kg) \leq \lambda_x(\sum_{n+1}^\infty |K_i g|) + \lambda_x(Kg - \sum_1^n K_i g),$$

and the second term on the right tends to zero as $n \to \infty$. Now $\sum_1^n |K_i g(x)|$ increases to $\sum_1^\infty |K_i g(x)|$ in X_0 whence, using (L 5) for λ_x,

$$\lambda_x[\sum_{n+1}^\infty |K_i g|] = \lim_{m\to\infty} \lambda_x[\sum_{n+1}^m |K_i g|]$$
$$\leq \lim_{m\to\infty} \sum_{n+1}^m \lambda_x(|K_i g|)$$
$$= \sum_{n+1}^\infty \lambda_x(|K_i g|)$$
$$\leq \lambda_y^*(g) \sum_{n+1}^\infty \lambda_{xy}(K_i) \to 0 \text{ as } n \to \infty.$$

Thus $\lambda_x(f - Kg) = 0$, $f - Kg \neq 0$ in a λ_x-null set and $\sum_1^n K_i g(x)$ converges to Kg(x) pointwise outside a λ_x-null set.

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COROLLARY. If $g \in L^{\lambda_y^*}$ and $K_i(x,y)$ is a Cauchy sequence in $L^{\lambda_{xy}}$ converging in norm to $K(x,y)$, if K_i, K are the bounded linear transformations with kernels $K_i(x,y)$, $K(x,y)$ then there is a subsequence K_{i_j} with $K_{i_j}g(x)$ (defined by (2)) converging pointwise to $Kg(x)$ outside a λ_x-null set.

We choose a subsequence $\{K_{i_j}(x,y)\}$ with

$$\lambda_{xy}(K_{i_1}) + \sum_{j=1}^{\infty} \lambda_{xy}(K_{i_{j+1}} - K_{i_j}) < \infty .$$

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