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ON A THEOREM OF MITCHELL

BY

E. R. BISHOP[1]

1. **Introduction.** In this note we apply a theorem of T. Mitchell on fixed points of contraction maps to a particular set of contraction maps of metric spaces. Also, part of Mitchell's theorem is extended to permit application, under suitable conditions, when the space involved is not compact.

We employ the following terminology due to Mitchell [1]. Let S be a semigroup; $m(S)$ the space of all bounded real functions on S with sup norm; and X a subset of $m(S)$. For all $s \in S$, the left translation l_s {right translation r_s} of $m(s)$ by s is given by $(l_s f)(s')=f(ss')$ $\{(r_s f)(s')=f(s's)\}$ where $f \in m(S)$ and $s' \in S$. A subspace X of $m(S)$ is left {right} translation invariant if $l_s f \in X$ $\{r_s f \in X\}$ for all $s \in S, f \in X$. X is translation invariant if it is both left and right translation invariant. Let X be a left translation invariant closed subalgebra of $m(S)$ that contains e, the constant 1 function on S. An element μ in X^* is a mean on S if $\|\mu\|=1$ and $\mu(e)=1$. A mean μ on X is left invariant if $\mu(l_s f)=\mu(f)$ for all $f \in X$; it is multiplicative if $\mu(f \cdot g) =\mu(f)\mu(g)$. Let Y be a compact Hausdorff space and η a homomorphism of S onto $\mathcal{S}$, a semigroup under composition of continuous maps of Y into itself. Let $Y'=\{y \in Y: Ty(C(Y)) \subseteq X\}$ where Ty is given by $(Tyh)(s)=h((\eta s)y)$ for $h \in C(Y)$, the space of continuous real functions on Y, and $s \in S$. The family $\mathcal{S}$ is an E-representation of S, X on Y if Y' is nonempty, a D-representation if Y' is dense in Y and an A-representation if $Y'=Y$. The pair S,X has the common fixed point property on compacta with respect to i-representations ($i=A$, D, or E) if for each compact Hausdorff space Y and for each i-representation of S, X on Y, there is in Y a common fixed point of the family $\mathcal{S}$.

THEOREM (Theorem 1 [1]). *Let S be a semigroup, X a translation invariant closed subalgebra of $m(S)$ that contains the constant functions. Then the following three conditions are equivalent:*

(1) *X has a left invariant mean.*

(2) *S,X has the common fixed point property on compacta with respect to E-representations.*

(3) *S,X has the common fixed point property on compacta with respect to D-representations.*

Each of the equivalent conditions (1), (2), *or* (3) *implies*

(4) *S,X has the common fixed point property on compacta with respect to A-representations.*

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2. A real valued function f is convex on the real interval U if the inequality

$$f(\lambda x+(1-\lambda)x') \leq \lambda f(x)+(1-\lambda)f(x')$$

holds for all pairs (x, x') of points of U and for all $\lambda \in [0, 1]$. By a convex modulus of continuity we mean a convex function α: $[0, \infty] \to [0, \infty]$ with $\alpha(0)=0$, $\lim_{x\to\infty} \alpha(x)/x=1$. We denote by I the identity map $\alpha(x)=x$. Let (Y, d) be a compact metric space; CV the set of convex moduli of continuity and S the multiplicative semigroup under functional composition of equivalence classes of CV defined by:

$$\alpha_1 \sim \alpha_2 \text{ iff } \alpha_1 \text{ is } 0(\alpha_2) \text{ and } \alpha_2 \text{ is } 0(\alpha_1) \text{ at } 0.$$

S contains the reals ≥ 1 by the injection $r \to [\alpha_r]$, where $[\alpha_r]$ is the class in CV containing functions equivalent to $\alpha_r(x)=x^r$ in a neighbourhood of 0 and suitably defined elsewhere.

For (Y, d) any metric space, $\alpha \in CV$, we say that a self-map t of Y satisfies the modulus of continuity α if for some $K>0$, $d(tx, ty) \leq K\alpha(d(x, y))$ for all x, y in Y. We note that if t satisfies the modulus of continuity α, it satisfies all equivalent moduli of continuity.

THEOREM 2.1. *Let (Y, d) be a compact metric space. For $[\alpha] \in S$, $[\alpha] \neq [I]$, let T_α denote all self-maps of Y satisfying moduli of continuity in $[\alpha]$; let T_1 denote the identity self-map of Y. Let $T \subset \bigcup_\alpha T_\alpha$ be a semigroup under functional composition such that $[\alpha] \to t_\alpha$, $t_\alpha \in T$, is a homomorphism of S onto T. Then T has a common fixed point in Y.*

Proof. By Theorem 1 [2], each t_α has a fixed point $p\alpha$ in Y. Either T consists only of the identity map or it is infinite, so there exists in Y an accumulation point p of the points $p\alpha$. Let $\mathscr{L}$ be the topology induced on S by the distance

$$\theta p([\alpha_1], [\alpha_2]) = d(p\alpha_1, p)+d(p\alpha_2, p).$$

Let X be the subalgebra of $\mathscr{L}$-continuous functions in $m(S)$ such that $f(\infty) =\lim_{x\to\infty} f(x)$ exists. X is translation-invariant since translation does not affect the above distance, and is closed in the sup norm. Let μ be the mean on X defined by $\mu(f)=f(\infty)$. μ is obviously translation-invariant and multiplicative.

For $g \in C(Y)$, consider the map $C(Y) \to X$ defined by

$$Tp\,(g(\alpha))=g(t_\alpha(p)).$$

For p as above, $Tp(g)$ is $\mathscr{L}$-continuous on S and so is in X. This gives an E-representation of S,X in Y, and the theorem follows.

3. We note: *If A is a function algebra on a set X with properties*:

(β) *If $f_1, \ldots, f_n$ are elements of A with no common zeroes, then there are elements $g_1, \ldots, g_n$ in A with $\sum g_i f_i=1$.*

(γ) *If $f \in A$ and f is not identically zero, there are a finite number of functions $f_1, \ldots, f_n$ in A which separate the zeroes of f.*

Then X is the spectrum of A and conversely. ([3, Proposition 5]).

Now let S be a semigroup, and Y a Hausdorff space. Let A be a subalgebra of $C(Y)$ having properties (β) and (γ). Let $\mathcal{S}$ be a semigroup of self-maps of Y such that $f \circ h \in A$ for $h \in \mathcal{S}, f \in A$. Let X be a translation invariant closed subalgebra of $m(S)$ which contains the constant functions.

THEOREM 3.1. *If X has a multiplicative left invariant mean μ then S,X has the common fixed point property on any Hausdorff space Y with respect to E-representations.*

Proof. Let η be a homomorphism of S onto $\mathcal{S}$ giving an E-representation of S,X on Y; i.e., there exists $z \in Y$ with $h(\eta s(z)) \in X$ for all $h \in C(Y)$, defining the map $T_z\colon A \to X$. Let T_z^* be the adjoint map, $T_z^*\colon X^* \to A^*$. Then

$$(T_z^*\mu)(1) = \mu(T_z(1)) = \mu(e) = 1.$$

Since T_z is easily seen to be multiplicative,

$$\begin{aligned}(T_z^*\mu)(hk) &= \mu(T_z(hk)) = \mu(T_zh\cdot T_zk)\\ &= \mu(T_zh)\mu(T_zk)\\ &= T_z^*\mu(h)T_z^*\mu(k), \quad h, k \in A.\end{aligned}$$

Thus $T_z^*\mu$ is a multiplicative nonzero linear functional on A, so it is an evaluation at some point $z' \in Y$. For $s \in S$, define $\theta_s\colon A \to A$ by $(\theta_s h)y = h(\eta s(y))$ for $y \in Y$. Then for $s' \in S$, $h \in A$;

$$\begin{aligned}T_z(\theta_s h)s' &= \theta_s h(\eta s'(z)) = h(\eta s(\eta s'(z)))\\ &= h(\eta(ss'(z)) = T_zh(ss') = l_s(T_zh(s'))\end{aligned}$$

where l_s is left translation by s.

So for $h \in A$, $s \in S$,

$$\begin{aligned}h(\eta s(z')) &= (\theta_s h)(z') = \mu(T_z(\theta_s h))\\ &= \mu(t_s(T_zh)) = \mu(T_zh) = h(z').\end{aligned}$$

Since A separates Y, $\eta s(z') = z'$ for all $s \in S$; so z' is the common fixed point of $\mathcal{S}$.

REFERENCES

1. T. Mitchell, *Function algebras, means and fixed points*, Trans. Amer. Math. Soc. **130** (1968), 117–126.

2. M. Edelstein, *On fixed and periodic points under contractive mappings* J. London Math. Soc. **37** (1962), 74–79.

3. H. L. Royden, *Function algebras*, Bull. Amer. Math. Soc. (4) **69** (1963), 281–298.

ACADIA UNIVERSITY,
WOLFVILLE, NOVA SCOTIA