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AN EQUATIONAL SPECTRUM GIVING CARDINALITIES OF ENDOMORPHISM MONOIDS[1]

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ABSTRACT. By determining the spectrum of a particular set of equations of type $\langle 2, 2, 0, 0, 0\rangle$ it is shown that a positive integer n is the cardinality of the endomorphism monoid of a universal algebra of the form $\mathfrak{A}\times\mathfrak{A}$ if and only if n is square.

It was shown in [2] that for any universal algebra $\mathfrak{U}$ the cardinality of End$(\mathfrak{U}\times\mathfrak{U})$ is square. Conversely, assuming the axiom of choice in the guise of the assertion that every infinite cardinal is its own square, one can readily deduce from the construction in Theorem 2.2 of [2] that every infinite cardinal is the power of the endomorphism monoid of $\mathfrak{U}\times\mathfrak{U}$ for a suitably chosen multi-unary algebra $\mathfrak{U}$. The goal of the present note is to establish this fact for finite non-zero squares as well.

By Theorem 1.3 of [2], the problem is equivalent to showing that the set of non-zero finite squares is contained in (hence equal to) the spectrum (i.e., the set of cardinalities of finite models) of the following set Σ of equations in two binary operation symbols, denoted by $*$ and juxtaposition, and three nullary operation symbols 1, d_0, and d_1.

$$\Sigma:\begin{cases} x(yz) = (xy)z \\ x1 = 1x = x \\ d_i d_j = d_i \qquad (i, j \in \{0, 1\}) \\ (x\, d_0) * (x\, d_1) = x \\ (x * y)\, d_0 = x\, d_0 \\ (x * y)\, d_1 = y\, d_1 \end{cases}$$

THEOREM. *For every positive integer n there is a multi-unary algebra* $\mathfrak{U}$ *such that* $|\mathrm{End}(\mathfrak{U}\times\mathfrak{U})| = n^2$.

Proof. As the theorem is trivial for $n=1$, assume $n>1$ and set $k=[n/2]$. Let M be any monoid of cardinality $2k$ containing an element t such that $t\neq t^2=e$ (the identity element) and there is a retraction ψ of M onto $\{e, t\}$. (E.g., take M to be the direct product of any k-element monoid with the two-element group.) If n is odd let N denote the monoid obtained from M by adjoining a new element 0 and

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extending multiplication in the usual way, $x0=0x=0$ for all $x \in N$. If n is even set $N=M$; in either case $|N|=n$. Let e' and t' respectively denote $\{e\}\psi^{-1}$ and $\{t\}\psi^{-1}$ if n is even, but if n is odd set $e'=\{e\}\psi^{-1} \cup \{0\}$ and $t'=\{t\}\psi^{-1} \cup \{0\}$.

Define on $N \times N$ an algebraic system of type $\langle 2, 2, 0, 0, 0\rangle$ as follows. Set $1=\langle e, e\rangle$, $d_0=\langle e, t\rangle$, $d_1=\langle t, e\rangle$. Letting x_0 and x_1 denote respectively the left and right components of an element x of $N \times N$, define a binary operation $*$ by setting $x * y=\langle x_0, y_1\rangle$ for all $x, y \in N \times N$. To define the remaining binary operation, first note that $N \times N=A \cup B \cup C \cup D$, where $A=e' \times e'$, $B=e' \times t'$, $C=t' \times e'$, and $D=t' \times t'$. Now define multiplication by stipulating that for all $x, y \in N \times N$,

$$xy = \begin{cases} \langle x_0y_0, x_1y_1\rangle & \text{if} \quad y \in A, \\ \langle x_0y_0, x_0y_1\rangle & \text{if} \quad y \in B, \\ \langle x_1y_0, x_1y_1\rangle & \text{if} \quad y \in C, \\ \langle x_1y_0, x_0y_1\rangle & \text{if} \quad y \in D. \end{cases}$$

(Note that if n is odd it is necessary to observe that this multiplication is well-defined.)

Since ψ is identity on $\{e, t\}$ we have $1 \in A$, $d_0 \in B$, and $d_1 \in C$. Verification that the equations in Σ are identities in the structure just defined is rather immediate except for the first equation, associativity of multiplication, whose verification requires sixteen cases, arising from the respective assignment of y and z to A, B, C, D. However, each of the cases is very easily checked once one knows that whenever $Y, Z \in \{A, B, C, D\}$ there is a unique $W \in \{A, B, C, D\}$ such that $yz \in W$ for all $y \in Y$ and $z \in Z$. Using the fact that ψ is an endomorphism of M it is easily shown that such a W exists and is given as the intersection of the Y-row and Z-column in the table

	A	B	C	D
A	A	B	C	D
B	B	B	B	B
C	C	C	C	C
D	D	C	B	A.

Thus Σ has a model of power n^2, and the proof is concluded.

The fact that every model of Σ (hence every monoid of the form End$(\mathfrak{U} \times \mathfrak{U})$ for a universal algebra $\mathfrak{U}$) has square cardinality was shown in [2] by observing that in any model M of Σ the map $x \to \langle xd_0, xd_1\rangle$ is a bijection of M onto $Md_0 \times Md_0$. An alternative proof can be obtained by noting that any non-trivial model of Σ is, with respect to $*$, a rectangular band admitting an anti-automorphism of order two, namely the map $x \to x(d_1 * d_0)$; a result of Evans [1] asserts that the cardinality of such a band is square.

Finally, we remark that the construction used in proving the theorem of this note bears some similarity to the proof of Theorem 3.1 of [2]. Moreover it can be shown that in the case where n is even the present result follows from the construction in Theorem 3.1.

REFERENCES

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