A GENERALIZATION OF THE YOUNG DIAGRAM

M. D. BURROW

1. Introduction. The method of A. Young for finding the set of primitive idempotents of the group algebra of the symmetric group is classical; it was first given by Frobenius **(4)** using results of Young **(10** and **11)**. A concise account can be found in **(9)** and a very detailed treatment in **(6)**.

From the purely algebraic point of view Young's method consists of finding pairs of subgroups R and C of the symmetric group S_n so that if

$$P = \sum_{r \in R} r, \qquad N = \sum_{c \in C} c\, \sigma(c),$$

where $\sigma(c) = \pm 1$ according as c is an even or odd permutation, then PN is a multiple of a primitive idempotent of the group algebra of S_n. This will be the case if R and C satisfy a condition of von Neumann. Below, in Lemma 1, we show that a more general formulation of his condition applicable to any group is possible in algebraic terms. An application of this new condition to the group GL$(2, q)$ is given in §§5–8 of this paper. In Lemma 2 we show that the condition is equivalent to a property of the representations of the group induced by the linear representations of R and C viz., that they have a single irreducible component in common, and neither induced representation contains this component more than once.

2. A Lemma on primitive idempotents.

LEMMA 1. *Let two subgroups R and C of a group G have representations of the first degree θ and ϕ respectively. If for any element $s \in G$ the condition*

$$s \in CR \rightleftarrows \theta(r) = \phi(c)$$

holds for every pair of elements $r \in R$, $c \in C$ for which $srs^{-1} = c$ then $e = PN$ is a multiple of a primitive idempotent, where

$$P = \sum_{r \in R} r\, \theta(r), \qquad N = \sum_{c \in C} c\, \phi(c).$$

Proof. First note that

$$Pr_1\theta(r_1) = \sum_{r \in R} r\, \theta(r)\, r_1\theta(r_1) = \sum_{r \in R} rr_1\theta(rr_1) = P,$$

where r_1 is any element of R. Similarly $Nc_1\phi(c_1) = N$. Consider the expression $PNsPN$. If $s \in CR$, $s = cr$ say, then

$$PNsPN = PNcrPN = \theta^{-1}(r)\, \phi^{-1}(c) PNPN = \theta^{-1}(r)\, \phi^{-1}(c)\,(PN)^2.$$

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On the other hand if $s \notin CR$ then the condition of the lemma implies the existence of a pair $r \in R$ and $c \in C$ such that $srs^{-1} = c$ and $\theta(r) \neq \phi(c)$. In this case

$$PNsPN = \theta(r)\, PNsrs^{-1}\, sPN = \theta(r)\, PNcsPN = \theta(r)\, \phi^{-1}(c)\, PNsPN.$$

Hence:

$$PNsPN(1 - \theta(r)\, \phi^{-1}(c)) = 0.$$

Since $\theta(r) \neq \phi(c)$, we have $PNsPN = 0$. Writing $e = PN$ we get

$$eAe = \Lambda e^2 \tag{2.1}$$

where A is the group algebra of G over the field of representation Λ. Note that $e^2 \neq 0$, otherwise $eAeA = 0$ and eA is a nilpotent right ideal, whereas the group algebra is semi-simple. Also $eA \neq 0$ otherwise $e = 0$ which is impossible. In fact the coefficient of the unit element I in PN is $\sum\theta(\dot{r})\, \phi(\dot{c})$, and the summation is over all $\dot{r}$, $\dot{c}$ for which $\dot{r}\dot{c} = I$, i.e., over all $\dot{r} \in R \cap C$. Now $I\dot{r}I^{-1} = \dot{c}^{-1}$ and since $I \in CR$ the condition of the lemma gives $\theta(\dot{r}) = \phi^{-1}(\dot{c})$. Hence the coefficient of I in PN is $\sum\theta(\dot{r})\, \theta^{-1}(\dot{r}) = R \cap C: 1 \neq 0$. By considering the expression $Ps^{-1}N$ and reasoning exactly as above we find that $PAN = \Lambda PN = \Lambda e$. Since $PAN \supset PNAPN = eAe$ we get

$$\Lambda e \supset eAe. \tag{2.2}$$

Combining equations (2.1) and (2.2) we have $e^2 = \lambda e$, so that e is a multiple of an idempotent. Besides it is seen from (1) that eAe is a field and hence by a well-known theorem e is a multiple of a primitive idempotent (**1**, p. 36).

3. A necessary and suffcient condition.

LEMMA 2. *The condition of Lemma 1 is satisfied if and only if the representations of G induced by the linear representations of R and C have one and only one irreducible component in common, and neither induced representation contains this component more than once.*

Proof. (a) First assume that the condition of Lemma 1 holds. Let

$$e_R = \frac{1}{R:1} P = \frac{1}{R:1} \sum r\, \theta(r),$$
$$e_C = \frac{1}{C:1} N = \frac{1}{C:1} \sum c\, \phi(c),$$

where R: 1 and C: 1 are the orders of the subgroups R and C respectively and the summations are taken over all $r \in R$ and $c \in C$. Since $Pr\, \theta(r) = P$ (proof of Lemma 1) we see that $P^2 = R$: 1. so that $e_R{}^2 = e_R$. Similarly $e_C{}^2 = e_C$. Then e_R and e_C are primitive idempotents of the subalgebras over R and C respectively (**5**, p. 46). We have

$$e_R = \sum_j e^j, \quad e_C = \sum_j \tilde{e}^j$$

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where e^j, $\tilde{e}^j$ are indempotents or 0 belonging to the jth Wedderburn component of A. Now because the condition of Lemma 1 holds:

$$\dim e_C A e_R = \dim \sum \tilde{e}^j A\, e^j = 1,$$

therefore $\tilde{e}^j A\, e^j = 0$ for all j except one, say $j = k$, and we have either $e^j = 0$ or $\tilde{e}^j = 0$ if $j \neq k$. Moreover $\dim \tilde{e}^k A e^k = 1$ so that e^k and $\tilde{e}^k$ are primitive idempotents; hence the right ideals $e_R A$ and $e_C A$ which give the representations of G induced by the linear representations θ and ϕ of R and C respectively have a single minimal right ideal in common.

(b) Assume that the induced representations of G have only one component in common, each containing it with multiplicity one. Let

$$e_R = e + \ldots, \quad e_C = \tilde{e} + \ldots,$$

where e and $\tilde{e}$ are from the same Wedderburn component and the decompositions have no other component in common. We may suppose that $(e_R e_C)^2 \neq 0$ since under present assumptions this condition can always be secured by transforming the group C with a suitable element g of G. Thus:

$$(e_R g e_C g^{-1})^2 = eg\tilde{e}g^{-1}eg\tilde{e}g^{-1} = (eg\tilde{e}g^{-1}e)\, g\tilde{e}g^{-1}$$

and the last expression in brackets cannot vanish for all $g \in G$ otherwise:

$$0 = \sum_g eg\tilde{e}g^{-1}e = e(\sum_g g\tilde{e}g^{-1})\, e = \lambda e^2 \neq 0,$$

the final step arising from the fact that the bracketted expression is a central element of the subalgebra to which e belongs. Hence with suitable choice of g: $eg\tilde{e}g^{-1}e \neq 0 \rightarrow eg\tilde{e} \neq 0$ and since e is a primitive idempotent $eg\tilde{e}g^{-1}e = \lambda_g\, e$. Returning to the first equation:

$$(e_R g e_C g^{-1})^2 = \lambda_g (eg\tilde{e})\, g^{-1} \neq 0.$$

Now $0 \neq (e_R e_C)^2 = e\tilde{e}e\tilde{e}$, so that $e_R e_C$ is a multiple of a primitive idempotent. Moreover $e_C e_R = \tilde{e}e \neq 0$. Also:

$$e_C\; s\; e_R = \tilde{e}se = \mu_s \tilde{e}e = \mu_s\, e_C\, e_R.$$

Because the last expression has only terms of the form cr, the same is true of the first. Therefore $s \notin CR \rightarrow \mu_s = 0$. On the other hand if $s \in CR$ then it is clear that $\mu_s \neq 0$. Let us suppose that $s \notin CR$ so that $e_C\; s\; e_R = 0$, i.e.:

$$\sum c\; s\; r\; \theta(r)\; \phi(c) = 0.$$

Consider terms of the form cs in the above; such exist, e.g. when $r = I$, the unit element. These terms occur only when $sr = c_r\, s$. Hence expressions with cs are

$$\sum{}'\; c\; c_r\; s\; \theta(r)\; \phi(c)$$

where the sum is now over c and such r for which $sr = c_r\, s$, i.e., for which $srs^{-1} = c_r$. We have then:

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$$\sum{}' c\, c_r\, s\, \theta(r)\, \phi(c) = \sum{}' cs\, \theta(r)\, \phi(c)\, \phi^{-1}(c_r).$$

Now Lemma 1 required that for $s \notin CR$ there should exist r, c_r with $srs^{-1} = c_r$ and $\theta(r) \neq \phi(c_r)$; hence to negate this condition of the lemma we assume that the equality always holds and we get

$$\left(\sum c\, \phi(c)\right) s = 0 \rightarrow e_C = 0.$$

which is impossible. Therefore the condition of Lemma 1 must be satisfied. We have now established Lemma 2.

4. Calculation of the character. The character of the representation corresponding to the idempotent derived from PN (Lemma 1) can be calculated by the formula

$$\chi(g) = \frac{n}{i(R \cap C: 1)} \sum \theta(r)\, \phi(c), \tag{4.1}$$

where χ is the character of the irreducible representation corresponding to the primitive idempotent formed from R and C; n is the degree of the irreducible representation; i is the index of the normalizer of the element g; r, c are elements of R and C and θ, ϕ are their respective signatures. The summation is taken over all r, c for which $rc \in \mathfrak{C}(g)$, the class of elements conjugate to g.

Proof of (4.1). In the first place,

$$\sum_{s \in G} s(PN)\, s^{-1}$$

is an element of the centre of the subalgebra *to which PN belongs*[1]; moreover the expression:

$$\sum_{t \in G} t\, \chi(t)$$

is the central idempotent of this subalgebra up to a multiple. Hence:

$$\lambda \sum_{t \in G} t\, \chi(t) = \sum_{s \in G} s(PN)\, s^{-1}.$$

Recalling that $PN = \sum rc\, \theta(r)\, \phi(c)$ and equating coefficients of g on both sides we get:

$$\lambda(\chi(g)) = \sum{}' \theta(r)\, \phi(c)$$

where the summation is over all r, c for which, for some s, $srcs^{-1} = g$. Now if this relation holds for a particular element s then it holds also for the element hs if h is an element of the normalizer $N(g)$ of g: $(hs)\, rc(hs)^{-1} = hgh^{-1} = g$. It follows that the contribution to the sum from each r, c for which $rc \in \mathfrak{C}(g)$ is repeated $N(g)$: 1 times. This permits us to write:

$$\lambda\chi(g) = (N(g):\ 1) \sum \theta(r)\, \phi(c), \tag{4.2}$$

[1] PN remains a multiple of a primitive idempotent even after extension of Λ to an algebraically closed field so that actually the centre of the Wedderburn component is of dimension 1.

the summation now being over all r, c such that $rc \in \mathfrak{C}(g)$. In particular if g is the identity element I of the group G then $N(g)\colon 1 = G\colon 1$ and $rc = I$ so that r and c must be from $R \cap C$ and the condition of Lemma 1 requires that $\theta(r) = \phi^{-1}(c)$. In consequence:

$$\lambda n = (G\colon 1)(R \cap C{:}1)$$

where $n = \chi(I)$ is the degree of the irreducible representation. Substitution for λ in (4.2) gives the result (4.1).

5. Application to GL(2, q). In the following paragraphs the preceding theory is used to find primitive idempotents of the group algebra of GL(2, q) as well as the actual bases for the corresponding irreducible representations. For this group there are **(7; 8)**

(a)	$q - 1$	irreducible representations of degree 1,
(b)	$q - 1$	irreducible representations of degree q,
(c)	$\frac{1}{2}(q-1)(q-2)$	irreducible representations of degree $q + 1$,
(d)	$\frac{1}{2}q(q-1)$	irreducible representations of degree $q - 1$.

In each of the cases (a), (b), and (c) we find bases for the complete matrix algebra of the Wedderburn component. The writer has not been able to obtain similar results for the representations of (d) by the present method in the general case. For GL(2, 5) whose factor group with respect to its centre is S_5 the R and C subgroups for a representation of degree $q - 1 = 4$ can be obtained from the appropriate Young tableau for S_5.

6. Primitive idempotents of the group algebra of GL(2, q). We now obtain a pair of subgroups R and C of GL(2, q) which satisfy Lemma 1. By varying the signatures of R and C different primitive idempotents are obtained which will be classified in the next paragraph. The condition of Lemma 1 will be trivially satisfied if $R \cap C = I$ and $(R\colon 1)(C\colon 1) = G\colon 1$, for then $G = CR$ and

$$sRs^{-1} \cap C = crRr^{-1}c^{-1} \cap C = cRc^{-1} \cap C = I,$$

so that

$$\phi(sRs^{-1} \cap C) = 1 = \theta(R \cap s^{-1}Cs).$$

The order of GL(2, q) is $q(q-1)(q^2-1)$; **(2)**. It is easy to find subgroups of orders $q(q-1)$ and $q^2 - 1$ having only the identity I in common; take for R the triangular subgroup

$$R = \left\{\begin{pmatrix} \alpha & \beta \\ & 1 \end{pmatrix}\right\}$$

where α is any non-zero mark of GF(q) and β is any mark of this Galois field of

$q = p^n$ elements. Then $R: 1 = q(q-1)$. If ρ is a primitive element of $\mathrm{GF}(q)$ and $\alpha = \rho^a$ then

$$\theta\begin{pmatrix}\alpha & \beta \\ & 1\end{pmatrix} = \epsilon^a,$$

with ϵ a root of $x^{q-1} = 1$ in the field of the representation, the field of complex numbers say, gives a representation of R of the first degree. Each root of this equation gives a distinct linear representation and we get them all in this way since

$$\left\{\begin{pmatrix}1 & \beta \\ & 1\end{pmatrix}\right\}$$

is the commutator of R and its index in the latter is $q - 1$.

For the subgroup C of order $C: 1 = q^2 - 1$ we take the cyclic group generated by an element of $\mathrm{GL}(2, q)$ similar to

$$\begin{pmatrix}\sigma & \\ & \sigma^q\end{pmatrix}$$

in which σ is a primitive root of the quadratic extension field $\mathrm{GF}(q^2)$. That is

$$C = \left\{T\begin{pmatrix}\sigma & \\ & \sigma^q\end{pmatrix}^m T^{-1}\right\} \tag{6.1}$$

where T is chosen so that the elements lie in $\mathrm{GL}(2, q)$. Now

$$\phi\left(T\begin{pmatrix}\sigma^m & \\ & \sigma^{mq}\end{pmatrix} T^{-1}\right) = \omega^m$$

where ω is a root of the equation $x^{q^2-1} = 1$ can clearly give all $q^2 - 1$ linear representations of the cyclic group C. Recalling the definition of P and N (Lemma 1) we see that

$$PN = \sum_{\beta} \sum_{a=1}^{q-1} \sum_{m=1}^{q^2-1} \begin{pmatrix}\rho^a & \beta \\ & 1\end{pmatrix} T\begin{pmatrix}\sigma^m & \\ & \sigma^{mq}\end{pmatrix} T^{-1} \epsilon^a \omega^m$$

is a multiple of a primitive indempotent for each choice of ϵ and ω.

7. Classification of the primitive idempotents. The primitive idempotents of the preceding section can be distinguished through the values of the corresponding irreducible characters on a suitable element of $\mathrm{GL}(2, q)$. We use for the calculation the formula (4.1).

Let us calculate $\chi(g_1)$ for

$$g_1 = \begin{pmatrix}\rho^{a_1} & \\ & \rho^{b_1}\end{pmatrix}, \qquad a_1 \neq b_1.$$

Here $N(g): 1 = q(q+1)$. Recall that $R \cap C: 1 = 1$. A simple choice for T in (6.1) can be obtained by assuming a matrix with unknown coefficients and then determining them so as to ensure that (6.1) lie in $GL(2, q)$. We shall use

$$T = \begin{pmatrix} 1 & 1 \\ \sigma & \sigma \end{pmatrix}.$$

With this choice of T we have, for C,

$$c = \left\{ \begin{pmatrix} (\sigma^{m+q} - \sigma^{mq+1})\,\Delta^{-1} & (\sigma^{mq} - \sigma^{m})\,\Delta^{-1} \\ -\,\sigma^{q+1}(\sigma^{mq} - \sigma^{m})\,\Delta^{-1} & (\sigma^{(m+1)q} - \sigma^{m+1})\,\Delta^{-1} \end{pmatrix} \right\}, \qquad \Delta = \sigma^{q} - \sigma,$$

so that

$$rc = \begin{pmatrix} \alpha & \beta \\ & 1 \end{pmatrix} \begin{pmatrix} (\sigma^{m+q} - \sigma^{mq+1})\,\Delta^{-1} & (\sigma^{mq} - \sigma^{m})\,\Delta^{-1} \\ -\sigma^{q+1}(\sigma^{mq} - \sigma^{m})\,\Delta^{-1} & (\sigma^{(m+1)q} - \sigma^{m+1})\,\Delta^{-1} \end{pmatrix}.$$

Since g_1 has two distinct eigenvalues the requirement that $rc \in \mathfrak{C}(g_1)$ will be satisfied if we make sure that trace (rc) = trace g_1 and that determinant (rc) = determinant g_1. These conditions yield

$$(7.1) \qquad \begin{gathered} \alpha(\sigma^{m+q} - \sigma^{mq+1}) - \sigma^{q+1}(\sigma^{mq} - \sigma^{m})\,\beta + \sigma^{(m+1)q} - \sigma^{m+1} = (\rho^{a_1} + \rho^{b_1})\Delta, \\ \alpha\,\sigma^{m(q+1)} = \rho^{a_1+b_1}. \end{gathered}$$

For fixed m the latter equation determines α: $\alpha = \rho^{a_1+b_1-m}$. Here σ and ρ have been so related that $\sigma^{q+1} = \rho$. Now β is uniquely determined by the former equation if $\sigma^m - \sigma^{mq} \neq 0$. In this case r is fully determined for a given c and

$$\phi(c) = \omega^{m}, \quad \theta(r) = \epsilon^{a_1+b_1-m}.$$

On the other hand if $\sigma^m - \sigma^{mq} = 0$ then β may be any of the q marks of $GF(q)$; also $m = t(q+1)$ where $1 \leqslant t \leqslant q-1$. Since $\sigma^{q+1} = \rho$ and $\rho^q = \rho$, the first equation of (7.1) gives

$$\rho^{a_1+b_1-2t}\,\rho^{t} + \rho^{t} = \rho^{a_1} + \rho^{b_1},$$

and after simplification

$$(\rho^{t})^2 - (\rho^{a_1} + \rho^{b_1})\,\rho^{t} + \rho^{a_1+b_1} = 0,$$

so that either $t = a_1$ or $t = b_1$. There are then just two possibilities for the element c determined by $m = a_1(q+1)$ and by $m = b_1(q+1)$. Each of these determines q possibilities for the element r. Also each value of m fixes the signature θ of the corresponding elements r through the second equation of (7.1), so that for one case

$$\phi(c) = \omega^{a_1(q+1)}, \quad \theta(r) = \epsilon^{b_1-a_1},$$

and for the other

$$\phi(c) = \omega^{b_1(q+1)}, \quad \theta(r) = \epsilon^{a_1-b_1}.$$

We are now able to write:

$$\sum \theta(r)\,\phi(c) = \sum_{m=1}^{q^2-1} \epsilon^{a_1+b_1-m}\,\omega^{m} + q[\epsilon^{b_1-a_1}\,\omega^{a_1(q+1)} + \epsilon^{a_1-b_1}\,\omega^{b_1(q+1)}];$$

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equation (4.1) now gives:

$$(7.2)\quad \chi(g_1) = \frac{n}{q(q+1)}\left\{ \sum_{m=1}^{q^2-1} \epsilon^{a_1+b_1-m}\omega^m + q(\epsilon^{b_1-a_1}\omega^{a_1(q+1)} + \epsilon^{a_1-b_1}\omega^{b_1(q+1)}) \right\},$$

where $m \not\equiv 0(\text{mod } q+1)$. The following cases can be considered:

Case I: $\omega = \epsilon$. Then $\chi(g_1) = n\epsilon^{a_1+b_1}$ and each of the $q-1$ roots ϵ of the equation $x^{q-1} = 1$ gives rise to a distinct character of degree n. Since g_1 is not a central element we know that $n = 1$ and that these are the $q-1$ linear characters **(7)**.

Case II: $$(\omega/\epsilon)^{q+1} = 1, \quad \omega \neq \epsilon.$$

Now $\chi(g_1) = (n/q)\,\epsilon^{a_1+b_1}$ and we have $q-1$ distinct characters, one for each ϵ. Their degree is $n = q$. We remark that each choice of ϵ gives a distinct character of degree q but that for fixed ϵ, ω can take q values. In this way we get q distinct idempotents associated with each irreducible representation of degree q. This remark will be useful in the next section.

Case III: $$(\omega/\epsilon)^{q+1} \neq 1.$$

In this case the summation term in equation (7.2) above is zero and we have:

$$\chi(g_1) = \frac{n}{q+1}(\epsilon^{b_1-a_1}\,\omega^{a_1(q+1)} + \epsilon^{a_1-b_1}\,\omega^{b_1(q+1)}).$$

Writing $\omega^{q+1} = \epsilon/\epsilon_1$, where ϵ_1 is a different root of the equation $x^{q-1} = 1$, we get finally:

$$\chi(g_1) = \frac{n}{q+1}(\epsilon^{b_1}\,\epsilon_1^{-a_1} + \epsilon^{a_1}\,\epsilon_1^{-b}).$$

In this formula ϵ can take $q-1$ values, to each of which ϵ_1 may take $q-2$ values, since $\epsilon_1 \neq \epsilon$. Hence values may be assigned to both in $(q-1)(q-2)$ ways; however, half of these lead to the same character as the other half. The results indicate that these are the $\frac{1}{2}(q-1)(q-2)$ irreducible characters of degree $q+1$. We note that ϵ and ϵ_1 fix the character but that ω is free to take $q+1$ values, giving rise to $q+1$ distinct idempotents belonging to the same irreducible representation of degree $q+1$.

We have now obtained, to within a multiple λ, primitive idempotents for all the irreducible representations of degrees 1, q, and $q+1$. The idempotents themselves can be determined since the trace $\chi_R(\lambda PN)$ in the regular representation is equal to the degree of the irreducible representation.

The irreducible representations of degree $q-1$ have not appeared. The writer has been unable to find them by other choices of R and C which have merely led to one or other of the representations already obtained.

8. Bases for the irreducible subalgebras. For the linear representations there is nothing to be discussed as each idempotent is already a basis and the linear characters are in fact representations.

Recalling Cases II and III of the previous section we see that for each of the irreducible representations of degree q or $q + 1$ there are as many distinct primitive idempotents as the degree n; these, together with an equal number of primitive idempotents obtained by reversing the roles of R and C, will be used to construct the n^2 basis elements of the matrix subalgebra.

We notice that in both cases II and III the signature ϵ remains fixed for all the equivalent idempotents; the changes in ω distinguish them. Thus the terms P in

$$\dot{e}_i = \lambda P N_i$$

are the same[2] for all the idempotents $\dot{e}_i$. The N_i stand for N under the different choices of ω. Now:

$$\dot{e}_i\dot{e}_j = \lambda^2 PN_iPN_j = \mu\lambda^2 PN_j = \nu\dot{e}_j.$$

The second step is from the fact that $PAN = \Lambda PN$ (§2). Thus

$$\dot{e}_i\dot{e}_i\dot{e}_j = \nu\dot{e}_i\dot{e}_j = \nu^2\dot{e}_j = \dot{e}_i\dot{e}_j = \nu\dot{e}_j$$

and hence

$$(\nu^2 - \nu)\,\dot{e}_j = 0,$$

implying that either $\nu = 0$ or $\nu = 1$, and so $\dot{e}_i\dot{e}_j = \dot{e}_j$ or 0. Similarly $\dot{e}_j\dot{e}_i = \dot{e}_i$ or 0.

LEMMA 3. $$\dot{e}_i\dot{e}_j = \dot{e}_j \rightleftarrows \dot{e}_j\dot{e}_i = \dot{e}_i.$$

Proof. If A is the group algebra then

$$\dot{e}_i\dot{e}_jA = \dot{e}_jA \subset \dot{e}_iA$$

and, since $\dot{e}_iA =$ is minimal, $\dot{e}_jA = \dot{e}_iA \rightarrow \dot{e}_i = \dot{e}_jx$, so that

$$\dot{e}_j\dot{e}_i = \dot{e}_j\dot{e}_jx = \dot{e}_jx = \dot{e}_i.$$

COROLLARY. $$\dot{e}_i\dot{e}_j = 0 \rightleftarrows \dot{e}_j\dot{e}_i = 0.$$

However this is not possible; for let $\dot{e}_i\dot{e}_j = 0 = \dot{e}_j\dot{e}_i$, then $\dot{e}_i$, $\dot{e}_j$ are primitive mutually orthogonal idempotents of a matrix algebra: we identify them with, say, e_{11} and e_{22}. Then

$$\dot{e}_iA\dot{e}_j = e_{11}Ae_{22} = \Lambda e_{12} \neq 0;$$

but

$$\dot{e}_iA\dot{e}_j = \lambda^2 PN_iAPN_j = \Lambda PN_j = \Lambda\dot{e}_j,$$

so that $e_{12} = \tau\dot{e}_j$ and this is impossible. Hence

$$\dot{e}_i\dot{e}_j = \dot{e}_j. \tag{8.1}$$

Now we interchange R and C, i.e., the group formerly taken for C will be used for R and vice versa. In terms of the original R and C the idempotents are now:

$$e = \lambda NP.$$

[2] λ also remains the same since the coefficient of I in PN_i is 1, so that $\chi_R(\dot{e}_i) = \lambda(G:1)$.

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Since the trace and determinant of an element cr is the same as that of the element rc the equations (7.1) are not changed and the features of cases II and III remain the same. Let $e_1, e_2, \ldots e_j, \ldots$ be the new system of equivalent idempotents belonging to a particular irreducible representation of degree q or $q+1$, so that $e_i = \lambda N_i P$. As for the $\dot{e}_i$, we prove for e_i in an entirely analogous way:

$$e_i e_j = e_i. \tag{8.2}$$

Lemma 4. *For the systems $\dot{e}_i$ and e_i of a particular representation:*

$$\begin{aligned} \dot{e}_i e_j &= 0, & i \neq j,\\ \dot{e}_i e_i &\neq 0, & \\ e_i \dot{e}_j &\neq 0, & \textit{for all } i, j. \end{aligned} \tag{8.3}$$

Proof. In the first place $N_i N_j$ must vanish since N_i, N_j are multiples of different smallest central idempotents of the group algebra of C. Thus $\dot{e}_i e_j = \lambda^2 P N_i N_j P = 0$. On the other hand

$$\dot{e}_i e_i = \lambda^2 P N_i N_i P = \lambda^2 (C\colon 1)\, P N_i P \neq 0,$$

otherwise on right multiplication by N_i we would get

$$\lambda^2 P N_i P N_i = \dot{e}_i^2 = \dot{e}_i = 0.$$

Moreover,

$$e_i \dot{e}_j = \lambda^2 N_i P P N_j = \lambda^2 (R\colon 1)\, N_i P N_j \neq 0,$$

otherwise on left multiplication by P we should get

$$\lambda^2 P N_i P N_j = 0 = \dot{e}_i \dot{e}_j = \dot{e}_j.$$

Relations (8.3) show $\dot{e}_i$ and e_j are distinct; for if $\dot{e}_i = e$

$$\dot{e}_i = \dot{e}_i \dot{e}_i = \dot{e}_i e_j = 0.$$

Again if $\dot{e}_i = e_i$ then

$$\dot{e}_i = \dot{e}_j \dot{e}_i = \dot{e}_j e_i = 0.$$

Lemma 5. *A matrix basis for the irreducible subalgebra, corresponding to a particular idempotent of degree q or $q+1$, is given by $E_{ij} = e_i \dot{e}_j$ with suitable normalization.*[3]

Proof.

$$\begin{aligned} E_{ij} E_{km} &= e_i \dot{e}_j e_k \dot{e}_m = 0, \qquad\qquad \text{if } j \neq k.\\ E_{ij} E_{jm} &= \lambda^3 e_i P N_j N_j P P N_m = \lambda^3 (C\colon 1)(R\colon 1)\, e_i P N_j P N_m\\ &= \lambda (C\colon 1)(R\colon 1)\, e_i \dot{e}_j \dot{e}_m = \lambda (C\colon 1)(R\colon 1)\, e_i \dot{e}_m\\ &= \lambda (C\colon 1)(R\colon 1)\, E_{im}. \end{aligned}$$

[3]The referee has kindly drawn the author's attention to an interesting paper by Frame **(3)** in which a pair of subgroups is used to give an irreducible representation of the group.

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The normalized system is thus $\tilde{E}_{ij} = E_{ij}/\lambda(C{:}\ 1)(R{:}\ 1)$, for then

$$\tilde{E}_{ij}\tilde{E}_{km} = 0, \qquad j \neq k,$$

and $\tilde{E}_{ij}\tilde{E}_{jk} = \tilde{E}_{ik}$. The λ is known from the regular trace. The $\tilde{E}_{ii}$ are primitive idempotents since

$$\tilde{E}_{ii}A\tilde{E}_{ii} = e_i\dot{e}_iAe_i\dot{e}_i = \Lambda e_i\dot{e}_i = \Lambda\tilde{E}_{ii},$$

the second step arising from the fact that e_i is a primitive idempotent. The $\tilde{E}_{ij}$ give bases for the actual construction of any of the irreducible representations of degree q or $q + 1$.

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McGill University